**DETERMINANTS OF 3 × 3 MATRICES** Determinants of $3 \times 3$ matrices are called **third-order determinants**. One method of evaluating third-order determinants is **expansion by minors**. The **minor** of an element is the determinant formed when the row and column containing that element are deleted.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{The minor of } a_1 \text{ is } \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{The minor of } b_1 \text{ is } \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{The minor of } c_1 \text{ is } \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

To use expansion by minors with third-order determinants, each member of one row is multiplied by its minor and its *position sign*, and the results are added together. The position signs alternate between positive and negative, beginning with a positive sign in the first row, first column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

**Key Concept** *Third-Order Determinant*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The definition of third-order determinants shows an expansion using the elements in the first row of the determinant. However, any row can be used.

## Example 2 Expansion by Minors

**Evaluate $\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix}$ using expansion by minors.**

Decide which row of elements to use for the expansion. For this example, we will use the first row.

$$\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix} = 2\begin{vmatrix} 5 & -4 \\ 9 & 0 \end{vmatrix} - 7\begin{vmatrix} -1 & -4 \\ 6 & 0 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 5 \\ 6 & 9 \end{vmatrix} \quad \text{Expansion by minors}$$

$$= 2(0 - (-36)) - 7(0 - (-24)) - 3(-9 - 30) \quad \text{Evaluate } 2 \times 2 \text{ determinants.}$$

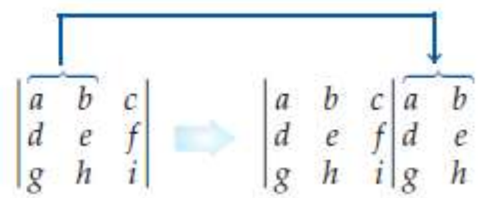
$$= 2(36) - 7(24) - 3(-39)$$

$$= 72 - 168 + 117 \quad \text{Multiply.}$$

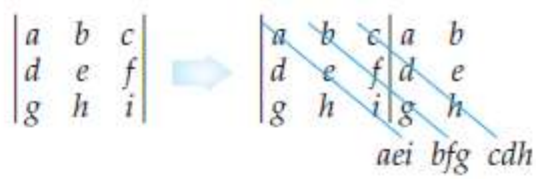
$$= 21 \quad \text{Simplify.}$$

Another method for evaluating a third-order determinant is by using diagonals.

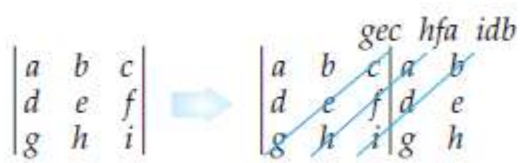
**Step 1** Begin by writing the first two columns on the right side of the determinant.




**Step 2** Next, draw diagonals from each element of the top row of the determinant downward to the right. Find the product of the elements on each diagonal.




Then, draw diagonals from the elements in the third row of the determinant upward to the right. Find the product of the elements on each diagonal.




**Step 3** To find the value of the determinant, add the products of the first set of diagonals and then subtract the products of the second set of diagonals. The sum is $aei + bfg + cdh - gec - hfa - idb$.

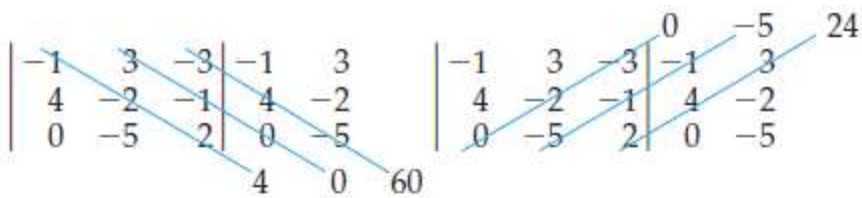
## Example 3 Use Diagonals

**Evaluate** $\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix}$ **using diagonals.**

**Step 1** Rewrite the first two columns to the right of the determinant.

$$\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ 4 & -2 \\ 0 & -5 \end{matrix}$$

**Step 2** Find the products of the elements of the diagonals.




**Step 3** Add the bottom products and subtract the top products.
$4 + 0 + 60 - 0 - (-5) - 24 = 45$

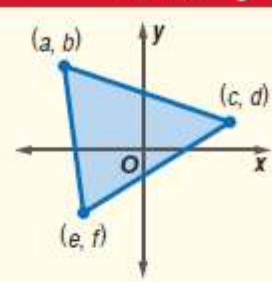
The value of the determinant is 45.

One very useful application of determinants is finding the areas of polygons. The formula below shows how determinants can be used to find the area of a triangle using the coordinates of the vertices.

## Key Concept — Area of a Triangle

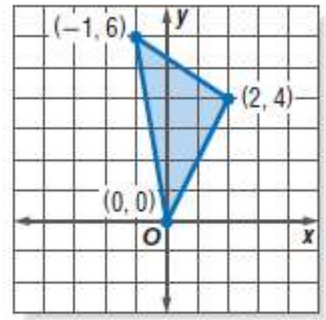
The area of a triangle having vertices at $(a, b)$, $(c, d)$, and $(e, f)$ is $|A|$, where

$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}.$$



## Example 4 Area of a Triangle

**GEOMETRY** **Find the area of a triangle whose vertices are located at (−1, 6), (2, 4), and (0, 0).**




Assign values to $a$, $b$, $c$, $d$, $e$, and $f$ and substitute them into the Area Formula. Then evaluate.

$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$ Area Formula

$$= \frac{1}{2}\begin{vmatrix} -1 & 6 & 1 \\ 2 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$ $(a, b) = (-1, 6), (c, d) = (2, 4), (e, f) = (0, 0)$

$$= \frac{1}{2}\left[-1\begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} - 6\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 0 & 0 \end{vmatrix}\right]$$ Expansion by minors

$$= \frac{1}{2}[-1(4 - 0) - 6(2 - 0) + 1(0 - 0)]$$ Evaluate $2 \times 2$ determinants.

$$= \frac{1}{2}[-4 - 12 + 0]$$ Multiply.

$$= \frac{1}{2}[-16] \text{ or } -8$$ Simplify.

Remember that the area of a triangle is the absolute value of $A$. Thus, the area is $|-8|$ or 8 square units.