

7.3 Partial Fractions

Target 8G: Decompose rational expressions into partial fractions

Review of Prior Concepts

Write the terms for partial fraction decomposition of:

$$\frac{-x^2+2x-5}{(x+1)(x^2+6x+9)} = \frac{-x^2+2x-5}{(x+1)(x+3)(x+3)} = \frac{-x^2+2x-5}{(x+1)(x+3)^2} = \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

↓
Factor
(x+3)(x+3)

Partial Fraction Decomposition

To decompose a fraction – write one fraction as the sum/difference of 2 or more fractions.

More examples

Decompose each fraction.

1. $\frac{-x^2+2x-5}{(x+1)(x^2+6x+9)} = \frac{-x^2+2x-5}{(x+1)(x+3)^2} = \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

LCD: (x+1)(x+3)²
Multiply every term by the LCD

~~(x+1)(x+3)~~² · $\frac{-x^2+2x-5}{(x+1)(x+3)^2} = \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

$$\begin{aligned} -x^2+2x-5 &= A(x+3)^2 + B(x+1)(x+3) + C(x+1) \\ -x^2+2x-5 &= A[(x+3)(x+3)] + B(x^2+4x+3) + Cx + C \\ -x^2+2x-5 &= A(x^2+6x+9) + Bx^2+4Bx+3B + Cx + C \\ -x^2+2x-5 &= \underline{Ax^2+6Ax+9A} + \underline{Bx^2+4Bx+3B} + \underline{Cx+C} \\ -x^2+2x-5 &= Ax^2+Bx^2+6Ax+4Bx+Cx+9A+3B+C \\ \underline{-1x^2+2x-5} &= \underline{(A+B)x^2} + \underline{(6A+4B+C)x} + \underline{9A+3B+C} \end{aligned}$$

Distribute

Rearrange like terms together

coeff of x² = coeff of x²
coeff of linear = coeff of linear
const term = constant term

Can also set up augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 6 & 4 & 1 & 2 \\ 9 & 3 & 1 & -5 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 10 \end{array} \right]$$

$$\begin{aligned} \textcircled{1} \quad -1 &= A+B \\ \textcircled{2} \quad 2 &= 6A+4B+C \xrightarrow{x=-1} -2 = -6A-4B-C \\ \textcircled{3} \quad -5 &= 9A+3B+C \xrightarrow{x=-1} -7 = 9A+3B+C \end{aligned}$$

$$\begin{aligned} \textcircled{1} \quad -1 &= A+B \\ -7 &= 3A-B \\ -8 &= 4A \\ \underline{-2} &= A \\ -1 &= A+B \\ -1 &= -2+B \\ \underline{+2} \quad \underline{+2} & \\ \underline{1} &= B \end{aligned}$$

Now sub in A=-2, B=1 in ②

$$\begin{aligned} 2 &= 6A+4B+C \\ 2 &= 6(-2)+4(1)+C \\ 2 &= -12+4+C \\ 2 &= -8+C \Rightarrow \underline{C=10} \end{aligned}$$

$$\therefore \frac{-x^2+2x-5}{(x+1)(x^2+6x+9)} = \frac{-2}{x+1} + \frac{1}{x+3} + \frac{10}{(x+3)^2}$$

2. $\frac{x}{x^2-1} = \frac{x}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

LCD: (x+1)(x-1)

~~(x+1)(x-1)~~ · $\frac{x}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

$x = A(x-1) + B(x+1)$

Let x = -1.
-1 = A(-1-1) + B(-1+1)
-1 = A(-2) + B·0

$\frac{-1}{-2} = \frac{-2A}{-2} \rightarrow \underline{\frac{1}{2} = A}$

Let x = 1.
1 = A(1-1) + B(1+1)
1 = A·0 + B·2
 $\frac{1}{2} = \frac{2B}{2} \rightarrow \underline{\frac{1}{2} = B}$

∴ $\frac{x}{x^2-1} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}$

$$\boxed{\frac{x}{x^2-1} = \frac{1}{2(x+1)} + \frac{1}{2(x-1)}}$$