Limits Involving (Approaching) Infinity: $\lim_{x\to\infty} f(x)$

Important Theorem: $\lim_{x\to\infty} \frac{1}{x} = 0$

Limits Involving Infinity (Principle of Dominance)
1. $\lim_{x\to\infty} \frac{x^a}{x^b}$, *if* $a<b$. Then the limit is equal to 0. (Look at the degree of the numerator and denominator, i.e., the greatest power/exponent of x of the polynomial in the numerator and the polynomial in the denominator.)
2. $\lim_{x\to\infty} \frac{Cx^a}{Dx^b}$, *if* $a=b$. Then the limit is a ratio of leading coefficients, $\frac{C}{D}$. (Look at the degree of the numerator and denominator, i.e., the greatest power/exponent of x of the polynomial in the numerator and the polynomial in the denominator.)
3. $\lim_{x\to\infty} \frac{x^a}{x^b}$, *if* $a>b$. Then the limit is equal to ∞ or $-\infty$. (Look at the degree of the numerator and denominator, i.e., the greatest power/exponent of x of the polynomial in the numerator and the polynomial in the denominator. **MUST** check the sign of ∞ by substituting into the rational function a sufficiently large value for every x.)

Directions: Show your work (write out your explanations) on a separate sheet of paper.

Problems:

1. $\lim_{x\to\infty} 7+\frac{1}{3x}-\frac{2}{x^2}$	2. $\lim_{x\to-\infty} \frac{4x+8}{5x}$	3. $\lim_{x\to\infty} \frac{3x-1000}{x+100}$	4. $\lim_{x\to-\infty} \frac{5x+5}{7x^2+1}$
5. $\lim_{x\to\infty} \frac{5x^2+2}{4x^2+7}$	6. $\lim_{x\to-\infty} \frac{3x^3+5}{5x^2+1}$	7. $\lim_{x\to\infty} \frac{2x^2-4x}{x+1}$	8. $\lim_{x\to-\infty} \frac{2x^2-4x}{x+1}$
9. $\lim_{x\to\infty} \frac{3x^3+2}{5x^2-1}$	10. $\lim_{x\to-\infty} \frac{3x^2+2}{4x^2-1}$	11. $\lim_{x\to\infty} \frac{x^2+2}{x-555}$	12. $\lim_{x\to-\infty} \frac{3-2x}{3x^3-1}$
13. $\lim_{x\to\infty} \frac{3-5x}{3x-1}$	14. $\lim_{x\to\infty} \frac{3-2x^2}{3x-1}$	15. $\lim_{x\to\infty} \frac{6x^2-2x-1}{2x^2+3x+2}$	16. $\lim_{x\to\infty} \frac{3x^3+2}{2x^2-9x^3+7}$
17. $\lim_{x\to-\infty} \frac{x}{x^2-1}$	18. $\lim_{x\to-\infty} \frac{8x^2+3x}{2x^2-1}$	19. $\lim_{x\to\infty} 10-\frac{2}{x^2}$	20. $\lim_{x\to-\infty} 4+\frac{3}{x}$
21. $\lim_{x\to-\infty} \frac{5x^2}{x+3}$	22. $\lim_{x\to\infty} \frac{1}{2}x-\frac{4}{x^2}$	23. $\lim_{x\to\infty} \frac{\sin x}{x}$	24. $\lim_{x\to\infty} \frac{\cos 2x}{3x}$

PA 9-1 Infinite Limits Wksht #1-24 all

(1) $\lim_{x\to\infty}\left(7+\cancel{\frac{1}{3x}}^{0}-\cancel{\frac{2}{x^2}}^{0}\right)=7+0-0=\boxed{7}$

Notice as $x\to\infty$, $\frac{1}{3x}\to 0$ and $\frac{2}{x^2}\to 0$

(2) $\lim_{x\to-\infty}\left(\frac{4x^1+8}{5x^1}\right)=\boxed{\frac{4}{5}}$

deg is 1

b/c deg Num. = deg Den., so limit = to ratio of leading coefficients

(3) $\lim_{x\to\infty}\left(\frac{3x^1-1000}{1x^1+100}\right)=\frac{3}{1}=\boxed{3}$

b/c deg Num. = deg Den.

(4) $\lim_{x\to-\infty}\left(\frac{5x^1+5}{7x^2+1}\right)=\boxed{0}$

b/c deg Num. < deg Den.

(5) $\lim_{x\to\infty}\left(\frac{5x^2+2}{4x^2+7}\right)=\boxed{\frac{5}{4}}$

b/c deg Num. = deg Den.

(6) $\lim_{x\to-\infty}\left(\frac{3x^3+5}{5x^2+1}\right)=\boxed{-\infty}$

b/c deg Num. > deg Den.

check sign of ∞

$\frac{3(-1000)^3+5}{5(-1000)^2+1}=\frac{-\#}{+\#}=-\#$

(7) $\lim_{x\to\infty}\left(\frac{2x^2-4x}{x^1+1}\right)=\boxed{\infty}$

b/c deg Num. > deg Den.

check sign of ∞

$\frac{2(1000)^2-4(1000)}{1000+1}=\frac{+\#}{+\#}=+\#$

(8) $\lim_{x\to-\infty}\left(\frac{2x^2-4x}{x^1+1}\right)=\boxed{-\infty}$

b/c deg Num. > deg Den.

$\frac{2(-1000)^2-4(-1000)}{-1000+1}=\frac{+\#}{-\#}=-\#$

(9) $\lim_{x\to\infty}\left(\frac{3x^3+2}{5x^2-1}\right)=\boxed{\infty}$

b/c deg Num. > deg Den.

(10) $\lim_{x\to-\infty}\left(\frac{3x^2+2}{4x^2-1}\right)=\boxed{\frac{3}{4}}$

b/c deg Num. = deg Den.

(11) $\lim_{x\to\infty}\left(\frac{x^2+2}{x^1-555}\right)=\boxed{\infty}$

b/c deg Num. > deg Den.

check sign of ∞

$\frac{1000^2+2}{1000-555}=\frac{+\#}{+\#}=+\#$

(12) $\lim_{x\to-\infty}\left(\frac{3-2x^1}{3x^3-1}\right)=\boxed{0}$

b/c deg Num. < deg Den.

(13) $\lim_{x\to\infty}\left(\frac{3-5x^1}{3x^1-1}\right)=\boxed{\frac{-5}{3}}$

b/c deg Num. = deg Den.

(14) $\lim_{x\to\infty}\left(\frac{3-2x^2}{3x-1}\right)=\boxed{-\infty}$

b/c deg Num. > deg Den.

check sign of ∞

$\frac{3-2(1000)^2}{3(1000)-1}=\frac{-\#}{+\#}=-\#$

(15) $\lim_{x\to\infty}\left(\frac{6x^2-2x-1}{2x^2+3x+2}\right)=\frac{6}{2}=\boxed{3}$

b/c deg Num. = deg Den.

(16) $\lim_{x\to\infty}\left(\frac{3x^3+2}{2x^2-9x^3+7}\right)=\frac{3}{-9}=\boxed{-\frac{1}{3}}$

b/c deg Num. = deg. Den.

(17) $\lim_{x\to-\infty}\left(\frac{x^1}{x^2-1}\right)=\boxed{0}$

b/c deg Num. < deg Den.

(18) $\lim_{x\to-\infty}\left(\frac{8x^2+3x}{2x^2-1}\right)=\frac{8}{2}=\boxed{4}$

b/c deg Num. = deg Den.

(19) $\lim_{x\to\infty}\left(10-\cancel{\frac{2}{x^2}}^{0}\right)=\lim_{x\to\infty}(10-0)=\boxed{10}$

Notice as $x\to\infty$, $\frac{2}{x^2}\to 0$

(20) $\lim_{x\to-\infty}\left(4+\cancel{\frac{3}{x}}^{0}\right)=\lim_{x\to-\infty}(4+0)=\boxed{4}$

Notice as $x\to-\infty$, $\frac{3}{x}\to 0$

(21) $\lim\limits_{x \to -\infty} \left(\dfrac{5x^2}{x^1 + 3} \right) = \boxed{-\infty}$

check sign of ∞

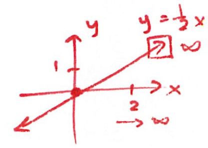
$$\frac{5(-1000)^2}{-1000+3} = \frac{+\#}{-\#} = -\#$$

b/c deg Num. > deg Den.

(22) $\lim\limits_{x \to \infty} \left(\frac{1}{2}x - \cancelto{0}{\frac{4}{x^2}} \right) = \boxed{\infty}$

Notice as $x \to \infty$, $\frac{4}{x^2} \to 0$

Notice $\lim\limits_{x \to \infty} \frac{1}{2}x = \infty$ b/c



(23) $\lim\limits_{x \to \infty} \left(\dfrac{\sin x}{x} \right) = \lim\limits_{x \to \infty} \left(\cancelto{0}{\frac{1}{x}} \cdot \sin x \right) = \lim\limits_{x \to 0} (0 \cdot \sin x) = \boxed{0}$

Notice $\frac{1}{x} \to 0$ as $x \to \infty$.

(24) $\lim\limits_{x \to \infty} \left(\dfrac{\cos 2x}{3x} \right) = \lim\limits_{x \to \infty} \left(\cancelto{0}{\frac{1}{3x}} \cdot \cos 2x \right) = \lim\limits_{x \to \infty} (0 \cdot \cos 2x) = \boxed{0}$

Notice $\frac{1}{3x} \to 0$ as $x \to \infty$.

PA9-3 p.781 # 1-25 odd

(1) $\lim_{x\to -1} x(x-1)^2 = (-1)[(-1)-1]^2$
$= (-1)(-2)^2$
$= (-1)(4)$
$= -4$

$\boxed{\therefore, \lim_{x\to -1} x(x-1)^2 = -4}$

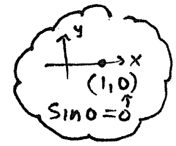
(3) $\lim_{x\to 2} (x^3-2x+3) = (2)^3-2(2)+3$
$= 8-4+3$
$= 7$

$\boxed{\therefore, \lim_{x\to 2} (x^3-2x+3) = 7}$

(5) $\lim_{x\to 2} \sqrt{x+5} = \sqrt{2+5}$
$= \sqrt{7}$

$\boxed{\therefore, \lim_{x\to 2} \sqrt{x+5} = \sqrt{7}}$

(7) $\lim_{x\to 0}(e^x \cdot \sin x) = e^0 \cdot \sin(0)$
$= (1)(0)$
$= 0$



$\boxed{\therefore, \lim_{x\to 0} (e^x \cdot \sin x) = 0}$

(9) $\lim_{x\to a} (x^2-2) = a^2-2$

$\boxed{\therefore, \lim_{x\to a} (x^2-2) = a^2-2}$

(11) a) $\lim_{x\to -3} \frac{x^2+7x+12}{x^2-9} = \frac{(-3)^2+7(-3)+12}{(-3)^2-9}$
$= \frac{9-21+12}{9-9}$
$= \frac{0}{0}$:(

Division by 0 is not allowed ... $\frac{0}{0}$ indetermin. form

b) $\lim_{x\to -3} \frac{x^2+7x+12}{x^2-9} = \lim_{x\to -3} \frac{(x+4)\cancel{(x+3)}}{(x-3)\cancel{(x+3)}}$
$= \lim_{x\to -3} \frac{x+4}{x-3}$
$= \frac{-3+4}{-3-3}$
$= \frac{1}{-6}$

$\boxed{\therefore, \lim_{x\to -3} \frac{x^2+7x+12}{x^2-9} = -\frac{1}{6}}$

(13) a) $\lim_{x\to -1} \frac{x^3+1}{x+1} = \frac{(-1)^3+1}{-1+1} = \frac{-1+1}{0} = \frac{0}{0}$:(

Indeterminant form

Division by 0 is NOT allowed

b) $\lim_{x\to -1} \frac{x^3+1}{x+1} = \lim_{x\to -1} \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{x+1}}$
$= \lim_{x\to -1} (x^2-x+1)$
$= (-1)^2-(-1)+1$
$= 1+1+1$
$= 3$

$\boxed{\therefore, \lim_{x\to -1} \frac{x^3+1}{x+1} = 3}$

(15) a) $\lim_{x\to -2} \frac{x^2-4}{x+2} = \frac{(-2)^2-4}{-2+2} = \frac{4-4}{0} = \frac{0}{0}$:(

Again, division by 0 NOT allowed

b) $\lim_{x\to -2} \frac{x^2-4}{x+2} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}}$
$= \lim_{x\to -2} x-2$
$= -2-2$
$= -4$

$\boxed{\therefore, \lim_{x\to -2} \frac{x^2-4}{x+2} = -4}$

(17) a) $\lim_{x\to 0} \sqrt{x-3} = \sqrt{0-3}$
$= \sqrt{-3}$

But $\sqrt{-3} \notin \mathbb{R}$, i.e., $\sqrt{-3} = i\sqrt{3}$ is a complex number, not a real #

$\boxed{\therefore, \lim_{x\to 0} \sqrt{x-3} \text{ D.N.E.}}$

(19) Recall that: $\lim_{x \to 0} \frac{\sin ax}{ax} = 1$

$$\lim_{x\to 0} \frac{\sin x}{2x^2 - x} = \lim_{x \to 0} \frac{\sin x}{x(2x-1)}$$

$$= \underbrace{\lim_{x\to 0} \frac{\sin x}{x}}_{} \cdot \lim_{x \to 0} \frac{1}{2x-1}$$

$$= (1)\left(\frac{1}{2\cdot 0 - 1}\right)$$

$$= (1)\left(\frac{1}{0-1}\right)$$

$$= (1)\left(\frac{1}{-1}\right)$$

$$= (1)(-1)$$

$$= -1$$

$$\boxed{\therefore, \lim_{x \to 0} \frac{\sin x}{2x^2 - x} = -1}$$

(21) $$\lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} \frac{\sin x \cdot \sin x}{x}$$

$$= \lim_{x \to 0}\left(\frac{\sin x}{x} \cdot \sin x\right)$$

$$= \underbrace{\lim_{x \to 0} \frac{\sin x}{x}}_{} \cdot \lim_{x \to 0} \sin x$$

$$= (1)(\sin(0))$$

$$= (1)(0)$$

$\sin 0 = 0$

$$= 0$$

$$\boxed{\therefore, \lim_{x \to 0} \frac{\sin^2 x}{x} = 0}$$

(23) $$\lim_{x \to 0} \frac{e^x - \sqrt{x}}{\log_4(x+2)} = \frac{e^0 - \sqrt{0}}{\log_4(0+2)}$$

$$= \frac{1 - 0}{\log_4 2}$$

$$= \frac{1}{\frac{1}{2}}$$

$$= 1 \cdot \frac{2}{1}$$

$$= 1 \cdot 2$$

$$= 2$$

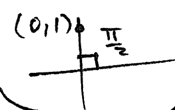
What is $\log_4 2 = ?$
Let $\log_4 2 = r$
$4^r = 2$
$(2^2)^r = 2$
$2^{2r} = 2^1$
$2r = 1$
$r = \frac{1}{2}$

$$\boxed{\therefore, \lim_{x \to 0} \frac{e^x - \sqrt{x}}{\log_4(x+2)} = 2}$$

(25) $$\lim_{x \to \pi/2} \frac{\ln(2x)}{\sin^2 x} = \frac{\ln\left(\cancel{2} \cdot \frac{\pi}{\cancel{2}}\right)}{\sin^2\left(\frac{\pi}{2}\right)}$$

$$= \frac{\ln(\pi)}{\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}$$

$\sin \frac{\pi}{2} = 1$



$$= \frac{\ln(\pi)}{(1)(1)}$$

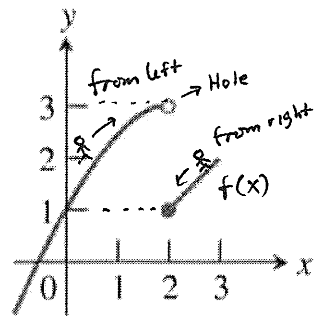
$$= \frac{\ln(\pi)}{1}$$

$$= \ln \pi$$

$$\boxed{\therefore, \lim_{x \to \pi/2} \frac{\ln(2x)}{\sin^2 x} = \ln \pi}$$

PA9-4 p. 781 # 27-33 odd, 37-45 odd

(27)

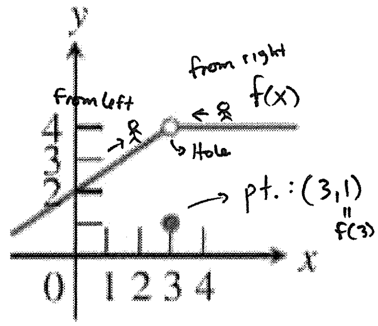


(a) $\lim_{x\to 2^-} f(x) = \boxed{3}$

(b) $\lim_{x\to 2^+} f(x) = \boxed{1}$

(c) $\boxed{\lim_{x\to 2} f(x) \text{ D.N.E. b/c } \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)}$

(29)

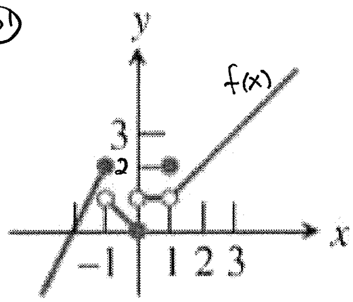


(a) $\lim_{x\to 3^-} f(x) = \boxed{4}$

(b) $\lim_{x\to 3^+} f(x) = \boxed{4}$

(c) $\boxed{\lim_{x\to 3} f(x) = 4 \text{ b/c } \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)}$

(31)



(a) $\lim_{x\to -1^+} f(x) = 1$, $\boxed{\text{True}}$

(b) $\lim_{x\to 0^-} f(x) = 0$, $\boxed{\text{True}}$

(c) $\lim_{x\to 0^-} f(x) = 1$, $\boxed{\text{False}}$

(d) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$, $\boxed{\text{False}}$
(Notice $\lim_{x\to 0^+} f(x) = 1$)

(e) $\lim_{x\to 0} f(x)$ exists, $\boxed{\text{False}}$
($\lim_{x\to 0} f(x)$ D.N.E. b/c $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$)

(f) $\lim_{x\to 0} f(x) = 0$, $\boxed{\text{False}}$
($\lim_{x\to 0} f(x)$ D.N.E. b/c $\lim_{x\to 0^-} f(x) = 0 \neq \lim_{x\to 0^+} f(x) = 1$)

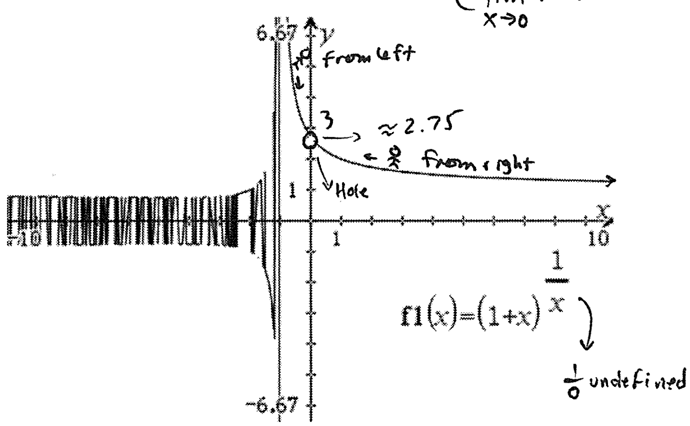
(g) $\lim_{x\to 0} f(x) = 1$, $\boxed{\text{False}}$

(h) $\lim_{x\to 1} f(x) = 1$, $\boxed{\text{True}}$
($\lim_{x\to 1} f(x) = 1$ b/c $\lim_{x\to 1^-} f(x) = 1 = \lim_{x\to 1^+} f(x)$)

(i) $\lim_{x\to 1} f(x) = 0$, $\boxed{\text{False}}$

(j) $\lim_{x\to 2^-} f(x) = 2$, $\boxed{\text{True}}$

(33)

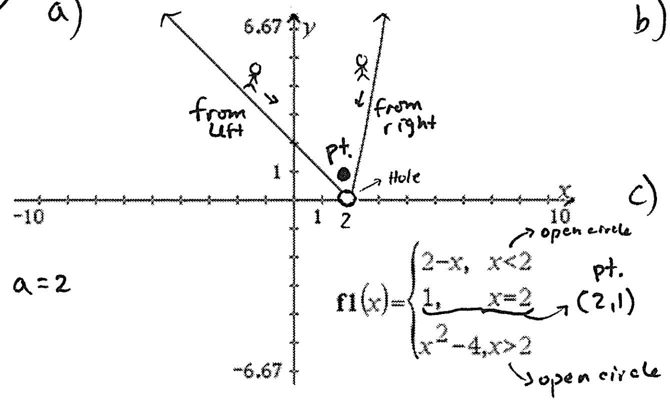


(a) $\lim_{x\to 0^-} f(x) \approx \boxed{2.75}$

(b) $\lim_{x\to 0^+} f(x) \approx \boxed{2.75}$

(c) $\boxed{\lim_{x\to 0} f(x) \approx 2.75 \text{ b/c } \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)}$

(37) a)



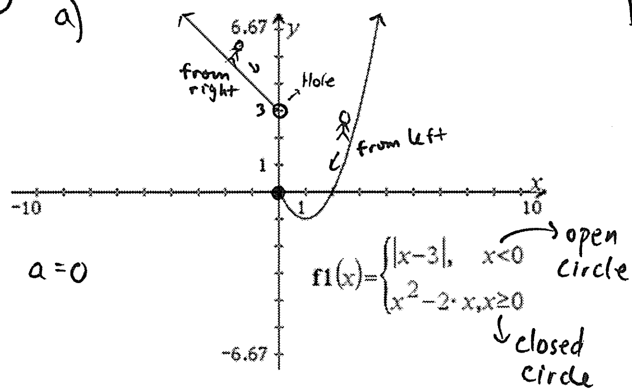
$a=2$

b) $\lim_{x\to 2^+} f(x) = \boxed{0}$

$\lim_{x\to 2^-} f(x) = \boxed{0}$

c) $\boxed{\text{Yes, } \lim_{x\to 2} f(x) = 0 \text{ b/c } \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)}$

(39) a)



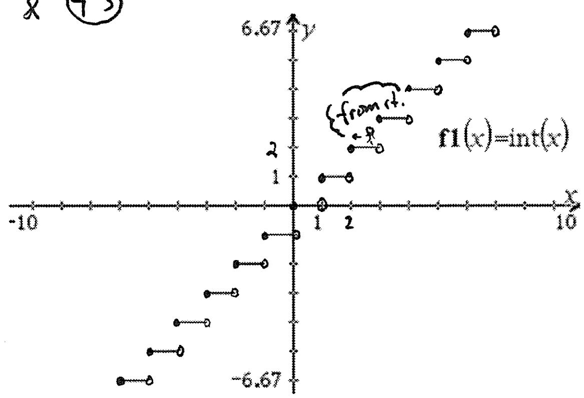
$a=0$

b) $\lim_{x\to 0^-} f(x) = \boxed{3}$

$\lim_{x\to 0^+} f(x) = \boxed{0}$

c) $\boxed{\text{No, } \lim_{x\to 0} f(x) \text{ D.N.E. b/c } \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)}$

(41) & (43)



$y = \text{int}(x) = [\![x]\!]$, step function (greatest int. fn.)

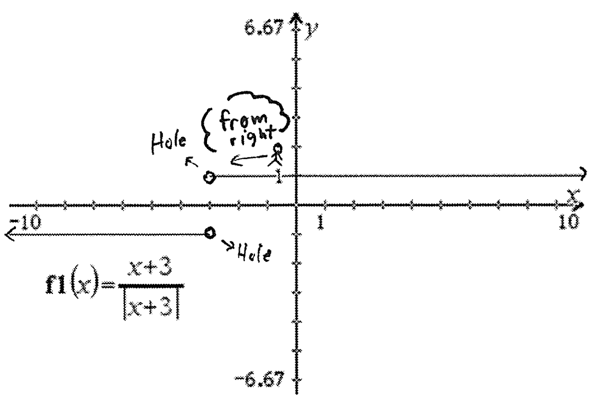
(41) $\lim_{x\to 2^+} \text{int}(x) = \boxed{2}$

(43) $\boxed{\lim_{x\to 0.0001} \text{int}(x) = 0 \text{ b/c } \lim_{x\to 0.0001^-} \text{int}(x) = 0 = \lim_{x\to 0.0001^+} \text{int}(x)}$

int(x) is the greatest integer less than or equal to x

Ex: $[\![1.2]\!] = 1$, $[\![1.99]\!] = 1$, $[\![2]\!] = 2$, etc.

(45)



$\lim_{x\to -3^+} \frac{x+3}{|x+3|} = \boxed{1}$

PA 9-5 Limits Numerically

Worksheet
Limits: A Numerical and Graphical Approach

1. Use your graphing calculator to graph $f(x) = \dfrac{\cos x}{x^2}$. Make a guess as to the value of $\lim\limits_{x\to 0} \dfrac{\cos x}{x^2}$. Construct a table of values for $f(-.1), f(-.01), f(-.001), f(-.0001), f(.1),$ $f(.01), f(.001), f(.0001)$. Estimate $\lim\limits_{x\to 0} \dfrac{\cos x}{x^2}$.

x	-.1	-.01	-.001	-.0001	0	.0001	.001	.01	.1
f(x)	99.5	9999.5	999999.5	99999999.5	DNE	99999999.5	999999.5	9999.5	99.5

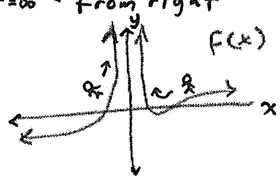
$\lim\limits_{x\to 0^-} \frac{\cos x}{x^2} = \infty$ from left

or undef.

$\lim\limits_{x\to 0^+} \frac{\cos x}{x^2} = \infty$ ← From right

As $x \to \infty$, the y-values are getting bigger and bigger, approching infinity :)

$$\boxed{\lim_{x\to 0} \frac{\cos x}{x^2} = \infty}$$



2. Graph $f(x) = x\dfrac{|x-1|}{x-1}$. What is the $\lim\limits_{x\to 1+} f(x)$ and $\lim\limits_{x\to 1-} f(x)$? Construct a table of values for $f(.9), f(.99), f(.999), f(1.001), f(1.01), f(1.1)$. What is the $\lim\limits_{x\to 1+} f(x)$ and $\lim\limits_{x\to 1-} f(x)$?

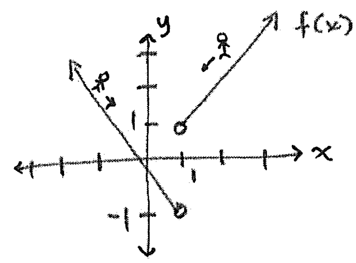
x	.9	.99	.999	1	1.001	1.01	1.1
f(x)	-.9	-.99	-.999	D.N.E. or undefined	1.001	1.01	1.1

From left

from right

$$\boxed{\lim_{x\to 1^-} f(x) = -1}$$

$$\boxed{\lim_{x\to 1^+} f(x) = 1}$$

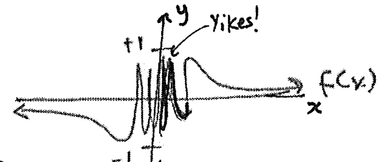


3. Using a graphing calculator, graph $f(x) = \sin\frac{1}{x}$. Does if look as if $\lim_{x\to0} f(x)$ exists? Construct a table of values for $f(-.1), f(-.01), f(-.001), f(-.0001), f(.1), f(.01), f(.001), f(.0001)$. What do you conclude about $\lim_{x\to0} f(x)$?

x	$-.1$	$-.01$	$-.001$	$-.0001$	0	$.0001$	$.001$	$.01$	$.1$
$f(x)$	$.544$	$.506$	$-.827$	$.306$	D.N.E. or undefined	$-.306$	$.827$	$-.506$	$-.544$

from left ⟶ ⟵ from right

$$\boxed{\lim_{x\to0} f(x) = \text{D.N.E.}}$$



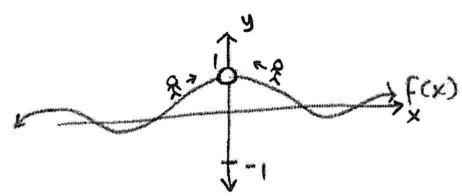
"oscillating"

4. Using a graphing calculator, graph $f(x) = \frac{\sin x}{x}$. Make a guess as to the $\lim_{x\to0} f(x)$. Construct a table of values for $f(-.1), f(-.01), f(-.001), f(-.0001), f(.1), f(.01), f(.001), f(.0001)$. Estimate $\lim_{x\to0} f(x)$.

x	$-.1$	$-.01$	$-.001$	$-.0001$	0	$.0001$	$.001$	$.01$	$.1$
$f(x)$	$.998$	$.999$	$.999$	$.999$	D.N.E. or undefined	$.999$	$.999$	$.999$	$.998$

from left ⟶ $\lim_{x\to0^-} f(x) = 1$ $\lim_{x\to0^+} f(x) = 1$ ⟵ from right

$$\boxed{\lim_{x\to0} \frac{\sin x}{x} = 1 \quad \text{b/c} \quad \lim_{x\to0^-} \frac{\sin x}{x} = 1 = \lim_{x\to0^+} \frac{\sin x}{x}}$$



Recall we already know the above fact (see notes). Here we confirm numerically.

PA 9-6 p.781 # 2, 6, 10, 12, 18, 22, 26, 28, 30, 32

(2) $\lim_{x\to 3}(x-1)^{12} = (3-1) = 2^{12} = 4096$

$\boxed{\therefore,\ \lim_{x\to 3}(x-1)^{12} = 4096}$

(6) $\lim_{x\to -2}(x-4)^{2/3} = (-2-4)^{2/3} = (-6)^{2/3} = \sqrt[3]{(-6)^2} = \sqrt[3]{36} = 3.302$

$\boxed{\therefore,\ \lim_{x\to -2}(x-4)^{2/3} = 3.302}$

(10) $\boxed{\lim_{x\to a}\frac{x^2-1}{x^2+1} = \frac{a^2-1}{a^2+1}}$

(12) b) $\lim_{x\to 3}\frac{x^2-9}{x^2+2x-15} = \lim_{x\to 3}\frac{(x+3)\cancel{(x-3)}}{(x+5)\cancel{(x-3)}}$

$= \lim_{x\to 3}\frac{x+3}{x+5}$

$= \frac{3+3}{3+5}$

$= \frac{6}{8}$

$= \frac{3}{4}$

$\boxed{\therefore,\ \lim_{x\to 3}\frac{x^2-9}{x^2+2x-15} = \frac{3}{4}}$

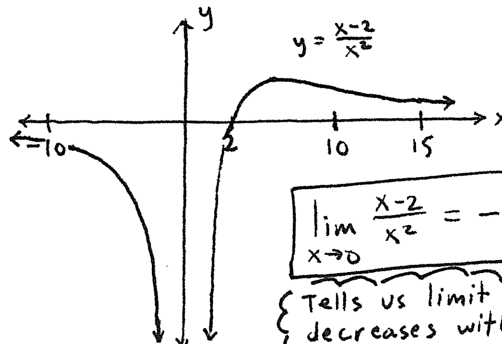
a) $\lim_{x\to 3}\frac{x^2-9}{x^2+2x-15} = \frac{(3)^2-9}{(3)^2+2(3)-15} = \frac{0}{0}$ ☹

$\boxed{\text{Division by 0 \underline{not} allowed}}$

(18) a) $\lim_{x\to 0}\frac{x-2}{x^2} = \frac{0-2}{0^2} = -\frac{2}{0}$ ☹

$\boxed{\text{Division by 0 \underline{not} allowed}}$

b) The best way to find this limit is graphically.



$\lim_{x\to 0^-}\frac{x-2}{x^2} = -\infty$

$\lim_{x\to 0^+}\frac{x-2}{x^2} = -\infty$

$\boxed{\lim_{x\to 0}\frac{x-2}{x^2} = -\infty}$ or $\boxed{\lim_{x\to 0}\frac{x-2}{x^2}\ \text{D.N.E.}}$

Tells us limit decreases without bound

Since limits are finite real numbers

(22) $\lim_{x\to 0}\frac{x+\sin x}{2x} = \lim_{x\to 0}\left(\frac{\cancel{x}}{2\cancel{x}} + \frac{1\cdot\sin x}{2x}\right)$

$= \lim_{x\to 0}\frac{1}{2} + \lim_{x\to 0}\frac{1}{2}\cdot\frac{\sin x}{x}$

Recall $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$= \frac{1}{2} + \frac{1}{2}(1)$

$= 1$

$\boxed{\therefore,\ \lim_{x\to 0}\frac{x+\sin x}{2x} = 1}$

(26) $\lim_{x\to 27}\frac{\sqrt{x+9}}{\log_3 x} = \frac{\sqrt{27+9}}{\log_3 27}$

$= \frac{\sqrt{36}}{\log_3 27}$

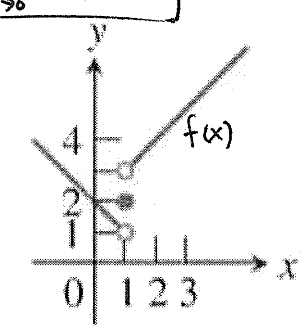
$= \frac{6}{3}$

$= 2$

What is $\log_3 27$?
Let $\log_3 27 = r$
$3^r = 27$
$3^r = 3^3$
$r = 3$
$\therefore, \log_3 27 = 3$

$\boxed{\therefore,\ \lim_{x\to 27}\frac{\sqrt{x+9}}{\log_3 27} = 2}$

(30)

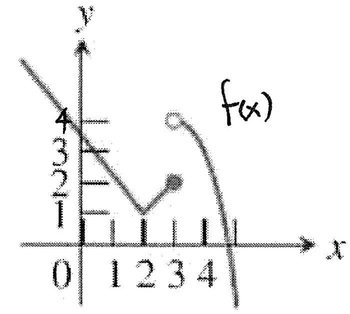


a) $\lim_{x\to 1^-} f(x) = \boxed{1}$

b) $\lim_{x\to 1^+} f(x) = \boxed{3}$

c) $\boxed{\lim_{x\to 3} f(x)\ \text{D.N.E}\quad \text{b/c}\ \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)}$

(28)



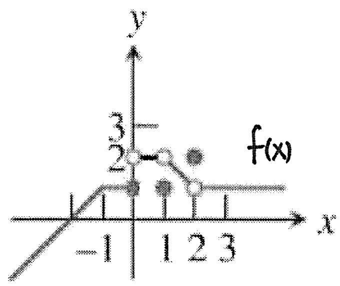
a) $\lim_{x\to 3^-} f(x) = 2$

b) $\lim_{x\to 3^+} f(x) = 4$

c) $\lim_{x\to 3} f(x)$ D.N.E

b/c $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

(32)



a) $\lim_{x\to -1^+} f(x) = 1$, $\boxed{\text{True}}$

b) $\lim_{x\to 2} f(x)$ D.N.E, $\boxed{\text{False}}$

$\left(\lim_{x\to 2} f(x) = 1 \text{ b/c } \lim_{x\to 2^-} f(x) = 1 = \lim_{x\to 2^+} f(x) \right)$

c) $\lim_{x\to 2} f(x) = 2$, $\boxed{\text{False}}$

d) $\lim_{x\to 1^-} f(x) = 2$, $\boxed{\text{True}}$

e) $\lim_{x\to 1^+} f(x) = 1$, $\boxed{\text{False}}$

$\left(\lim_{x\to 1^+} f(x) = 2 \right)$

f) $\lim_{x\to 1} f(x)$ D.N.E., $\boxed{\text{False}}$

$\left(\lim_{x\to 1} f(x) = 2 \text{ b/c } \lim_{x\to 1^-} f(x) = 2 = \lim_{x\to 1^+} f(x) \right)$

g) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$, $\boxed{\text{False}}$

$\left(\lim_{x\to 0^+} f(x) = 2 \text{ and } \lim_{x\to 0^-} f(x) = 1 \right)$

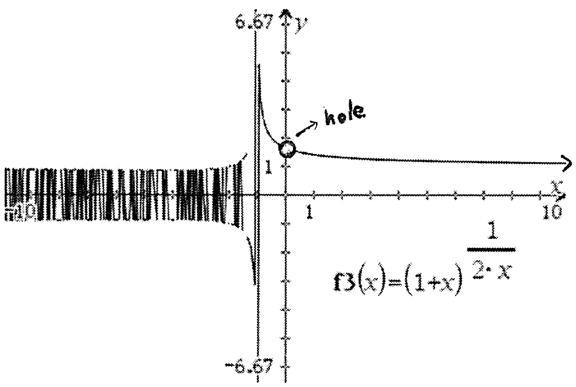
h) $\lim_{x\to c} f(x)$ exists for every c in $(-1, 1)$, $\boxed{\text{False}}$

(For every c except $x = 0$)

(i) $\lim_{x\to c} f(x)$ exist for every c in $(1, 3)$, $\boxed{\text{True}}$

PA 9-8 p. 781 # 34, 38, 40, 46, 50, 56, 58, 66, 72

(34)



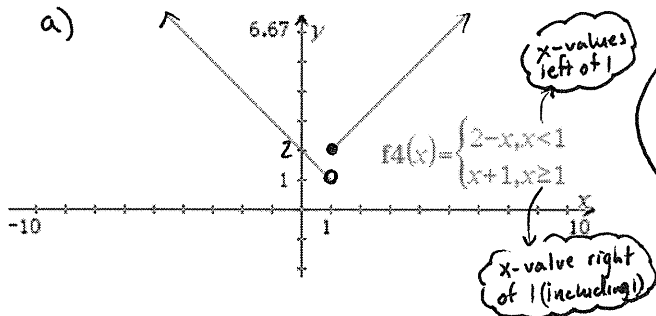
a) $\lim_{x\to 0^-} f(x) \approx \boxed{1.65}$

b) $\lim_{x\to 0^+} f(x) \approx \boxed{1.65}$

Check numerically

c) $\boxed{\lim_{x\to 0} f(x) \approx 1.65 \quad b/c \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)}$

(38) a)



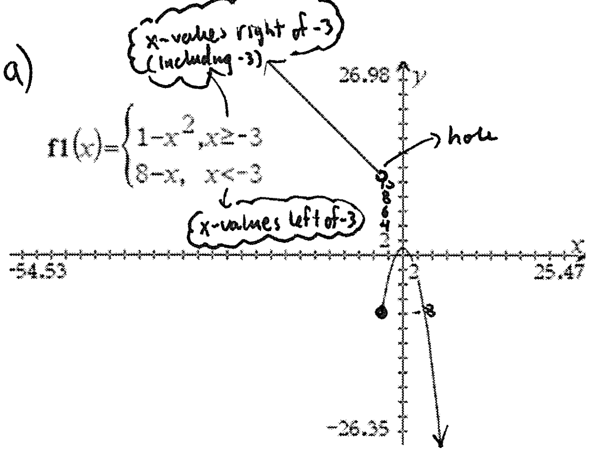
b) $\boxed{\lim_{x\to 1^-} f(x) = 1 \quad \& \quad \lim_{x\to 1^+} f(x) = 2}$

c) $\boxed{\lim_{x\to 1} f(x) \quad \text{D.N.E.} \quad b/c \quad \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2-x) = 2-1 = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x+1) = 1+1 = 2$

(40) a)



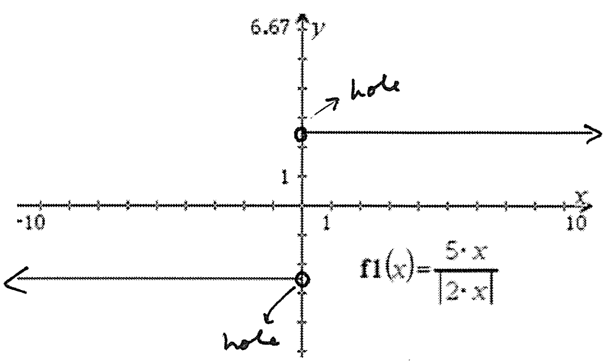
b) $\boxed{\lim_{x\to -3^-} f(x) = 11 \quad \& \quad \lim_{x\to -3^+} f(x) = -8}$

c) $\boxed{\lim_{x\to -3} f(x) \quad \text{D.N.E.} \quad b/c \quad \lim_{x\to -3^-} f(x) \neq \lim_{x\to -3^+} f(x)}$

$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^-}(1-x^2) = 1-(-3)^2 = 1-9 = -8$

$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-}(8-x) = 8-(-3) = 11$

(46)



$\boxed{\lim_{x\to 0^-} \frac{5x}{|2x|} = -\frac{5}{2} = -2.5}$

(50) $y = \frac{x}{1+2x}$

a) $\boxed{\lim_{x\to\infty} \frac{x}{1+2x} = \frac{1}{2}}$

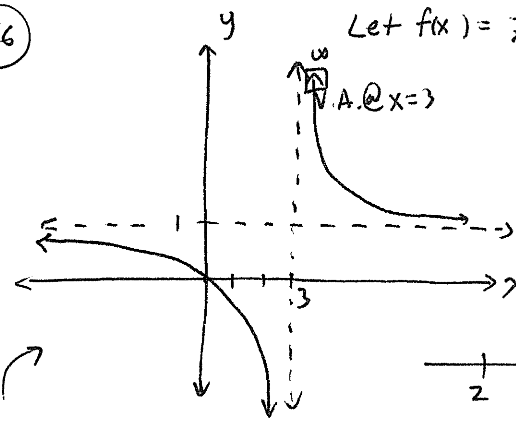
b/c deg Num. = deg Den.

b) $\boxed{\lim_{x\to-\infty} \frac{x}{1+2x} = \frac{1}{2}}$

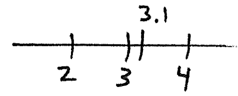
b/c deg Num. = deg Den.

(56) Let $f(x) = \frac{x}{x-3}$ $\quad x-3=0$, $x=3$

$\boxed{\therefore \text{ V.A. @ } x=3}$



$\boxed{\lim_{x\to3^+} \frac{x}{x-3} = \infty}$



x	$\times$	$\times$	$\times$	3	3.001	3.01	3.1
$f(x)$	$\times$	$\times$	$\times$	Undef.	3001	301	31

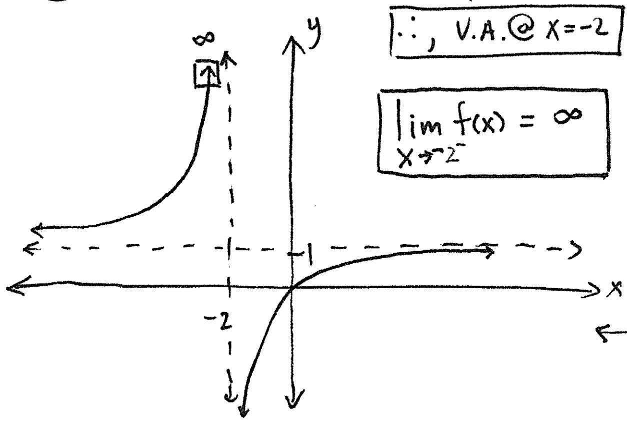
← from the right

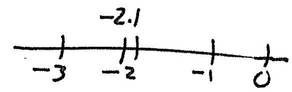
$\lim_{x\to3^+} \frac{x}{x-3} = \infty$

(58) Let $f(x) = \frac{x}{x+2}$ $\quad x+2=0$, $x=-2$

$\boxed{\therefore \text{ V.A. @ } x=-2}$

$\boxed{\lim_{x\to-2^-} f(x) = \infty}$



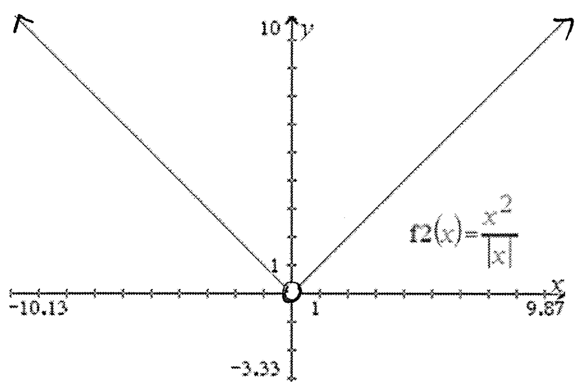


x	-2.1	-2.01	-2.001	-2	$\times$	$\times$	$\times$
$f(x)$	21	201	2001	undef.	$\times$	$\times$	$\times$

from left →

$\lim_{x\to-2^-} f(x) = \infty$

(66)

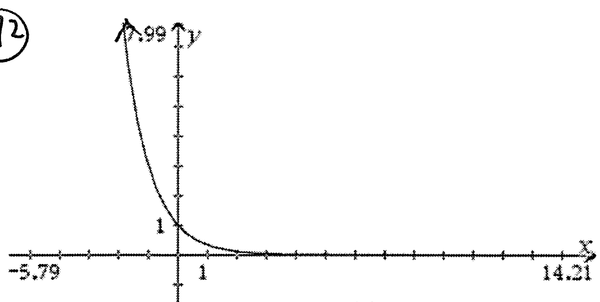


$\lim_{x\to0^-} \frac{x^2}{|x|} = 0$

$\lim_{x\to0^+} \frac{x^2}{|x|} = 0$

$\therefore \lim_{x\to0} \frac{x^2}{|x|} = 0$ b/c $\lim_{x\to0^-} \frac{x^2}{|x|} = \lim_{x\to0^+} \frac{x^2}{|x|}$

(72)



$f1(x)=3^{-x}$

$\boxed{\lim_{x\to\infty} 3^{-x} = 0}$

PA9-9 p.762 # 3, 7, 9, 11, 23, 25, 29

(3) $s'(t) = \lim_{h\to 0} \dfrac{s(t+h) - s(t)}{h}$ $\quad s(t) = 3t - 5$ at $t = 4$

$= \lim_{h\to 0} \dfrac{3(t+h) - 5 - (3t-5)}{h}$

$= \lim_{h\to 0} \dfrac{\cancel{3t} + 3h \cancel{-5} \cancel{-3t} \cancel{+5}}{h}$

$= \lim_{h\to 0} \dfrac{3\cancel{h}}{\cancel{h}}$

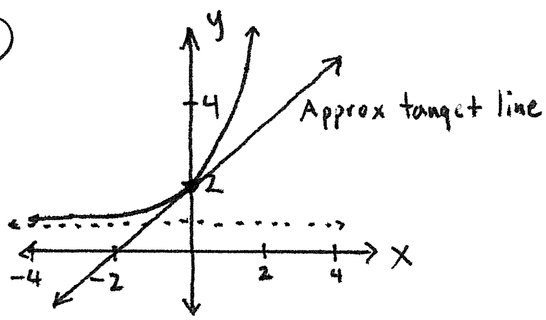
$= \lim_{h\to 0} 3$

$= 3$

$\therefore$, $s'(t) = 3$ @ $t = 4$

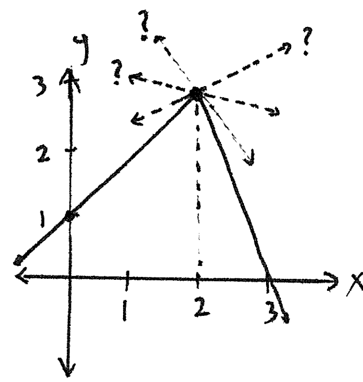
$\therefore$, The instantaneous velocity @ $t = 4$ is 3.

(7)



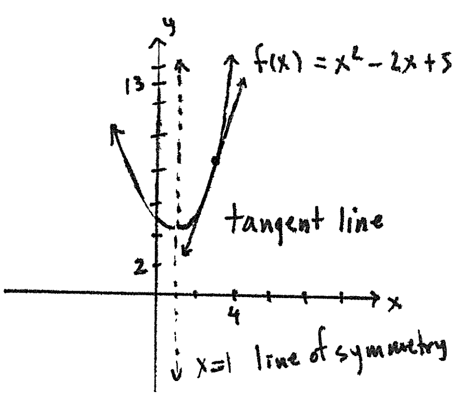
$\therefore$, The slope of the tangent line at $x = 0$ is about 1.

(9)



No tangent line exists because of the sharp point @ $x = 2$

(11)



$\therefore$, The derivative of $f(x)$ at $x = 3$ is ≈ 4.

(23) $f(x) = 1 - x^2$ at $x = 2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1-(x+h)^2 - (1-x^2)}{h}$$

$$= \lim_{h\to 0} \frac{1-[(x+h)(x+h)] - 1 + x^2}{h}$$

$$= \lim_{h\to 0} \frac{1-(x^2+xh+xh+h^2) - 1 + x^2}{h}$$

$$= \lim_{h\to 0} \frac{1-(x^2+2xh+h^2) - 1 + x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1} \cancel{-x^2} - 2xh - h^2 \cancel{-1} \cancel{+x^2}}{h}$$ Factor out h

$$= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (-2x-h)$$

$$= -2x - 0$$

$$= -2x$$

$$\boxed{\therefore,\ f'(x) = -2x \ \& \quad \therefore,\ f'(2) = -2(2) = -4}$$

(25) $f(x) = 3x^2 + 2$ at $x = -2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{3(x+h)^2 + 2 - (3x^2+2)}{h}$$

$$= \lim_{h\to 0} \frac{3[(x+h)(x+h)] + 2 - 3x^2 - 2}{h}$$

$$= \lim_{h\to 0} \frac{3(x^2+2xh+h^2) + 2 - 3x^2 - 2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{+2} \cancel{-3x^2} \cancel{-2}}{h}$$ Factor out h

$$= \lim_{h\to 0} \frac{\cancel{h}(6x+3h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (6x+3h)$$

$$= 6x + 3(0)$$

$$= 6x$$

$$\boxed{\therefore,\ f'(x) = 6x \ \& \quad \therefore,\ f'(-2) = 6(-2) = -12}$$

(29) $f(x) = 2 - 3x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2 - 3(x+h) - (2-3x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2} \cancel{-3x} - 3h \cancel{-2} \cancel{+3x}}{h}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}}$$

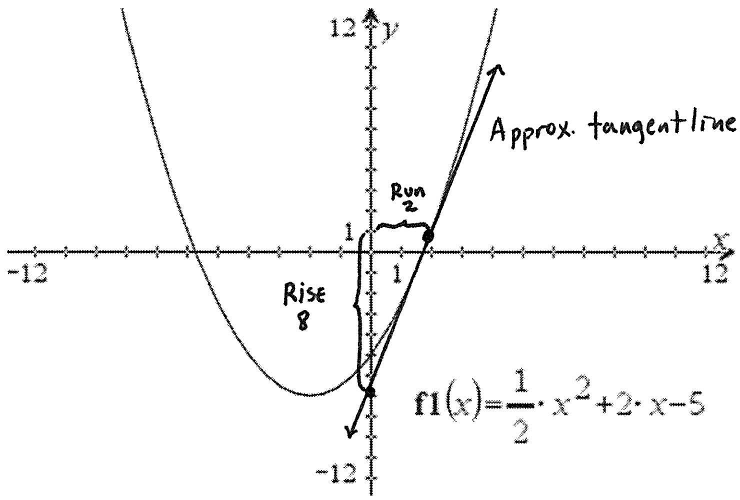
$$= \lim_{h\to 0} (-3)$$

$$= -3$$

$$\boxed{\therefore,\ f'(x) = -3}$$

PA 9-10 p.762 # 12, 13, 24, 26, 28, 31

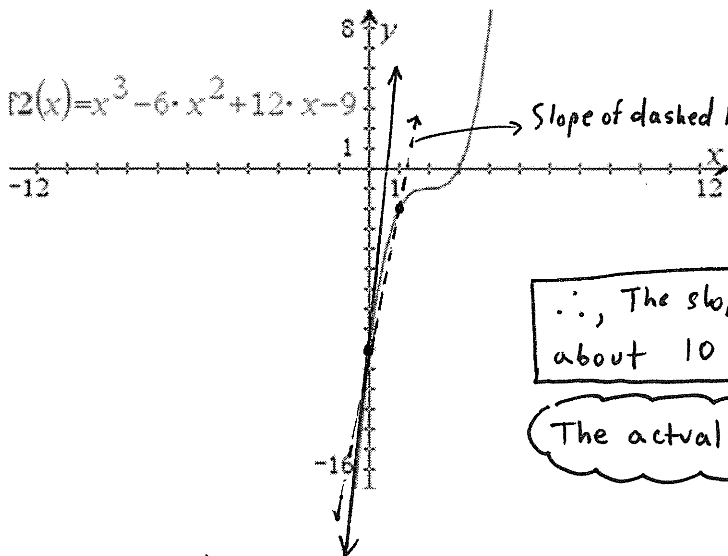
(12)



$\frac{\text{RISE}}{\text{RUN}} \approx \frac{8}{2} = 4$

$\boxed{\therefore, \text{ The slope of the tangent line at } x = 2 \text{ is about } 4}$

(13)



$\boxed{\therefore, \text{ The slope of the tangent line at } x = 2 \text{ is about } 10 \text{ or } 11 \text{ or } 12 \text{ (hard to tell)}}$

The actual slope is $f'(x) = 3(0)^2 - 12(0) + 12 = \boxed{12}$

(24) $f(x) = 2x + \frac{1}{2}x^2$ at $x = 2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$(x+h)^2 = (x+h)(x+h) = x^2 + xh + xh + h^2$
$= x^2 + 2xh + h^2$

$$= \lim_{h\to 0} \frac{2(x+h) + \frac{1}{2}(x+h)^2 - (2x + \frac{1}{2}x^2)}{h}$$

$$= \lim_{h\to 0} \frac{2x + 2h + \frac{1}{2}(x^2 + 2xh + h^2) - 2x - \frac{1}{2}x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x} + 2h + \cancel{\frac{1}{2}x^2} + xh + \frac{1}{2}h^2 - \cancel{2x} - \cancel{\frac{1}{2}x^2}}{h}$$

$$= \lim_{h\to 0} \frac{2h + xh + \frac{1}{2}h^2}{h}$$ factor out h

$$= \lim_{h\to 0} \frac{\cancel{h}(2 + x + \frac{1}{2}h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2 + x + \frac{1}{2}h$$

$= 2 + x + \frac{1}{2}(0)$

$= 2 + x$

$\boxed{\begin{array}{l} \therefore, \ f'(x) = 2 + x \quad \text{or} \quad x + 2 \quad \& \\ \therefore, \ f'(2) = 2 + 2 = 4 \end{array}}$

(26) $f(x) = x^2 - 3x + 1$ at $x = 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - 3(x+h) + 1 - (x^2 - 3x + 1)}{h}$$

$(x+h)^2 = (x+h)(x+h) = x^2 + xh + hx + h^2 = x^2 + 2xh + h^2$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h \cancel{+1} - \cancel{x^2} + \cancel{3x} \cancel{-1}}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h}$$

Factor out h

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x + h - 3$$

$$= 2x + 0 - 3$$

$$= 2x - 3$$

$$\boxed{\therefore,\ f'(x) = 2x - 3 \ \& \quad f'(1) = 2(1) - 3 = -1}$$

(28) $f(x) = \frac{1}{x+2}$ at $x = -1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)+2} - \frac{1}{x+2}}{h}$$

$\frac{1}{x+h+2} - \frac{1}{x+2}$ Get common denominator

$$= \frac{1}{(x+h+2)} \cdot \frac{(x+2)}{(x+2)} - \frac{1}{x+2} \cdot \frac{(x+h+2)}{(x+h+2)}$$

$$= \frac{x+2 - (x+h+2)}{(x+h+2)(x+2)} = \frac{\cancel{x} + \cancel{2} - \cancel{x} - h - \cancel{2}}{(\ + h + 2)(x+2)}$$

$$= \frac{-h}{(x+h+2)(x+2)}$$

$$= \lim_{h\to 0} \frac{\frac{-h}{(x+h+2)(x+2)}}{\frac{h}{1}}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{(x+h+2)(x+2)} \cdot \frac{1}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h+2)(x+2)}$$

$$= \frac{-1}{(x+0+2)(x+2)}$$

$$= \frac{-1}{(x+2)(x+2)}$$

$$= \frac{-1}{(x+2)^2}$$

$$\boxed{\therefore,\ f'(x) = \frac{-1}{(x+2)^2} \ \& \quad f'(-1) = \frac{-1}{(-1+2)^2} = \frac{-1}{(1)^2} = -\frac{1}{1} = -1}$$

(31) $f(x) = 3x^2 + 2x - 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{3(x+h)^2 + 2(x+h) - 1 - (3x^2 + 2x - 1)}{h}$$

$$(x+h)^2 = (x+h)(x+h)$$
$$= x^2 + xh + xh + h^2$$
$$= x^2 + 2xh + h^2$$

$$= \lim_{h\to 0} \frac{3(x^2 + 2xh + h^2) + 2x + 2h - 1 - 3x^2 - 2x + 1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 + \cancel{2x} + 2h \cancel{-1} \cancel{-3x^2} \cancel{-2x} \cancel{+1}}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2 + 2h}{h}$$

factor out h

$$= \lim_{h\to 0} \frac{\cancel{h}(6x + 3h + 2)}{\cancel{h}}$$

$$= \lim_{h\to 0} 6x + 3h + 2$$

$$= 6x + 3(0) + 2$$

$$= 6x + 2$$

$$\boxed{\therefore,\ f'(x) = 6x + 2}$$