UNIT 4 Notes

Introduction

In this unit, students explore various topics of discrete mathematics, including arithmetic and geometric sequences and series, as well as recursion and fractals. They also apply the Binomial Theorem, and prove statements using mathematical induction.

The unit concludes with an investigation of probability and statistics, including permutations, combinations, and the normal distribution. Finally, students apply their mathematical skills in a simulation, as well as to sampling situations and to testing hypotheses.

Assessment Options

Unit 4 Test Pages 773–774 of the *Chapter 12 Resource Masters* may be used as a test or review for Unit 4. This assessment contains both multiple-choice and short answer items.

TestCheck and Worksheet Builder

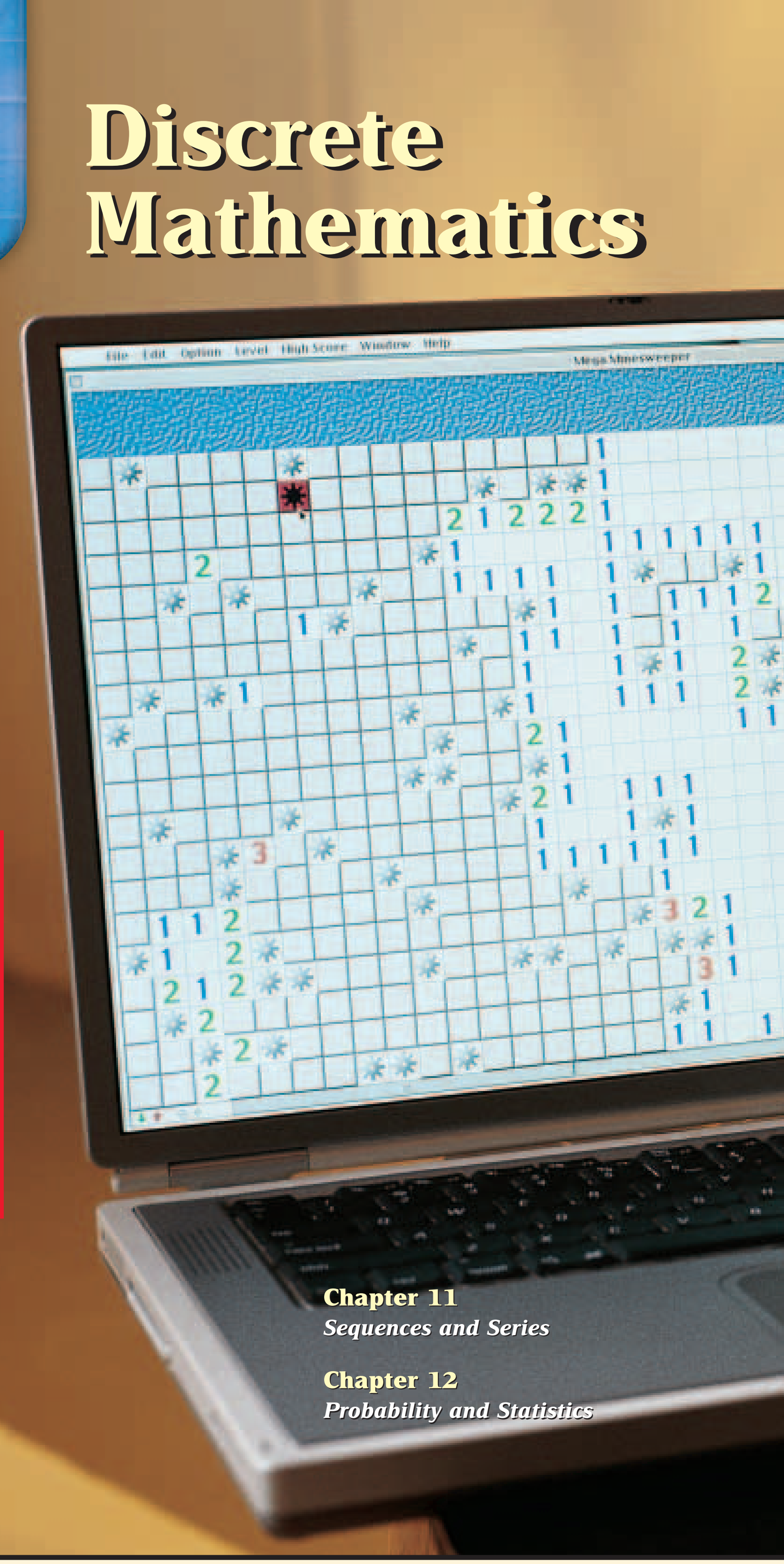
This CD-ROM can be used to create additional unit tests and review worksheets.

UNIT 4

Discrete Mathematics

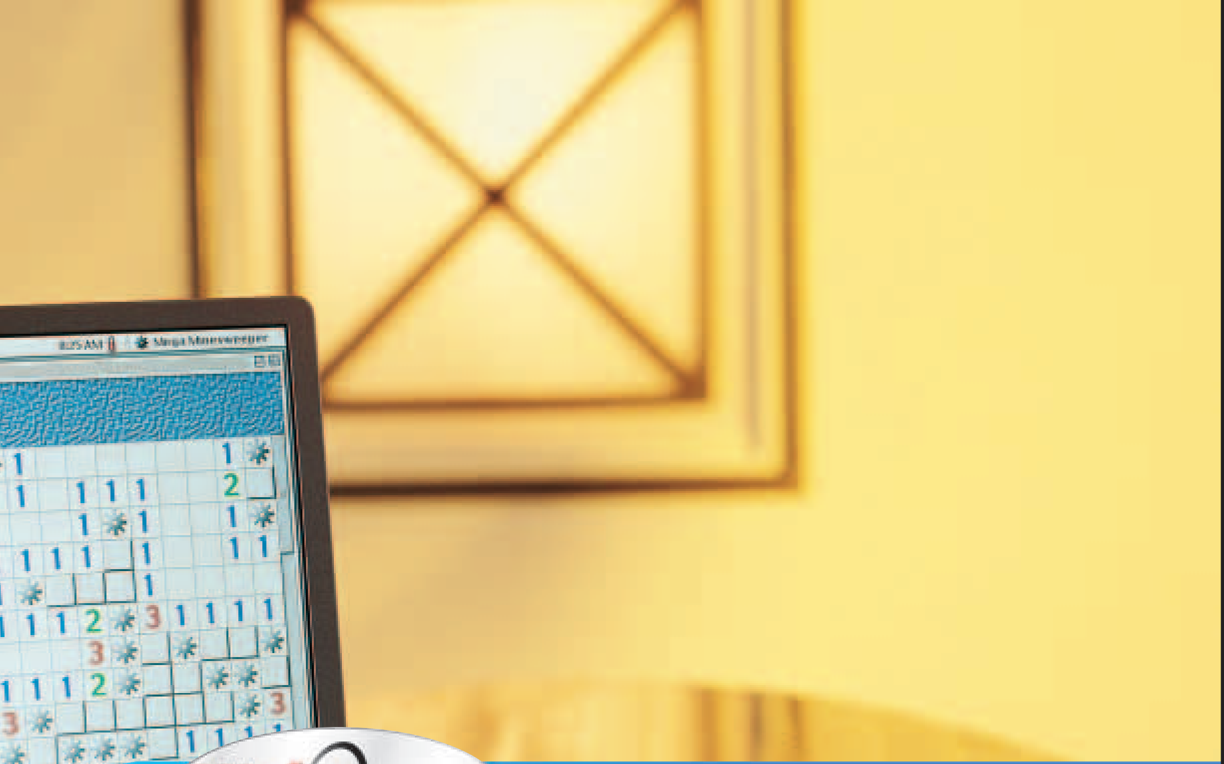
Discrete mathematics is the branch of mathematics that involves finite or discontinuous quantities. In this unit, you will learn about sequences, series, probability, and statistics.

Richard Kaye
Professor of Mathematics
University of Birmingham



Chapter 11
Sequences and Series

Chapter 12
Probability and Statistics



WebQuest Internet Project

'Minesweeper': Secret to Age-Old Puzzle?

Source: *USA TODAY,* November 3, 2000

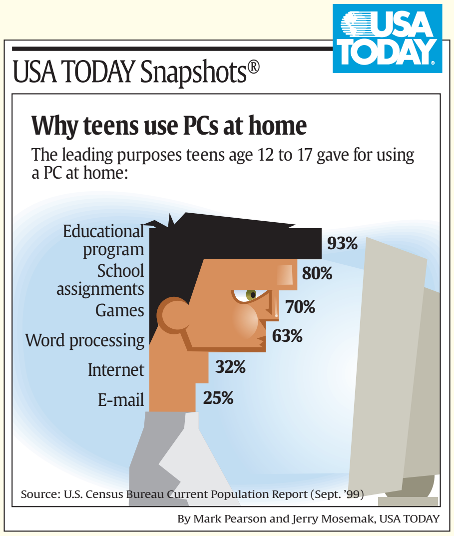
"*Minesweeper*, a seemingly simple game included on most personal computers, could help mathematicians crack one of the field's most intriguing problems. The buzz began after Richard Kaye, a mathematics professor at the University of Birmingham in England, started playing *Minesweeper*. After playing the game steadily for a few weeks, Kaye realized that *Minesweeper*, if played on a much larger grid, has the same mathematical characteristics as other problems deemed insolvable." In this project, you will research a mathematician of the past and his or her role in the development of discrete mathematics.

Log on to www.algebra2.com/webquest. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 4.

Lesson	11-7	12-1
Page	616	635

USA TODAY Snapshots®



Why teens use PCs at home

The leading purposes teens age 12 to 17 gave for using a PC at home:

Source: U.S. Census Bureau Current Population Report (Sept. '99)

By Mark Pearson and Jerry Mosemak, USA TODAY

Teaching Suggestions

Have students study the USA TODAY Snapshot®.

- Ask students to name some historical mathematicians (such as Pythagoras, Fibonacci, and Descartes).
- Have students compare their own use of a PC at home to the percents shown in the graph.
- Point out to students that use of the Internet is quickly becoming the main tool for doing research on a given topic.

Additional USA TODAY Snapshots® appearing in Unit 4:

Chapter 11	Yosemite visitors peak in '96 (p. 604)
Chapter 12	Getting ready for bed (p. 658)

WebQuest Internet Project

A WebQuest is an online project in which students do research on the Internet, gather data, and make presentations using word processing, graphing, page-making, or presentation software. In each chapter, students advance to the next step in their WebQuest. At the end of Chapter 12, the project culminates with a presentation of their findings.

Teaching suggestions and sample answers are available in the *WebQuest and Project Resources.*

Chapter 11

Sequences and Series

Chapter Overview and Pacing

LESSON OBJECTIVES	PACING (days)			
	Regular		Block	
	Basic/ Average	Advanced	Basic/ Average	Advanced
11-1 **Arithmetic Sequences** *(pp. 578–582)* • Use arithmetic sequences. • Find arithmetic means.	1	1	0.5	0.5
11-2 **Arithmetic Series** *(pp. 583–587)* • Find sums of arithmetic series. • Use sigma notation.	1	1	0.5	0.5
11-3 **Geometric Sequences** *(pp. 588–592)* • Use geometric sequences. • Find geometric means.	1	1	0.5	0.5
11-4 **Geometric Series** *(pp. 593–598)* ***Preview:*** Limits • Find sums of geometric series. • Find specific terms of geometric series.	2 (with 11-4 Preview)	1	0.5	0.5
11-5 **Infinite Geometric Series** *(pp. 599–604)* • Find the sum of an infinite geometric series. • Write repeating decimals as fractions.	1	2 (with 11-6 Preview)	0.5	1
11-6 **Recursion and Special Sequences** *(pp. 605–611)* ***Preview:*** Amortizing Loans • Recognize and use special sequences. • Iterate functions. ***Follow-Up:*** Fractals	2	3 (with 11-6 Follow-Up	1	1
11-7 **The Binomial Theorem** *(pp. 612–617)* • Use Pascal's triangle to expand powers of binomials. • Use the Binomial Theorem to expand powers of binomials.	1	1	0.5	0.5
11-8 **Proof and Mathematical Induction** *(pp. 618–621)* • Prove statements by using mathematical induction. • Disprove statements by finding a counterexample.	2	1	1	0.5
Study Guide and **Practice Test** *(pp. 622–627)* **Standardized Test Practice** *(pp. 628–629)*	1	1	0.5	0.5
Chapter Assessment	1	1	0.5	0.5
TOTAL	**13**	**13**	**6**	**6**

Pacing suggestions for the entire year can be found on pages T20–T21.

Timesaving Tools
TeacherWorks™
All-In-One Planner and Resource Center
See pages T12–T13.

Chapter Resource Manager

CHAPTER 11 RESOURCE MASTERS									
Study Guide and Intervention	Practice (Skills and Average)	Reading to Learn Mathematics	Enrichment	Assessment	Applications*	5-Minute Check Transparencies	Interactive Chalkboard	Alge2PASS: Tutorial Plus (lessons)	Materials
631–632	633–634	635	636			11-1	11-1	20	isometric dot paper, wooden or plastic cubes
637–638	639–640	641	642	693	GCS 48, SM 133–138	11-2	11-2		graphing calculator
643–644	645–646	647	648		SC 21	11-3	11-3		
649–650	651–652	653	654	693, 695	SC 22	11-4	11-4	21	(*Preview:* graphing calculator)
655–656	657–658	659	660			11-5	11-5		
661–662	663–664	665	666	694	GCS 47	11-6	11-6		(*Preview:* spreadsheet software) penny, nickel, dime, cardboard (*Follow-Up:* isometric dot paper)
667–668	669–670	671	672			11-7	11-7		colored pens or pencils
673–674	675–676	677	678	694		11-8	11-8		
				679–692, 696–698					

**Key to Abbreviations:* GCS = Graphing Calculator and Speadsheet Masters,
SC = School-to-Career Masters,
SM = Science and Mathematics Lab Manual

Chapter 11

Mathematical Connections and Background

Continuity of Instruction

Prior Knowledge

Students have used formulas for area, volume, and other attributes, and they have used notation such as subscripts, superscripts, and factorials. They have used the output of one function as input for another function, and they have explored proof by deriving particular properties from other properties.

This Chapter

Students study and apply formulas for arithmetic sequences and series, for finite geometric sequences and series, and for infinite geometric series. They use sigma notation and factorial notation to write concise forms of formulas, especially the formula for the Binomial Theorem. Students continue to use formulas and notation as they explore two-part recursive formulas and the three-step process of mathematical induction.

Future Connections

Students will continue to use subscripts, factorials and sigma notation in later math topics, and they will continue to see sequences and series. They will translate between recursive formulas and non-recursive formulas (“explicit formulas”), and they frequently will revisit the powerful idea of mathematical induction.

11-1 Arithmetic Sequences

Throughout this chapter students study notation and formulas. They see how notation allows a formula to be written in a concise form, and they investigate how one formula can be related to or contain another formula. The first lesson uses arithmetic sequences to explore notation and formulas. In an *arithmetic sequence,* each term after the first is found by adding a constant, called the *common difference,* to the previous term. (Students may need to be told that when “arithmetic” is used as an adjective, the accent is on the next-to-last syllable.) Subscripts indicate a particular order for terms, and the formula $a_n = a_1 + (n - 1)d$ is seen as a concise way to represent the nth term.

11-2 Arithmetic Series

This lesson begins with two ideas, that the average of the first n terms in an arithmetic sequence is the mean of the first and nth terms, and that the sum of the first n terms is the number n times the average of these terms. This leads to two formulas for the sum S_n of the terms of an arithmetic series. For one formula, a_1 and $a_1 + (n - 1)d$ are used as the first and nth terms; a different formula uses a_1 and a_n as the first and nth terms. As in the previous lesson, students investigate the formulas by solving problems that ask them to find the value S_n, n, a_1, a_n, or d from given information. The lesson also introduces sigma notation. For example, $\sum_{n=0}^{6}(x^2 + 2)$ represents a sum of terms where x is replaced by each of the seven values 0, 1, 2, 3, 4, 5, and 6. So $\sum_{n=0}^{6}(x^2 + 2) = (0^2 + 2) + (1^2 + 2) + (2^2 + 2) + (3^2 + 2) + (4^2 + 2) + (5^2 + 2) + (6^2 + 2)$. The value of this series is 105.

11-3 Geometric Sequences

This lesson introduces sequences whose terms have a constant ratio r; that is, each successive term is the product of r and the previous term. Using r to represent that common ratio, the formula $a_n = a_1 \cdot r^{n-1}$, which includes both subscripts and superscripts, is a concise way to represent the nth term. After geometric means are described as numbers that form a geometric sequence, students use the formula for the nth term of a geometric sequence to find a given number of geometric means between two given numbers. A *geometric mean* is a number or numbers that are missing terms between two nonsuccessive terms of a geometric sequence.

11-4 Geometric Series

The formula for the sum of the first n terms of a geometric series is derived by using several ideas, each expressed concisely with subscripts and exponents.

(1) The $(n + 1)$st term of a geometric sequence is a_1r^n.

(2) If you multiply the 1st through nth terms of a geometric series by the common ratio, the result is the 2nd through $(n + 1)$st terms.

(3) The difference $S_n - rS_n$ can be written as the equation $S_n - rS_n = a_1 - a_1r^n$.

Dividing each side of that equation by $1 - r$ results in a formula for S_n, the sum of an geometric series. Another formula for S_n can be derived by substituting a_nr for a_1r^n.

11-5 Infinite Geometric Series

A formula for an infinite series can be simpler than a formula for a finite series. If the common ratio r is such that $|r| < 1$, then as n get larger the value of r^n approaches 0. As a result, the sum of an infinite geometric series can be written as a formula that has no exponents, and the sum is completely determined by the first term and the common ratio. Repeating decimals can be expressed by an infinite geometric series as well as by a fraction.

11-6 Recursion and Special Sequences

This lesson introduces a different kind of formula for the terms of a sequence. The formulas have two parts. One part gives specific values for the first one or more terms of the sequence. The second part describes the "next" term as a function of previous terms. (The formulas from the earlier lessons are often called explicit formulas.) The lesson also introduces a situation in which a function rule and the first term of a sequence are given. Each term of the sequence, as an input value, yields the next term in the sequence. This situation is called iterating a function or generating a sequence using iteration.

11-7 The Binomial Theorem

Previously learned formulas are used to develop the Binomial Theorem. The expansion of the expression $(a + b)^n$ for nonnegative values of n involves finding the coefficient and the exponents for a and b for each term. The expansions show many patterns: the sum of the exponents for a and b is n; the coefficients are the entries in Pascal's triangle; and the coefficients are functions of the exponents.

One way to write the Binomial Theorem is to describe each coefficient as a fraction. Another way is to use factorial notation for the coefficients. And another way, using sigma notation as well as factorial notation, illustrates how notation can provide a concise way to write a complex formula. Students should be familiar with factorials, finding powers of monomials, and using sigma notation.

11-8 Proof and Mathematical Induction

Mathematical induction is a powerful idea in mathematics. Much of higher mathematics uses this principle for verification of conjectures. This lesson examines series, divisibility, and finding a counterexample to show that a formula is not true. Students can relate mathematical induction to the two-part recursive formulas of a previous lesson. One part of mathematical induction is to show that a particular property is true for a particular number (often for the number 1). The second part is to prove that if the property holds for some positive integer, then the property holds for the "next" integer. Completing both parts is a proof that the property holds for all positive integers.

Another way to look at mathematical induction is to consider the set S of positive integers for which some property is true. If you can show that 1 is in S and, for any integer k in S, that the integer following k is in S, then S contains all positive integers. Since S is the set of integers for which the property is true, then the property is true for all positive integers.

Chapter 11

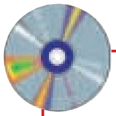
DAILY INTERVENTION and Assessment

	Type	Student Edition	Teacher Resources	Technology/Internet
INTERVENTION	Ongoing	Prerequisite Skills, pp. 577, 582, 587, 592, 598, 604, 610, 617 Practice Quiz 1, p. 592 Practice Quiz 2, p. 617	5-Minute Check Transparencies Quizzes, *CRM* pp. 693–694 Mid-Chapter Test, *CRM* p. 695 Study Guide and Intervention, *CRM* pp. 631–632, 637–638, 643–644, 649–650, 655–656, 661–662, 667–668, 673–674	Alge2PASS: Tutorial Plus www.algebra2.com/self_check_quiz www.algebra2.com/extra_examples
	Mixed Review	pp. 582, 587, 592, 598, 604, 610, 617, 621	Cumulative Review, *CRM* p. 696	
	Error Analysis	Find the Error, pp. 590, 602	Find the Error, *TWE* pp. 590, 602 Unlocking Misconceptions, *TWE* pp. 579, 600 Tips for New Teachers, *TWE* pp. 582, 587, 592, 598, 604, 610, 617, 620	
	Standardized Test Practice	pp. 582, 587, 588, 591, 592, 598, 603, 610, 616, 621, 627, 628–629	*TWE* p. 589 Standardized Test Practice, *CRM* pp. 697–698	Standardized Test Practice CD-ROM www.algebra2.com/standardized_test
ASSESSMENT	Open-Ended Assessment	Writing in Math, pp. 582, 587, 592, 598, 603, 610, 616, 621 Open Ended, pp. 580, 586, 590, 596, 602, 608, 615, 619	Modeling: *TWE* p. 592 Speaking: *TWE* pp. 582, 587, 610 Writing: *TWE* pp. 598, 604, 617 Open-Ended Assessment, *CRM* p. 691	
	Chapter Assessment	Study Guide, pp. 622–626 Practice Test, p. 627	Multiple-Choice Tests (Forms 1, 2A, 2B), *CRM* pp. 679–684 Free-Response Tests (Forms 2C, 2D, 3), *CRM* pp. 685–690 Vocabulary Test/Review, *CRM* p. 692	TestCheck and Worksheet Builder (see below) MindJogger Videoquizzes www.algebra2.com/vocabulary_review www.algebra2.com/chapter_test

Key to Abbreviations: TWE = Teacher Wraparound Edition; CRM = Chapter Resource Masters

Additional Intervention Resources

The Princeton Review's *Cracking the SAT & PSAT*
The Princeton Review's *Cracking the ACT*
ALEKS



TestCheck and Worksheet Builder

This **networkable** software has three modules for intervention and assessment flexibility:

- **Worksheet Builder** to make worksheet and tests
- **Student Module** to take tests on screen (optional)
- **Management System** to keep student records (optional)

Special banks are included for SAT, ACT, TIMSS, NAEP, and End-of-Course tests.

Intervention Technology

Alge2PASS: Tutorial Plus CD-ROM offers a complete, self-paced algebra curriculum.

Algebra 2 Lesson	Alge2PASS Lesson	
11-1	**20**	*Finding the Missing Number in a Sequence*
11-4	**21**	*Sequences and Series*

ALEKS is an online mathematics learning system that adapts assessment and tutoring to the student's needs. Subscribe at www.k12aleks.com.

Intervention at Home

Log on for student study help.

- For each lesson in the Student Edition, there are Extra Examples and Self-Check Quizzes.
  www.algebra2.com/extra_examples
  www.algebra2.com/self_check_quiz
- For chapter review, there is vocabulary review, test practice, and standardized test practice.
  www.algebra2.com/vocabulary_review
  www.algebra2.com/chapter_test
  www.algebra2.com/standardized_test

For more information on Intervention and Assessment, see pp. T8–T11.

Reading and Writing in Mathematics

Glencoe Algebra 2 provides numerous opportunities to incorporate reading and writing into the mathematics classroom.

Student Edition

- Foldables Study Organizer, p. 577
- Concept Check questions require students to verbalize and write about what they have learned in the lesson. (pp. 580, 586, 590, 596, 602, 608, 615, 619, 622)
- Writing in Math questions in every lesson, pp. 582, 587, 592, 598, 603, 610, 616, 621
- Reading Study Tip, pp. 606, 619
- WebQuest, p. 616

Teacher Wraparound Edition

- Foldables Study Organizer, pp. 577, 622
- Study Notebook suggestions, pp. 580, 585, 590, 596, 602, 605, 608, 611, 615, 619
- Modeling activities, p. 592
- Speaking activities, pp. 582, 587, 610
- Writing activities, pp. 598, 604, 617
- Differentiated Instruction, (Verbal/Linguistic), p. 615
- ELL Resources, pp. 576, 581, 586, 591, 597, 603, 609, 615, 616, 621, 622

Additional Resources

- Vocabulary Builder worksheets require students to define and give examples for key vocabulary terms as they progress through the chapter. (*Chapter 11 Resource Masters*, pp. vii-viii)
- Reading to Learn Mathematics master for each lesson (*Chapter 11 Resource Masters*, pp. 635, 641, 647, 653, 659, 665, 671, 677)
- *Vocabulary PuzzleMaker* software creates crossword, jumble, and word search puzzles using vocabulary lists that you can customize.
- *Teaching Mathematics with Foldables* provides suggestions for promoting cognition and language.
- *Reading and Writing in the Mathematics Classroom*
- *WebQuest and Project Resources*

For more information on Reading and Writing in Mathematics, see pp. T6–T7.

Chapter 11 Notes

What You'll Learn

Have students read over the list of objectives and make a list of any words with which they are not familiar.

Why It's Important

Point out to students that this is only one of many reasons why each objective is important. Others are provided in the introduction to each lesson.

Lesson	NCTM Standards	Local Objectives
11-1	1, 6, 7, 8, 9	
11-2	1, 6, 7, 8, 9, 10	
11-3	1, 2, 6, 7, 8, 9, 10	
11-4 Preview	1, 6	
11-4	1, 2, 6, 7, 8, 9, 10	
11-5	1, 2, 6, 7, 8, 9, 10	
11-6 Preview	1, 6, 8, 9, 10	
11-6	1, 2, 6, 7, 8, 9, 10	
11-6 Follow-Up	1, 3, 4, 6, 7, 8	
11-7	1, 2, 6, 8, 9, 10	
11-8	2, 6, 7, 8	

Key to NCTM Standards:

1=Number & Operations, 2=Algebra, 3=Geometry, 4=Measurement, 5=Data Analysis & Probability, 6=Problem Solving, 7=Reasoning & Proof, 8=Communication, 9=Connections, 10=Representation

Chapter 11 Sequences and Series

What You'll Learn

- **Lessons 11-1 through 11-5** Use arithmetic and geometric sequences and series.
- **Lesson 11-6** Use special sequences and iterate functions.
- **Lesson 11-7** Expand powers by using the Binomial Theorem.
- **Lesson 11-8** Prove statements by using mathematical induction.

Key Vocabulary

- arithmetic sequence (p. 578)
- arithmetic series (p. 583)
- sigma notation (p. 585)
- geometric sequence (p. 588)
- geometric series (p. 594)

Why It's Important

Many number patterns found in nature and used in business can be modeled by sequences, which are lists of numbers. Some sequences are classified by the method used to predict the next term from the previous term(s). When the terms of a sequence are added, a series is formed. *In Lesson 11-2, you will learn how the number of seats in the rows of an amphitheater can be modeled using a series.*

Vocabulary Builder

ELL

The Key Vocabulary list introduces students to some of the main vocabulary terms included in this chapter. For a more thorough vocabulary list with pronunciations of new words, give students the Vocabulary Builder worksheets found on pages vii and viii of the *Chapter 11 Resource Masters*. Encourage them to complete the definition of each term as they progress through the chapter. You may suggest that they add these sheets to their study notebooks for future reference when studying for the Chapter 11 test.

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 11.

For Lessons 11-1 and 11-3 — Solve Equations

Solve each equation. *(For review, see Lessons 1-3 and 5-5.)*

1. $36 = 12 + 4x$ **6**

2. $-40 = 10 + 5x$ **−10**

3. $12 - 3x = 27$ **−5**

4. $162 = 2x^4$ **±3**

5. $\frac{1}{8} = 4x^5$ $\frac{1}{2}$

6. $3x^3 + 4 = -20$ **−2**

For Lessons 11-1 and 11-5 — Graph Functions

Graph each function. *(For review, see Lesson 2-1.)* **7–10. See pp. 629A–629F.**

7. $\{(1, 1), (2, 3), (3, 5), (4, 7), (5, 9)\}$

8. $\{(1, -20), (2, -16), (3, -12), (4, -8), (5, -4)\}$

9. $\left\{(1, 64), (2, 16), (3, 4), (4, 1), \left(5, \frac{1}{4}\right)\right\}$

10. $\left\{(1, 2), (2, 3), \left(3, \frac{7}{2}\right), \left(4, \frac{15}{4}\right), \left(5, \frac{31}{8}\right)\right\}$

For Lessons 11-1 through 11-5, 11-8 — Evaluate Expressions

Evaluate each expression for the given value(s) of the variable(s). *(For review, see Lesson 1-1.)*

11. $x + (y - 1)z$ if $x = 3$, $y = 8$, and $z = 2$ **17**

12. $\frac{x}{2}(y + z)$ if $x = 10$, $y = 3$, and $z = 25$ **140**

13. $a \cdot b^{c-1}$ if $a = 2$, $b = \frac{1}{2}$, and $c = 7$ $\frac{1}{32}$

14. $\frac{a(1 - bc)^2}{1 - b}$ if $a = -2$, $b = 3$, and $c = 5$ **196**

15. $\frac{a}{1 - b}$ if $a = \frac{1}{2}$, and $b = \frac{1}{6}$ $\frac{3}{5}$

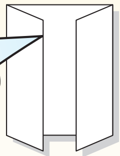
16. $\frac{n(n + 1)}{2}$ if $n = 10$ **55**

FOLDABLES™ Study Organizer

Make this Foldable to record information about sequences and series. Begin with one sheet of 11" by 17" paper and four sheets of notebook paper.

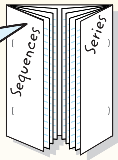
Step 1 Fold and Cut

Fold the short sides of the 11" by 17" paper to meet in the middle.



Step 2 Staple and Label

Fold the notebook paper in half lengthwise. Insert two sheets of notebook paper in each tab and staple edges. Label with lesson numbers.



Reading and Writing As you read and study the chapter, fill the journal with examples for each lesson.

Getting Started

This section provides a review of the basic concepts needed before beginning Chapter 11. Page references are included for additional student help.

Prerequisite Skills in the Getting Ready for the Next Lesson section at the end of each exercise set review a skill needed in the next lesson.

For Lesson	Prerequisite Skill
11-2	Evaluating Expressions (p. 582)
11-3	Evaluating Expressions (p. 587)
11-4	Evaluating Expressions (p. 592)
11-5	Evaluating Expressions (p. 598)
11-6	Evaluating Functions (p. 604)
11-8	Evaluating Expressions (p. 617)

FOLDABLES™ Study Organizer

For more information about Foldables, see *Teaching Mathematics with Foldables.*

Questioning and Organizing Data Before beginning each lesson, ask students to preview each lesson and write several questions about what they see on each of the lesson tabs of their Foldable. Encourage students to write different types of questions including factual, open-ended, analytical, and test-like questions. As students read and work through the lesson, ask them to record the answers to their questions in their journal. Students can add questions to their Foldable that arise during reading, taking notes, or doing homework.

11-1 Lesson Notes

1 Focus

5-Minute Check Transparency 11-1 Use as a quiz or review of Chapter 10.

Mathematical Background notes are available for this lesson on p. 576C.

How are arithmetic sequences related to roofing?

Ask students:

- What other sequences have you seen before? **Answers will vary, but some may recall the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, … .**
- How can you find the next 5 numbers in the shingles sequence? **Add 1 to each successive row.**

11-1 Arithmetic Sequences

What You'll Learn

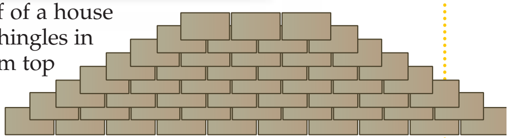
- Use arithmetic sequences.
- Find arithmetic means.

Vocabulary

- sequence
- term
- arithmetic sequence
- common difference
- arithmetic means

How are arithmetic sequences related to roofing?

A roofer is nailing shingles to the roof of a house in overlapping rows. There are three shingles in the top row. Since the roof widens from top to bottom, one additional shingle is needed in each successive row.

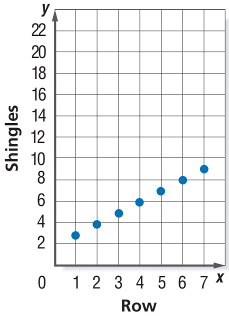


Row	1	2	3	4	5	6	7
Shingles	3	4	5	6	7	8	9

Study Tip

Sequences The numbers in a sequence may not be ordered. For example, the numbers 33, 25, 36, 40, 36, 66, 63, 50, … are a sequence that represents the number of home runs Sammy Sosa hit in each year beginning with 1993.

ARITHMETIC SEQUENCES The numbers 3, 4, 5, 6, …, representing the number of shingles in each row, are an example of a sequence of numbers. A **sequence** is a list of numbers in a particular order. Each number in a sequence is called a **term**. The first term is symbolized by a_1, the second term is symbolized by a_2, and so on.



The graph represents the information from the table above. A sequence is a function whose domain is the set of positive integers. You can see from the graph that a sequence is a discrete function.

Many sequences have patterns. For example, in the sequence above for the number of shingles, each term can be found by adding 1 to the previous term. A sequence of this type is called an arithmetic sequence. An **arithmetic sequence** is a sequence in which each term after the first is found by adding a constant, called the **common difference** d, to the previous term.

Example 1 Find the Next Terms

Find the next four terms of the arithmetic sequence 55, 49, 43, … .

Find the common difference d by subtracting two consecutive terms.

$49 - 55 = -6$ and $43 - 49 = -6$ So, $d = -6$.

Now add -6 to the third term of the sequence, and then continue adding -6 until the next four terms are found.

$$43 \xrightarrow[+(-6)]{} 37 \xrightarrow[+(-6)]{} 31 \xrightarrow[+(-6)]{} 25 \xrightarrow[+(-6)]{} 19$$

The next four terms of the sequence are 37, 31, 25, and 19.

There is a pattern in the way the terms of an arithmetic sequence are formed. It is possible to develop a formula for each term of an arithmetic sequence in terms of the first term a_1 and the common difference d. Look at the sequence in Example 1.

Resource Manager

Workbook and Reproducible Masters

Chapter 11 Resource Masters

- Study Guide and Intervention, pp. 631–632
- Skills Practice, p. 633
- Practice, p. 634
- Reading to Learn Mathematics, p. 635
- Enrichment, p. 636

Teaching Algebra With Manipulatives Masters, pp. 282, 283

Transparencies

5-Minute Check Transparency 11-1
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lesson 20
Interactive Chalkboard

Sequence	**numbers**	55	49	43	37	…	
	symbols	a_1	a_2	a_3	a_4	…	a_n
Expressed in Terms of d and the First Term	**numbers**	$55 + 0(-6)$	$55 + 1(-6)$	$55 + 2(-6)$	$55 + 3(-6)$	…	$55 + (n - 1)(-6)$
	symbols	$a_1 + 0 \cdot d$	$a_1 + 1 \cdot d$	$a_1 + 2 \cdot d$	$a_1 + 3 \cdot d$	…	$a_1 + (n - 1)d$

The following formula generalizes this pattern for any arithmetic sequence.

Key Concept *nth Term of an Arithmetic Sequence*

The nth term a_n of an arithmetic sequence with first term a_1 and common difference d is given by

$$a_n = a_1 + (n - 1)d,$$

where n is any positive integer.

Example 2 *Find a Particular Term*

CONSTRUCTION Refer to the information at the left. Assuming that the arithmetic sequence continues, how much would it cost to rent the crane for twelve months?

Explore Since the difference between any two successive costs is \$15,000, the costs form an arithmetic sequence with common difference 15,000.

Plan You can use the formula for the nth term of an arithmetic sequence with $a_1 = 75{,}000$ and $d = 15{,}000$ to find a_{12}, the cost for twelve months.

Solve

$a_n = a_1 + (n - 1)d$ Formula for nth term

$a_{12} = 75{,}000 + (12 - 1)15{,}000$ $n = 12, a_1 = 75{,}000, d = 15{,}000$

$a_{12} = 240{,}000$ Simplify.

It would cost \$240,000 to rent the crane for twelve months.

Examine You can find terms of the sequence by adding 15,000. a_5 through a_{12} are 135,000, 150,000, 165,000, 180,000, 195,000, 210,000, 225,000, and 240,000. Therefore, \$240,000 is correct.

More About. . .

Construction

The table below shows typical costs for a construction company to rent a crane for one, two, three, or four months.

Months	Cost (\$)
1	75,000
2	90,000
3	105,000
4	120,000

Source: www.howstuffworks.com

Example 3 *Write an Equation for the nth Term*

Write an equation for the nth term of the arithmetic sequence 8, 17, 26, 35, … .

In this sequence, $a_1 = 8$ and $d = 9$. Use the nth term formula to write an equation.

$a_n = a_1 + (n - 1)d$ Formula for nth term

$a_n = 8 + (n - 1)9$ $a_1 = 8, d = 9$

$a_n = 8 + 9n - 9$ Distributive Property

$a_n = 9n - 1$ Simplify.

An equation is $a_n = 9n - 1$.

Study Tip

Arithmetic Sequences

An equation for an arithmetic sequence is always linear.

www.algebra2.com/extra_examples

DAILY INTERVENTION

Unlocking Misconceptions

Subscripts Make sure that all students understand that the subscript in a_n names a term and that it is not an exponent.

2 Teach

ARITHMETIC SEQUENCES

In-Class Examples

Power Point®

1 Find the next four terms of the arithmetic sequence $-8, -6, -4, \ldots$. **−2, 0, 2, 4**

2 **CONSTRUCTION** Use the information in Example 2 to find the cost to rent the crane for 24 months. **\$420,000**

Teaching Tip Ask students to use the formula for the nth Term of an Arithmetic Sequence to show why doubling n does not result in doubling a_n.

3 Write an equation for the nth term of the arithmetic sequence $-8, -6, -4, \ldots$. $a_n = 2n - 10$

Teaching Tip Ask students to read the Study Tip and explain why the equation is always linear. **There is no power greater than 1, and two variables are not multiplied together.**

Interactive Chalkboard

PowerPoint® Presentations

This CD-ROM is a customizable Microsoft® PowerPoint® presentation that includes:

- Step-by-step, dynamic solutions of each In-Class Example from the Teacher Wraparound Edition
- Additional, Your Turn exercises for each example
- The 5-Minute Check Transparencies
- Hot links to Glencoe Online Study Tools

ARITHMETIC MEANS

In-Class Example

Power Point®

4 Find the three arithmetic means between 21 and 45. **27, 33, 39**

Teaching Tip Ask students to create their own questions similar to Example 4. Lead them to see that, for the value of d to be an integer, the number of arithmetic means must evenly divide the difference between the first and last terms given.

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 11.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective
- **Arithmetic Sequences:** 15–51
- **Arithmetic Means:** 52–55

Odd/Even Assignments

Exercises 15–40, 43–48, and 52–55 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 15, 17, 23, 25, 29, 31, 33, 37–43 odd, 47, 49–51, 53, 55–67

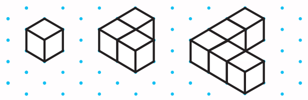
Average: 15–47 odd, 49–51, 53, 55–67

Advanced: 16–48 even, 49–51, 52, 54, 56–64 (optional: 65–67)

Algebra Activity

Arithmetic Sequences

Study the figures below. The length of an edge of each cube is 1 centimeter.



Model and Analyze **1. See margin.**

1. Based on the pattern, draw the fourth figure on a piece of isometric dot paper.
2. Find the volumes of the four figures. **1 cm³, 3 cm³, 5 cm³, 7 cm³**
3. Suppose the number of cubes in the pattern continues. Write an equation that gives the volume of Figure n. $V_n = 2n - 1$
4. What would the volume of the twelfth figure be? **23 cm³**

ARITHMETIC MEANS Sometimes you are given two terms of a sequence, but they are not successive terms of that sequence. The terms between any two nonsuccessive terms of an arithmetic sequence are called **arithmetic means**. In the sequence below, 41, 52, and 63 are the three arithmetic means between 30 and 74.

$$19,\ 30,\ \underbrace{41,\ 52,\ 63}_{\text{3 arithmetic means between 30 and 74}},\ 74,\ 85,\ 96,\ \ldots$$

Example 4 Find Arithmetic Means

Find the four arithmetic means between 16 and 91.

You can use the nth term formula to find the common difference. In the sequence 16, __?__, __?__, __?__, __?__, 91, …, a_1 is 16 and a_6 is 91.

$a_n = a_1 + (n - 1)d$	Formula for the nth term
$a_6 = 16 + (6 - 1)d$	$n = 6, a_1 = 16$
$91 = 16 + 5d$	$a_6 = 91$
$75 = 5d$	Subtract 16 from each side.
$15 = d$	Divide each side by 5.

Now use the value of d to find the four arithmetic means.

$$16 \xrightarrow{+\,15} 31 \xrightarrow{+\,15} 46 \xrightarrow{+\,15} 61 \xrightarrow{+\,15} 76$$

The arithmetic means are 31, 46, 61, and 76. **CHECK** $76 + 15 = 91$ ✓

Study Tip

Alternate Method

You may prefer this method. The four means will be $16 + d$, $16 + 2d$, $16 + 3d$, and $16 + 4d$. The common difference is $d = 91 - (16 + 4d)$ or $d = 15$.

Check for Understanding

Concept Check

1. **Explain** why the sequence 4, 5, 7, 10, 14, … is not arithmetic. **See margin.**
2. **Find** the 15th term in the arithmetic sequence −3, 4, 11, 18, … . **95**
3. **OPEN ENDED** Write an arithmetic sequence with common difference −5.

3. Sample answer: 1, −4, −9, −14, …

Guided Practice

Find the next four terms of each arithmetic sequence.

4. 12, 16, 20, … **24, 28, 32, 36**
5. 3, 1, −1, … **−3, −5, −7, −9**

Find the first five terms of each arithmetic sequence described.

6. $a_1 = 5, d = 3$ **5, 8, 11, 14, 17**
7. $a_1 = 14, d = -2$ **14, 12, 10, 8, 6**

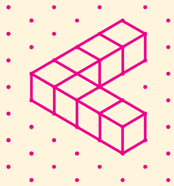
Algebra Activity

Materials: isometric dot paper
- Point out that this activity does not stack cubes but keeps them on a plane.
- Suggest that students explore by repeating this activity using a different initial arrangement of three cubes.

Answer

Algebra Activity

1.



GUIDED PRACTICE KEY

Exercises	Examples
4–7	1
8–11, 14	2
12	3
13	4

12. $a_n = 11n - 37$

8. Find a_{13} for the arithmetic sequence −17, −12, −7, … . **43**

Find the indicated term of each arithmetic sequence.

9. $a_1 = 3, d = -5, n = 24$ **−112**

10. $a_1 = -5, d = 7, n = 13$ **79**

11. Complete: 68 is the __?__ th term of the arithmetic sequence −2, 3, 8, … . **15**

12. Write an equation for the nth term of the arithmetic sequence −26, −15, −4, 7, … .

13. Find the three arithmetic means between 44 and 92. **56, 68, 80**

Application

14. **ENTERTAINMENT** A basketball team has a halftime promotion where a fan gets to shoot a 3-pointer to try to win a jackpot. The jackpot starts at $5000 for the first game and increases $500 each time there is no winner. Ken has tickets to the fifteenth game of the season. How much will the jackpot be for that game if no one wins by then? **$12,000**

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
15–28, 49	1
29–45, 51	2
46–48, 50	3
52–55	4

Extra Practice

See page 851.

Find the next four terms of each arithmetic sequence.

15. 9, 16, 23, … **30, 37, 44, 51**

16. 31, 24, 17, … **10, 3, −4, −11**

17. −6, −2, 2, … **6, 10, 14, 18**

18. −8, −5, −2, … **1, 4, 7, 10**

★19. $\frac{1}{3}, 1, \frac{5}{3}, \ldots$ $\mathbf{\frac{7}{3}, 3, \frac{11}{3}, \frac{13}{3}}$

★20. $\frac{18}{5}, \frac{16}{5}, \frac{14}{5}, \ldots$ $\mathbf{\frac{12}{5}, 2, \frac{8}{5}, \frac{6}{5}}$

★21. 6.7, 6.3, 5.9, … **5.5, 5.1, 4.7, 4.3**

★22. 1.3, 3.8, 6.3, … **8.8, 11.3, 13.8, 16.3**

Find the first five terms of each arithmetic sequence described.

23. $a_1 = 2, d = 13$ **2, 15, 28, 41, 54**

24. $a_1 = 41, d = 5$ **41, 46, 51, 56, 61**

25. $a_1 = 6, d = -4$ **6, 2, −2, −6, −10**

26. $a_1 = 12, d = -3$ **12, 9, 6, 3, 0**

★27. $a_1 = \frac{4}{3}, d = -\frac{1}{3}$ $\mathbf{\frac{4}{3}, 1, \frac{2}{3}, \frac{1}{3}, 0}$

★28. $a_1 = \frac{5}{8}, d = \frac{3}{8}$ $\mathbf{\frac{5}{8}, 1, \frac{11}{8}, \frac{7}{4}, \frac{17}{8}}$

29. Find a_8 if $a_n = 4 + 3n$. **28**

30. If $a_n = 1 - 5n$, what is a_{10}? **−49**

Find the indicated term of each arithmetic sequence.

31. $a_1 = 3, d = 7, n = 14$ **94**

32. $a_1 = -4, d = -9, n = 20$ **−175**

33. $a_1 = 35, d = 3, n = 101$ **335**

34. $a_1 = 20, d = 4, n = 81$ **340**

★35. $a_1 = 5, d = \frac{1}{3}, n = 12$ $\mathbf{\frac{26}{3}}$

★36. $a_1 = \frac{5}{2}, d = -\frac{3}{2}, n = 11$ $\mathbf{-\frac{25}{2}}$

37. a_{12} for −17, −13, −9, … **27**

38. a_{12} for 8, 3, −2, … **−47**

39. a_{21} for 121, 118, 115, … **61**

40. a_{43} for 5, 9, 13, 17, … **173**

41. **GEOLOGY** Geologists estimate that the continents of Europe and North America are drifting apart at a rate of an average of 12 miles every 1 million years, or about 0.75 inch per year. If the continents continue to drift apart at that rate, how many inches will they drift in 50 years? (*Hint*: $a_1 = 0.75$) **37.5 in.**

More About. . .

Tower of Pisa

Upon its completion in 1370, the Leaning Tower of Pisa leaned about 1.7 meters from vertical. Today, it leans about 5.2 meters from vertical.

Source: Associated Press

42. **TOWER OF PISA** To prove that objects of different weights fall at the same rate, Galileo dropped two objects with different weights from the Leaning Tower of Pisa in Italy. The objects hit the ground at the same time. When an object is dropped from a tall building, it falls about 16 feet in the first second, 48 feet in the second second, and 80 feet in the third second, regardless of its weight. How many feet would an object fall in the tenth second? **304 ft**

www.algebra2.com/self_check_quiz

Answer

1. The differences between the terms are not constant.

Enrichment, p. 636

Fibonacci Sequence

Leonardo Fibonacci first discovered the sequence of numbers named for him while studying rabbits. He wanted to know how many pairs of rabbits would be produced in n months, starting with a single pair of newborn rabbits. He made the following assumptions.

1. Newborn rabbits become adults in one month.
2. Each pair of rabbits produces one pair each month.
3. No rabbits die.

Let F_n represent the number of pairs of rabbits at the end of n months. If you begin with one pair of newborn rabbits, $F_0 = F_1 = 1$. This pair of rabbits would produce one pair at the end of the second month, so $F_2 = 1 + 1$, or 2. At the end of the third month, the first pair of rabbits would produce another pair. Thus, $F_3 = 2 + 1$, or 3.

The chart below shows the number of rabbits each month for several months.

Study Guide and Intervention, p. 631 (shown) and p. 632

Arithmetic Sequences An **arithmetic sequence** is a sequence of numbers in which each **term** after the first term is found by adding the **common difference** to the preceding term.

nth Term of an Arithmetic Sequence	$a_n = a_1 + (n - 1)d$, where a_1 is the first term, d is the common difference, and n is any positive integer

Example 1 **Find the next four terms of the arithmetic sequence 7, 11, 15, … .**

Find the common difference by subtracting two consecutive terms.

11 − 7 = 4 and 15 − 11 = 4, so $d = 4$.

Now add 4 to the third term of the sequence, and then continue adding 4 until the four terms are found. The next four terms of the sequence are 19, 23, 27, and 31.

Example 2 **Find the thirteenth term of the arithmetic sequence with $a_1 = 21$ and $d = -6$.**

Use the formula for the nth term of an arithmetic sequence with $a_1 = 21$, $n = 13$, and $d = -6$.

$a_n = a_1 + (n - 1)d$ Formula for nth term
$a_{13} = 21 + (13 - 1)(-6)$ $n = 13, a_1 = 21, d = -6$
$a_{13} = -51$ Simplify.

The thirteenth term is −51.

Example 3 **Write an equation for the nth term of the arithmetic sequence −14, −5, 4, 13, … .**

In this sequence $a_1 = -14$ and $d = 9$. Use the formula for a_n to write an equation.

$a_n = a_1 + (n - 1)d$ Formula for the nth term
$= -14 + (n - 1)9$ $a_1 = -14, d = 9$
$= -14 + 9n - 9$ Distributive Property
$= 9n - 23$ Simplify.

Exercises

Find the next four terms of each arithmetic sequence.

1. 106, 111, 116, … **121, 126, 131, 136**
2. −28, −31, −34, … **−37, −40, −43, −46**
3. 207, 194, 181, … **168, 155, 142, 129**

Find the first five terms of each arithmetic sequence described.

4. $a_1 = 101, d = 9$ **101, 110, 119, 128, 137**
5. $a_1 = -60, d = 4$ **−60, −56, −52, −48, −44**
6. $a_1 = 210, d = -40$ **210, 170, 130, 90, 50**

Find the indicated term of each arithmetic sequence.

7. $a_1 = 4, d = 6, n = 14$ **82**
8. $a_1 = -4, d = -2, n = 12$ **−26**
9. $a_1 = 80, d = -8, n = 21$ **−80**
10. a_{10} for 0, −3, −6, −9, … **−27**

Write an equation for the nth term of each arithmetic sequence.

11. 18, 25, 32, 39, … **7*n* + 11**
12. −110, −85, −60, −35, … **25*n* − 135**
13. 6.2, 8.1, 10.0, 11.9, … **1.9*n* + 4.3**

Skills Practice, p. 633 and Practice, p. 634 (shown)

Find the next four terms of each arithmetic sequence.

1. 5, 8, 11, … **14, 17, 20, 23**
2. −4, −6, −8, … **−10, −12, −14, −16**
3. 100, 93, 86, … **79, 72, 65, 58**
4. −24, −19, −14, … **−9, −4, 1, 6**
5. $\frac{7}{2}, 6, \frac{17}{2}, 11, \ldots$ $\mathbf{\frac{27}{2}, 16, \frac{37}{2}, 21}$
6. 4.8, 4.1, 3.4, … **2.7, 2, 1.3, 0.6**

Find the first five terms of each arithmetic sequence described.

7. $a_1 = 7, d = 7$ **7, 14, 21, 28, 35**
8. $a_1 = -8, d = 2$ **−8, −6, −4, −2, 0**
9. $a_1 = -12, d = -4$ **−12, −16, −20, −24, −28**
10. $a_1 = \frac{1}{2}, d = \frac{1}{2}$ $\mathbf{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}}$
11. $a_1 = -\frac{5}{6}, d = -\frac{1}{3}$ $\mathbf{-\frac{5}{6}, -\frac{7}{6}, -\frac{3}{2}, -\frac{11}{6}, -\frac{13}{6}}$
12. $a_1 = 10.2, d = -5.8$ **10.2, 4.4, −1.4, −7.2, −13**

Find the indicated term of each arithmetic sequence.

13. $a_1 = 5, d = 3, n = 10$ **32**
14. $a_1 = 9, d = 3, n = 29$ **93**
15. a_{18} for −6, −7, −8, … . **−23**
16. a_{37} for 124, 119, 114, … . **−56**
17. $a_1 = \frac{9}{5}, d = -\frac{3}{5}, n = 10$ $\mathbf{-\frac{18}{5}}$
18. $a_1 = 14.25, d = 0.15, n = 31$ **18.75**

Complete the statement for each arithmetic sequence.

19. 166 is the __?__ th term of 30, 34, 38, … **35**
20. 2 is the __?__ th term of $\frac{3}{5}, \frac{4}{5}, 1, \ldots$ **8**

Write an equation for the nth term of each arithmetic sequence.

21. −5, −3, −1, 1, … $\mathbf{a_n = 2n - 7}$
22. −8, −11, −14, −17, … $\mathbf{a_n = -3n - 5}$
23. 1, −1, −3, −5, … $\mathbf{a_n = -2n + 3}$
24. −5, 3, 11, 19, … $\mathbf{a_n = 8n - 13}$

Find the arithmetic means in each sequence.

25. −5, __?__, __?__, __?__, 11 **−1, 3, 7**
26. 82, __?__, __?__, __?__, 18 **66, 50, 34**

27. **EDUCATION** Trevor Koba has opened an English Language School in Isehara, Japan. He began with 26 students. If he enrolls 3 new students each week, in how many weeks will he have 101 students? **26 wk**

28. **SALARIES** Yolanda interviewed for a job that promised her a starting salary of $32,000 with a $1250 raise at the end of each year. What will her salary be during her sixth year if she accepts the job? **$38,250**

Reading to Learn Mathematics, p. 635

ELL

Pre-Activity **How are arithmetic sequences related to roofing?**

Read the introduction to Lesson 11-1 at the top of page 578 in your textbook.

Describe how you would find the number of shingles needed for the fifteenth row. (Do not actually calculate this number.) Explain why your method will give the correct answer. **Sample answer: Add 3 times 14 to 2. This works because the first row has 2 shingles and 3 more are added 14 times to go from the first row to the fifteenth row.**

Reading the Lesson

1. Consider the formula $a_n = a_1 + (n - 1)d$.
   - a. What is this formula used to find? **a particular term of an arithmetic sequence**
   - b. What do each of the following represent?
     - a_n: **the *n*th term**
     - a_1: **the first term**
     - n: **a positive integer that indicates which term you are finding**
     - d: **the common difference**
2. Consider the equation $a_n = -3n + 5$.
   - a. What does this equation represent? **Sample answer: It gives the *n*th term of an arithmetic sequence with first term 2 and common difference −3.**
   - b. Is the graph of this equation a straight line? Explain your answer. **Sample answer: No; the graph is a set of points that fall on a line, but the points do not fill the line.**
   - c. The functions represented by the equations $a_n = -3n + 5$ and $f(x) = -3x + 5$ are alike in that they have the same formula. How are they different? **Sample answer: They have different domains. The domain of the first function is the set of positive integers. The domain of the second function is the set of all real numbers.**

Helping You Remember

3. A good way to remember something is to explain it to someone else. Suppose that your classmate Shala has trouble remembering the formula $a_n = a_1 + (n - 1)d$ correctly. She thinks that the formula should be $a_n = a_1 + nd$. How would you explain to her that she should use $(n - 1)d$ rather than nd in the formula? **Sample answer: Each term after the first in an arithmetic sequence is found by adding *d* to the previous term. You would add *d* once to get to the second term, twice to get to the third term, and so on. So *d* is added *n* − 1 times, not *n* times, to get the *n*th term.**

4 Assess

Open-Ended Assessment

Speaking Have students explain what an arithmetic sequence is and how to find a specified term without repeatedly adding the common difference.

Tips for New Teachers

Intervention Make sure students understand that an arithmetic sequence is a list of numbers that share a certain characteristic, but not all lists of numbers are arithmetic sequences. This will prepare them for Lesson 11-3 on geometric sequences.

Getting Ready for Lesson 11-2

PREREQUISITE SKILL Students will use sigma notation in Lesson 11-2. This will involve their evaluating variable expressions for different values as they find values in a series. Use Exercises 65–67 to determine your students' familiarity with evaluating variable expressions for given values.

Answer

57. Arithmetic sequences can be used to model the numbers of shingles in the rows on a section of roof. Answers should include the following.
    - One additional shingle is needed in each successive row.
    - One method is to successively add 1 to the terms of the sequence: $a_8 = 9 + 1$ or 10, $a_9 = 10 + 1$ or 11, $a_{10} = 11 + 1$ or 12, $a_{11} = 12 + 1$ or 13, $a_{12} = 13 + 1$ or 14, $a_{13} = 14 + 1$ or 15, $a_{14} = 15 + 1$ or 16, $a_{15} = 16 + 1$ or 17. Another method is to use the formula for the nth term: $a_{15} = 3 + (15 - 1)1$ or 17.

Complete the statement for each arithmetic sequence.

43. 170 is the __?__ term of −4, 2, 8, … . **30th**

44. 124 is the __?__ term of −2, 5, 12, … . **19th**

★**45.** −14 is the __?__ term of $2\frac{1}{5}, 2, 1\frac{4}{5}, \ldots$. **82nd**

Write an equation for the *n*th term of each arithmetic sequence.

46. 7, 16, 25, 34, … $a_n = 9n - 2$

47. 18, 11, 4, −3, … $a_n = -7n + 25$

48. −3, −5, −7, −9, … $a_n = -2n - 1$

GEOMETRY **For Exercises 49–51, refer to the first three arrays of numbers below.**

49. Make drawings to find the next three numbers in this pattern.

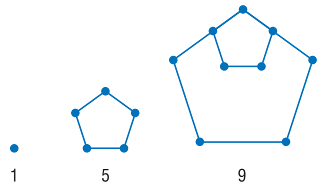
49. 13, 17, 21; See pp. 629A–629F for drawings.

50. Write an equation representing the nth number in this pattern.

50. $p_n = 4n - 3$

51. Is 397 a number in this pattern? Explain.

51. Yes; it corresponds to $n = 100$.



Find the arithmetic means in each sequence.

52. 55, __?__, __?__, __?__, 115 **70, 85, 100**

53. 10, __?__, __?__, −8 **4, −2**

54. −8, __?__, __?__, __?__, __?__, 7

54. −5, −2, 1, 4

55. 3, __?__, __?__, __?__, __?__, __?__, 27

55. 7, 11, 15, 19, 23

56. **CRITICAL THINKING** The numbers x, y, and z are the first three terms of an arithmetic sequence. Express z in terms of x and y. $z = 2y - x$

57. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

How are arithmetic sequences related to roofing?

Include the following in your answer:
- the words that indicate that the numbers of shingles in the rows form an arithmetic sequence, and
- explanations of at least two ways to find the number of shingles in the fifteenth row.

Standardized Test Practice

58. What number follows 20 in this arithmetic sequence? **B**

8, 11, 14, 17, 20, …

Ⓐ 5 Ⓑ 23 Ⓒ 26 Ⓓ 29

59. Find the first term in the arithmetic sequence. **B**

_____, $8\frac{1}{3}, 7, 5\frac{2}{3}, 4\frac{1}{3}, \ldots$

Ⓐ 3 Ⓑ $9\frac{2}{3}$ Ⓒ $10\frac{1}{3}$ Ⓓ 11

Maintain Your Skills

Mixed Review

60. **COMPUTERS** Suppose a computer that costs $3000 new is only worth $600 after 3 years. What is the average annual rate of depreciation? *(Lesson 10-6)* **about 26.7%**

Solve each equation. *(Lesson 10-5)*

61. $3e^x - 2 = 0$ **−0.4055**

62. $e^{3x} = 4$ **0.4621**

63. $\ln(x + 2) = 5$ **146.4132**

64. If y varies directly as x and $y = 5$ when $x = 2$, find y when $x = 6$. *(Lesson 9-4)* **15**

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression for the given values of the variable.** *(To review **evaluating expressions**, see Lesson 1-1.)*

65. $3n - 1$; $n = 1, 2, 3, 4$ **2, 5, 8, 11**

66. $6 - j$; $j = 1, 2, 3, 4$ **5, 4, 3, 2**

67. $4m + 7$; $m = 1, 2, 3, 4, 5$ **11, 15, 19, 23, 27**

DAILY INTERVENTION — Differentiated Instruction

Kinesthetic Have students use wooden or plastic cubes (or ones they make themselves out of paper with a net for a cube drawn on it) to model the Algebra Activity in this lesson.

11-2 Arithmetic Series

What You'll Learn

- Find sums of arithmetic series.
- Use sigma notation.

How do arithmetic series apply to amphitheaters?

The first amphitheaters were built for contests between gladiators. Modern amphitheaters are usually used for the performing arts. Amphitheaters generally get wider as the distance from the stage increases. Suppose a small amphitheater can seat 18 people in the first row and each row can seat 4 more people than the previous row.

Vocabulary

- series
- arithmetic series
- sigma notation
- index of summation

Study Tip

Indicated Sum

The sum of a series is the result when the terms of the series are added. An *indicated sum* is the expression that illustrates the series, which includes the terms + or −.

ARITHMETIC SERIES The numbers of seats in the rows of the amphitheater form an arithmetic sequence. To find the number of people who could sit in the first four rows, add the first four terms of the sequence. That sum is 18 + 22 + 26 + 30 or 96. A **series** is an indicated sum of the terms of a sequence. Since 18, 22, 26, 30 is an arithmetic sequence, 18 + 22 + 26 + 30 is an **arithmetic series**. Below are some more arithmetic sequences and the corresponding arithmetic series.

Arithmetic Sequence	Arithmetic Series
5, 8, 11, 14, 17	5 + 8 + 11 + 14 + 17
−9, −3, 3	−9 + (−3) + 3
$\frac{3}{8}, \frac{8}{8}, \frac{13}{8}, \frac{18}{8}$	$\frac{3}{8} + \frac{8}{8} + \frac{13}{8} + \frac{18}{8}$

S_n represents the sum of the first n terms of a series. For example, S_4 is the sum of the first four terms. For the series 5 + 8 + 11 + 14 + 17, S_4 is 5 + 8 + 11 + 14 or 38.

To develop a formula for the sum of any arithmetic series, consider the series below.

$$S_9 = 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60$$

Suppose we write S_9 in two different orders and add the two equations.

$$\begin{aligned} S_9 &= 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60 \\ (+)\ S_9 &= 60 + 53 + 46 + 39 + 32 + 25 + 18 + 11 + 4 \\ \hline 2S_9 &= 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 \\ 2S_9 &= 9(64) \\ S_9 &= \frac{9}{2}(64) \end{aligned}$$

Note that the sum had 9 terms.

The first and last terms of the sum are 64.

An arithmetic sequence S_n has n terms, and the sum of the first and last terms is $a_1 + a_n$. Thus, the formula $S_n = \frac{n}{2}(a_1 + a_n)$ represents the sum of any arithmetic series.

Key Concept *Sum of an Arithmetic Series*

The sum S_n of the first n terms of an arithmetic series is given by

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d] \text{ or } S_n = \frac{n}{2}(a_1 + a_n).$$

11-2 Lesson Notes

1 Focus

5-Minute Check Transparency 11-2 Use as a quiz or review of Lesson 11-1.

Mathematical Background notes are available for this lesson on p. 576C.

How do arithmetic series apply to amphitheaters?

Ask students:

- What is the value of a_1 in the sequence of seats? **18**
- What is the value of d? **4**
- How would you determine the number of people who could be seated in 15 rows? **Answers will vary.**

Resource Manager

Workbook and Reproducible Masters

Chapter 11 Resource Masters

- Study Guide and Intervention, pp. 637–638
- Skills Practice, p. 639
- Practice, p. 640
- Reading to Learn Mathematics, p. 641
- Enrichment, p. 642
- Assessment, p. 693

Graphing Calculator and Spreadsheet Masters, p. 48

Science and Mathematics Lab Manual, pp. 133–138

Transparencies

5-Minute Check Transparency 11-2

Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

ARITHMETIC SERIES

In-Class Examples

Power Point®

1 Find the sum of the first 20 even numbers, beginning with 2. **420**

Teaching Tip Discuss the difference between a sequence and a series, and ask students to suggest ways to remember which is which.

2 **RADIO** Refer to Example 2 in the Student Edition. Suppose the radio station decided to give away another $124,000 during the next month, using the same plan. How much should they give away on the first day of September, rounded to the nearest cent? **$2683.33**

Teaching Tip Remind students that September is one day shorter than August.

3 Find the first four terms of an arithmetic series in which $a_1 = 14$, $a_n = 29$, and $S_n = 129$. **14, 17, 20, 23**

Answer

Graphing Calculator Investigation

1. The index of summation is always replaced by specific values, so the letter that is used does not affect the value of the sum.

Example 1 Find the Sum of an Arithmetic Series

Find the sum of the first 100 positive integers.

The series is 1 + 2 + 3 + … + 100. Since you can see that $a_1 = 1$, $a_{100} = 100$, and $d = 1$, you can use either sum formula for this series.

Method 1		Method 2
$S_n = \frac{n}{2}(a_1 + a_n)$	Sum formula	$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$
$S_{100} = \frac{100}{2}(1 + 100)$	$n = 100$, $a_1 = 1$, $a_{100} = 100$, $d = 1$	$S_{100} = \frac{100}{2}[2(1) + (100 - 1)1]$
$S_{100} = 50(101)$	Simplify.	$S_{100} = 50(101)$
$S_{100} = 5050$	Multiply.	$S_{100} = 5050$

The sum of the first 100 positive integers is 5050.

Example 2 Find the First Term

RADIO **A radio station considered giving away $4000 every day in the month of August for a total of $124,000. Instead, they decided to increase the amount given away every day while still giving away the same total amount. If they want to increase the amount by $100 each day, how much should they give away the first day?**

You know the values of n, S_n, and d. Use the sum formula that contains d.

$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$	Sum formula
$S_{31} = \frac{31}{2}[2a_1 + (31 - 1)100]$	$n = 31$, $d = 100$
$124{,}000 = \frac{31}{2}(2a_1 + 3000)$	$S_{31} = 124{,}000$
$8000 = 2a_1 + 3000$	Multiply each side by $\frac{2}{31}$.
$5000 = 2a_1$	Subtract 3000 from each side.
$2500 = a_1$	Divide each side by 2.

The radio station should give away $2500 the first day.

More About. . .

Radio

99.0% of teens ages 12–17 listen to the radio at least once a week. 79.1% listen at least once a day.

Source: Radio Advertising Bureau

Sometimes it is necessary to use both a sum formula and the formula for the nth term to solve a problem.

Example 3 Find the First Three Terms

Find the first three terms of an arithmetic series in which $a_1 = 9$, $a_n = 105$, and $S_n = 741$.

Step 1 Since you know a_1, a_n, and S_n, use $S_n = \frac{n}{2}(a_1 + a_n)$ to find n.

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$741 = \frac{n}{2}(9 + 105)$$
$$741 = 57n$$
$$13 = n$$

Step 2 Find d.

$$a_n = a_1 + (n - 1)d$$
$$105 = 9 + (13 - 1)d$$
$$96 = 12d$$
$$8 = d$$

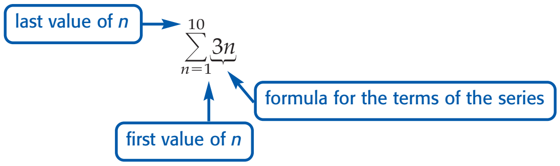
Step 3 Use d to determine a_2 and a_3.

$a_2 = 9 + 8$ or 17 $\qquad$ $a_3 = 17 + 8$ or 25

The first three terms are 9, 17, and 25.

SIGMA NOTATION

Writing out a series can be time-consuming and lengthy. For convenience, there is a more concise notation called **sigma notation**. The series $3 + 6 + 9 + 12 + \ldots + 30$ can be expressed as $\sum_{n=1}^{10} 3n$. This expression is read *the sum of 3n as n goes from 1 to 10.*



The variable, in this case n, is called the **index of summation**.

To generate the terms of a series given in sigma notation, successively replace the index of summation with consecutive integers between the first and last values of the index, inclusive. For the series above, the values of n are 1, 2, 3, and so on, through 10.

Example 4 Evaluate a Sum in Sigma Notation

Evaluate $\sum_{j=5}^{8} (3j - 4)$.

Method 1

Find the terms by replacing j with 5, 6, 7, and 8. Then add.

$$\begin{aligned}\sum_{j=5}^{8} (3j - 4) &= [3(5) - 4] + [3(6) - 4] + [3(7) - 4] + [3(8) - 4]\\ &= 11 + 14 + 17 + 20\\ &= 62\end{aligned}$$

Method 2

Since the sum is an arithmetic series, use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. There are 4 terms, $a_1 = 3(5) - 4$ or 11, and $a_4 = 3(8) - 4$ or 20.

$$S_4 = \frac{4}{2}(11 + 20)$$

$$S_4 = 62$$

The sum of the series is 62.

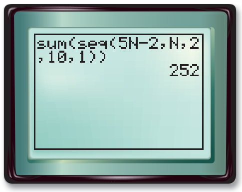
You can use the sum and sequence features on a graphing calculator to find the sum of a series.

Graphing Calculator Investigation

Sums of Series

The calculator screen shows the evaluation of $\sum_{N=2}^{10} (5N - 2)$. The first four entries for seq(are

- the formula for the general term of the series,
- the index of summation,
- the first value of the index, and
- the last value of the index, respectively.



```
sum(seq(5N-2,N,2
,10,1))
                 252
```

The last entry is always 1 for the types of series that we are considering.

Think and Discuss

1. Explain why you can use any letter for the index of summation.
2. Evaluate $\sum_{n=1}^{8} (2n - 1)$ and $\sum_{j=5}^{12} (2j - 9)$. **Make a conjecture** as to their relationship and explain why you think it is true.

Study Tip

Sigma Notation

There are many ways to represent a given series.

$$\begin{aligned}&\sum_{r=4}^{9} (r - 3)\\ &= \sum_{s=2}^{7} (s - 1)\\ &= \sum_{j=0}^{5} (j + 1)\end{aligned}$$

Study Tip

Graphing Calculators

On the TI-83 Plus, sum(is located on the LIST MATH menu. The function seq(is located on the LIST OPS menu.

1. See margin.

2. 64; They represent the same series. Any series can be written in many ways using sigma notation.

www.algebra2.com/extra_examples

SIGMA NOTATION

In-Class Example Power Point®

4 Evaluate $\sum_{k=3}^{10} (2k + 1)$. **112**

Teaching Tip Help students become comfortable with sigma notation by having them read aloud several expressions written in this notation. Explain that sigma is simply the upper case letter S in the Greek alphabet. Ask them what other mathematical notation uses Greek letters. **Sample answer: π**

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 11.
- keep a list of study tips for the graphing calculator, including the one in this lesson.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises…

Organization by Objective

- **Arithmetic Series:** 15–32, 39–45
- **Sigma Notation:** 33–38

Odd/Even Assignments

Exercises 15–26, 29–38, and 41–44 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 15–23 odd, 27–35 odd, 39–45 odd, 46–50, 54–65

Average: 15–45 odd, 46–50, 54–65 (optional: 51–53)

Advanced: 16–46 even, 47–62 (optional: 63–65)

Graphing Calculator Investigation

When the calculator is in **Seq** mode, the variable will automatically be n rather than x. To select **Seq** mode, press MODE, move the cursor down to **FUNC** and over to **Seq** and press ENTER.

Study Guide and Intervention, p. 637 (shown) and p. 638

Arithmetic Series An **arithmetic series** is the sum of consecutive terms of an arithmetic sequence.

Sum of an Arithmetic Series	The sum S_n of the first n terms of an arithmetic series is given by the formula $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$

Example 1 **Find S_n for the arithmetic series with $a_1 = 14$, $a_n = 101$, and $n = 30$.**

Use the sum formula for an arithmetic series.

$$\begin{aligned} S_n &= \tfrac{n}{2}(a_1 + a_n) && \text{Sum formula}\\ S_{30} &= \tfrac{30}{2}(14 + 101) && n = 30,\ a_1 = 14,\ a_n = 101\\ &= 15(115) && \text{Simplify.}\\ &= 1725 && \text{Multiply.} \end{aligned}$$

The sum of the series is 1725.

Example 2 **Find the sum of all positive odd integers less than 180.**

The series is 1 + 3 + 5 + … + 179.

Find n using the formula for the nth term of an arithmetic sequence.

$$\begin{aligned} a_n &= a_1 + (n-1)d && \text{Formula for } n\text{th term}\\ 179 &= 1 + (n-1)2 && a_n = 179,\ a_1 = 1,\ d = 2\\ 179 &= 2n - 1 && \text{Simplify.}\\ 180 &= 2n && \text{Add 1 to each side.}\\ n &= 90 && \text{Divide each side by 2.} \end{aligned}$$

Then use the sum formula for an arithmetic series.

$$\begin{aligned} S_n &= \tfrac{n}{2}(a_1 + a_n) && \text{Sum formula}\\ S_{90} &= \tfrac{90}{2}(1 + 179) && n = 90,\ a_1 = 1,\ a_n = 179\\ &= 45(180) && \text{Simplify.}\\ &= 8100 && \text{Multiply.} \end{aligned}$$

The sum of all positive odd integers less than 180 is 8100.

Exercises

Find S_n for each arithmetic series described.

1. $a_1 = 12$, $a_n = 100$, $n = 12$ **672**
2. $a_1 = 50$, $a_n = -50$, $n = 15$ **0**
3. $a_1 = 60$, $a_n = -136$, $n = 50$ **−1900**
4. $a_1 = 20$, $d = 4$, $a_n = 112$ **1584**
5. $a_1 = 180$, $d = -8$, $a_n = 68$ **1860**
6. $a_1 = -8$, $d = -7$, $a_n = -71$ **−395**
7. $a_1 = 42$, $n = 8$, $d = 6$ **504**
8. $a_1 = 4$, $n = 20$, $d = 2\frac{1}{2}$ **555**
9. $a_1 = 32$, $n = 27$, $d = 3$ **1917**

Find the sum of each arithmetic series.

10. 8 + 6 + 4 + … + −10 **−10**
11. 16 + 22 + 28 + … + 112 **1088**
12. −45 + (−41) + (−37) + … + 35 **−105**

Find the first three terms of each arithmetic series described.

13. $a_1 = 12$, $a_n = 174$, $S_n = 1767$ **12, 21, 30**
14. $a_1 = 80$, $a_n = -115$, $S_n = -245$ **80, 65, 50**
15. $a_1 = 6.2$, $a_n = 12.6$, $S_n = 84.6$ **6.2, 7.0, 7.8**

Skills Practice, p. 639 and Practice, p. 640 (shown)

Find S_n for each arithmetic series described.

1. $a_1 = 16$, $a_n = 98$, $n = 13$ **741**
2. $a_1 = 3$, $a_n = 36$, $n = 12$ **234**
3. $a_1 = -5$, $a_n = -26$, $n = 8$ **−124**
4. $a_1 = 5$, $n = 10$, $a_n = -13$ **−40**
5. $a_1 = 6$, $n = 15$, $a_n = -22$ **−120**
6. $a_1 = -20$, $n = 25$, $a_n = 148$ **1600**
7. $a_1 = 13$, $d = -6$, $n = 21$ **−987**
8. $a_1 = 5$, $d = 4$, $n = 11$ **275**
9. $a_1 = 5$, $d = 2$, $a_n = 33$ **285**
10. $a_1 = -121$, $d = 3$, $a_n = 5$ **−2494**
11. $d = 0.4$, $n = 10$, $a_n = 3.8$ **20**
12. $d = -\frac{2}{3}$, $n = 16$, $a_n = 44$ **784**

Find the sum of each arithmetic series.

13. 5 + 7 + 9 + 11 + … + 27 **192**
14. −4 + 1 + 6 + 11 + … + 91 **870**
15. 13 + 20 + 27 + … + 272 **5415**
16. 89 + 86 + 83 + 80 + … + 20 **1308**
17. $\sum_{n=1}^{4}(1 - 2n)$ **−16**
18. $\sum_{j=1}^{6}(5 + 3n)$ **93**
19. $\sum_{n=1}^{5}(9 - 4n)$ **−15**
20. $\sum_{k=4}^{10}(2k + 1)$ **105**
21. $\sum_{n=3}^{8}(5n - 10)$ **105**
22. $\sum_{n=1}^{101}(4 - 4n)$ **−20,200**

Find the first three terms of each arithmetic series described.

23. $a_1 = 14$, $a_n = -85$, $S_n = -1207$ **14, 11, 8**
24. $a_1 = 1$, $a_n = 19$, $S_n = 100$ **1, 3, 5**
25. $n = 16$, $a_n = 15$, $S_n = -120$ **−30, −27, −24**
26. $n = 15$, $a_n = 5\frac{4}{5}$, $S_n = 45$ $\frac{1}{5}, \frac{3}{5}, 1$

27. **STACKING** A health club rolls its towels and stacks them in layers on a shelf. Each layer of towels has one less towel than the layer below it. If there are 20 towels on the bottom layer and one towel on the top layer, how many towels are stacked on the shelf? **210 towels**

28. **BUSINESS** A merchant places $1 in a jackpot on August 1, then draws the name of a regular customer. If the customer is present, he or she wins the $1 in the jackpot. If the customer is not present, the merchant adds $2 to the jackpot on August 2 and draws another name. Each day the merchant adds an amount equal to the day of the month. If the first person to win the jackpot wins $496, on what day of the month was her or his name drawn? **August 31**

Reading to Learn Mathematics, p. 641

ELL

Pre-Activity **How do arithmetic series apply to amphitheaters?**

Read the introduction to Lesson 11-2 at the top of page 583 in your textbook.

Suppose that an amphitheater can seat 50 people in the first row and that each row thereafter can seat 9 more people than the previous row. Using the vocabulary of arithmetic sequences, describe how you would find the number of people who could be seated in the first 10 rows. (Do not actually calculate the sum.) **Sample answer: Find the first 10 terms of an arithmetic sequence with first term 50 and common difference 9. Then add these 10 terms.**

Reading the Lesson

1. What is the relationship between an arithmetic sequence and the corresponding arithmetic series? **Sample answer: An arithmetic sequence is a list of terms with a common difference between successive terms. The corresponding arithmetic series is the sum of the terms of the sequence.**

2. Consider the formula $S_n = \frac{n}{2}(a_1 + a_n)$. Explain the meaning of this formula in words. **Sample answer: To find the sum of the first *n* terms of an arithmetic sequence, find half the number of terms you are adding. Multiply this number by the sum of the first term and the *n*th term.**

3. a. What is the purpose of sigma notation? **Sample answer: to write a series in a concise form**

   b. Consider the expression $\sum_{i=2}^{12}(4i - 2)$.

   This form of writing a sum is called **sigma notation**.

   The variable i is called the **index of summation**.

   The first value of i is **2**.

   The last value of i is **12**.

   How would you read this expression? **The sum of 4*i* −2 as *i* goes from 2 to 12.**

Helping You Remember

4. A good way to remember something is to relate it to something you already know. How can your knowledge of how to find the average of two numbers help you remember the formula $S_n = \frac{n}{2}(a_1 + a_n)$? **Sample answer: Rewrite the formula as $S_n = n \cdot \frac{a_1 + a_n}{2}$. The average of the first and last terms is given by the expression $\frac{a_1 + a_n}{2}$. The sum of the first *n* terms is the average of the first and last terms multiplied by the number of terms.**

Check for Understanding

Concept Check

1–3. See margin.

1. **Explain** the difference between a sequence and a series.
2. **OPEN ENDED** Write an arithmetic series for which $S_5 = 10$.
3. **OPEN ENDED** Write the series 7 + 10 + 13 + 16 using sigma notation.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	**Examples**
4–9	1, 2
10, 11	4
12, 13	3
14	1

Find S_n for each arithmetic series described.

4. $a_1 = 4$, $a_n = 100$, $n = 25$ **1300**
5. $a_1 = 40$, $n = 20$, $d = -3$ **230**
6. $a_1 = 132$, $d = -4$, $a_n = 52$ **1932**
7. $d = 5$, $n = 16$, $a_n = 72$ **552**

Find the sum of each arithmetic series.

8. 5 + 11 + 17 + … + 95 **800**
9. 38 + 35 + 32 + … + 2 **260**
10. $\sum_{n=1}^{7}(2n + 1)$ **63**
11. $\sum_{k=3}^{7}(3k + 4)$ **95**

Find the first three terms of each arithmetic series described.

12. $a_1 = 11$, $a_n = 110$, $S_n = 726$ **11, 20, 29**
13. $n = 8$, $a_n = 36$, $S_n = 120$ **−6, 0, 6**

Application

14. **WORLD CULTURES** The African-American festival of *Kwanzaa* includes a ritual involving candles. The first night, a candle is lit and then blown out. The second night, a new candle and the candle from the previous night are lit and blown out. This pattern of lighting a new candle and relighting all the candles from the previous nights is continued for seven nights. Use a formula from this lesson to find the total number of candle lightings during the festival. **28**

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
15–32, 39, 40, 45	1, 2
33–38	4
41–44	3

Extra Practice

See page 851.

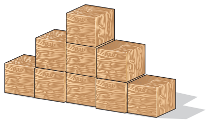
Find S_n for each arithmetic series described.

15. $a_1 = 7$, $a_n = 79$, $n = 8$ **344**
16. $a_1 = 58$, $a_n = -7$, $n = 26$ **663**
17. $a_1 = 43$, $n = 19$, $a_n = 115$ **1501**
18. $a_1 = 76$, $n = 21$, $a_n = 176$ **2646**
19. $a_1 = 7$, $d = -2$, $n = 9$ **−9**
20. $a_1 = 3$, $d = -4$, $n = 8$ **−88**
21. $a_1 = 5$, $d = \frac{1}{2}$, $n = 13$ **104**
22. $a_1 = 12$, $d = \frac{1}{3}$, $n = 13$ **182**
23. $d = -3$, $n = 21$, $a_n = -64$ **−714**
24. $d = 7$, $n = 18$, $a_n = 72$ **225**

★ 25. $d = \frac{1}{5}$, $n = 10$, $a_n = \frac{23}{10}$ **14**

★ 26. $d = -\frac{1}{4}$, $n = 20$, $a_n = -\frac{53}{12}$ $-\frac{245}{6}$

27. **TOYS** Jamila is making a triangular wall with building blocks. The top row has one block, the second row has three, the third has five, and so on. How many rows can she make with a set of 100 blocks? **10 rows**



28. **CONSTRUCTION** A construction company will be fined for each day it is late completing its current project. The daily fine will be $4000 for the first day and will increase by $1000 each day. Based on its budget, the company can only afford $60,000 in total fines. What is the maximum number of days it can be late? **8 days**

Find the sum of each arithmetic series.

29. 6 + 13 + 20 + 27 + … + 97 **721**
30. 7 + 14 + 21 + 28 + … + 98 **735**
31. 34 + 30 + 26 + … + 2 **162**
32. 16 + 10 + 4 + … + (−50) **−204**
33. $\sum_{n=1}^{6}(2n + 11)$ **108**
34. $\sum_{n=1}^{5}(2 - 3n)$ **−35**
35. $\sum_{k=7}^{11}(42 - 9k)$ **−195**
36. $\sum_{t=19}^{23}(5t - 3)$ **510**

★ 37. $\sum_{i=1}^{300}(7i - 3)$ **315,150**

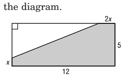
★ 38. $\sum_{k=1}^{150}(11 + 2k)$ **24,300**

Enrichment, p. 642

Geometric Puzzlers

For the problems on this page, you will need to use the Pythagorean Theorem and the formulas for the area of a triangle and a trapezoid.

1. A rectangle measures 5 by 12 units. The upper left corner is cut off as shown in the diagram.

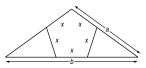


   a. Find the area $A(x)$ of the shaded pentagon. $A(x) = 60 - (5 - x)(6 - x)$

   b. Find x and $2x$ so that $A(x)$ is a maximum. What happens to the

2. A triangle with sides of lengths a, a, and b is isosceles. Two triangles are cut off so that the remaining pentagon has five equal sides of length x. The value of x can be found using this equation.

$$(2b - a)x^2 + (4a^2 - b^2)(2x - a) = 0$$



   a. Find x when $a = 10$ and $b = 12$. $x \approx 4.46$

Answers

1. In a series, the terms are added. In a sequence, they are not.
2. Sample answer: 0 + 1 + 2 + 3 + 4
3. Sample answer: $\sum_{n=1}^{4}(3n + 4)$

39. Find the sum of the first 1000 positive even integers. **1,001,000**

★ **40.** What is the sum of the multiples of 3 between 3 and 999, inclusive? **166,833**

Find the first three terms of each arithmetic series described.

41. $a_1 = 17, a_n = 197, S_n = 2247$ **42.** $a_1 = -13, a_n = 427, S_n = 18{,}423$

43. $n = 31, a_n = 78, S_n = 1023$ **44.** $n = 19, a_n = 103, S_n = 1102$

41. 17, 26, 35
42. −13, −8, −3
43. −12, −9, −6
44. 13, 18, 23

45. AEROSPACE On the Moon, a falling object falls just 2.65 feet in the first second after being dropped. Each second it falls 5.3 feet farther than in the previous second. How far would an object fall in the first ten seconds after being dropped? **265 ft**

CRITICAL THINKING **State whether each statement is *true* or *false*. Explain.**

46. Doubling each term in an arithmetic series will double the sum.

47. Doubling the number of terms in an arithmetic series, but keeping the first term and common difference the same, will double the sum.

46. True; for any series, $2a_1 + 2a_2 + 2a_3 + \ldots + 2a_n = 2(a_1 + a_2 + a_3 + \ldots + a_n)$.

47. False; for example, $7 + 10 + 13 + 16 = 46$, but $7 + 10 + 13 + 16 + 19 + 22 + 25 + 28 = 140$.

48. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See pp. 629A–629F.**

How do arithmetic series apply to amphitheaters?

Include the following in your answer:

- explanations of what the sequence and the series that can be formed from the given numbers represent, and
- two ways to find the amphitheater capacity if it has ten rows of seats.

Standardized Test Practice

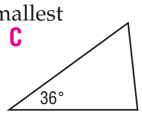
49. $18 + 22 + 26 + 30 + \ldots + 50 = ?$ **C**

Ⓐ 146 Ⓑ 272 Ⓒ 306 Ⓓ 340

50. The angles of a triangle form an arithmetic sequence. If the smallest angle measures 36°, what is the measure of the largest angle? **C**

Ⓐ 60° Ⓑ 72°

Ⓒ 84° Ⓓ 144°



Graphing Calculator

Use a graphing calculator to find the sum of each arithmetic series.

51. $\sum_{n=21}^{75} (2n + 5)$ **5555** **52.** $\sum_{n=10}^{50} (3n - 1)$ **3649** **53.** $\sum_{n=20}^{60} (4n + 3)$ **6683**

Maintain Your Skills

Mixed Review **Find the indicated term of each arithmetic sequence.** *(Lesson 11-1)*

54. $a_1 = 46, d = 5, n = 14$ **111** **55.** $a_1 = 12, d = -7, n = 22$ **−135**

56. RADIOACTIVITY The decay of Radon-222 can be modeled by the equation $y = ae^{-0.1813t}$, where t is measured in days. What is the half-life of Radon-222? *(Lesson 10-6)* **about 3.82 days**

Solve each equation by completing the square. *(Lesson 6-4)*

57. $x^2 + 9x + 20.25 = 0$ **58.** $9x^2 + 96x + 256 = 0$ **59.** $x^2 - 3x - 20 = 0$

57. $-\frac{9}{2}$
58. $-\frac{16}{3}$
59. $\frac{3 \pm \sqrt{89}}{2}$

Simplify. *(Lesson 5-6)*

60. $5\sqrt{3} - 4\sqrt{3}$ $\sqrt{3}$ **61.** $\sqrt{26} \cdot \sqrt{39} \cdot \sqrt{14}$ $26\sqrt{21}$ **62.** $(\sqrt{10} - \sqrt{6})(\sqrt{5} + \sqrt{3})$ $2\sqrt{2}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Evaluate the expression $a \cdot b^{n-1}$ for the given values of a, b, and n.** *(To review **evaluating expressions**, see Lesson 1-1.)*

63. $a = 1, b = 2, n = 5$ **16** **64.** $a = 2, b = -3, n = 4$ **−54** **65.** $a = 18, b = \frac{1}{3}, n = 6$ $\frac{2}{27}$

www.algebra2.com/self_check_quiz

4 Assess

Open-Ended Assessment

Speaking Have students explain what sigma notation means, and why it is a useful way to write a series.

Tips for New Teachers

Intervention Make sure that all students can demonstrate understanding of sigma notation by asking them to write out the terms of a series described in sigma notation.

Getting Ready for Lesson 11-3

PREREQUISITE SKILL Students will find terms in geometric sequences in Lesson 11-3. This will involve their evaluating variable expressions for different values as they find values in a sequence. Use Exercises 63–65 to determine your students' familiarity with evaluating variable expressions for given values.

Assessment Options

Quiz (Lessons 11-1 and 11-2) is available on p. 693 of the *Chapter 11 Resource Masters*.

DAILY INTERVENTION

Differentiated Instruction

Auditory/Musical Have musical students explore and explain how the keys from octave to octave on a piano might relate to a sequence such as A_1, A_2, A_3.

11-3 Lesson Notes

1 Focus

5-Minute Check Transparency 11-3 Use as a quiz or review of Lesson 11-2.

Mathematical Background notes are available for this lesson on p. 576C.

How do geometric sequences apply to a bouncing ball?

Ask students:

- Why is this sequence not an arithmetic sequence? **There is no common difference between terms.**
- Compare a common difference and a common ratio. **The first involves addition; the second, multiplication.**

11-3 Geometric Sequences

What You'll Learn

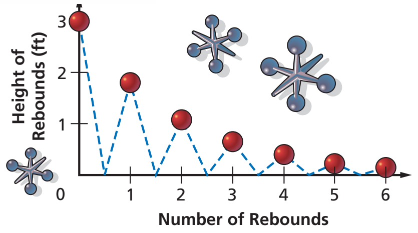
- Use geometric sequences.
- Find geometric means.

Vocabulary

- geometric sequence
- common ratio
- geometric means

How do geometric sequences apply to a bouncing ball?

If you have ever bounced a ball, you know that when you drop it, it never rebounds to the height from which you dropped it. Suppose a ball is dropped from a height of three feet, and each time it falls, it rebounds to 60% of the height from which it fell. The heights of the ball's rebounds form a sequence.



GEOMETRIC SEQUENCES The height of the first rebound of the ball is 3(0.6) or 1.8 feet. The height of the second rebound is 1.8(0.6) or 1.08 feet. The height of the third rebound is 1.08(0.6) or 0.648 feet. The sequence of heights, 1.8, 1.08, 0.648, …, is an example of a **geometric sequence**. A geometric sequence is a sequence in which each term after the first is found by multiplying the previous term by a constant r called the **common ratio**.

As with an arithmetic sequence, you can label the terms of a geometric sequence as a_1, a_2, a_3, and so on. The nth term is a_n and the previous term is a_{n-1}. So, $a_n = r(a_{n-1})$. Thus, $r = \frac{a_n}{a_{n-1}}$. That is, the common ratio can be found by dividing any term by its previous term.

Standardized Test Practice

A B C D

Example 1 Find the Next Term

Multiple-Choice Test Item

Find the missing term in the geometric sequence: 8, 20, 50, 125, ___.

Ⓐ 75 Ⓑ 200 Ⓒ 250 Ⓓ 312.5

Read the Test Item

Since $\frac{20}{8} = 2.5$, $\frac{50}{20} = 2.5$, and $\frac{125}{50} = 2.5$, the sequence has a common ratio of 2.5.

Solve the Test Item

To find the missing term, multiply the last given term by 2.5: $125(2.5) = 312.5$.

The answer is D.

The Princeton Review

Test-Taking Tip

Since the terms of this sequence are increasing, the missing term must be greater than 125. You can immediately eliminate 75 as a possible answer.

You have seen that each term of a geometric sequence can be expressed in terms of r and its previous term. It is also possible to develop a formula that expresses each term of a geometric sequence in terms of r and the first term a_1. Study the patterns shown in the table on the next page for the sequence 2, 6, 18, 54, … .

Resource Manager

Workbook and Reproducible Masters

Chapter 11 Resource Masters

- Study Guide and Intervention, pp. 643–644
- Skills Practice, p. 645
- Practice, p. 646
- Reading to Learn Mathematics, p. 647
- Enrichment, p. 648

School-to-Career Masters, p. 21

Transparencies

5-Minute Check Transparency 11-3

Answer Key Transparencies

Technology

Interactive Chalkboard

Sequence	**numbers**	2	6	18	54	…	
	symbols	a_1	a_2	a_3	a_4	…	a_n
Expressed in Terms of *r* and the Previous Term	**numbers**	2	2(3)	6(3)	18(3)	…	
	symbols	a_1	$a_1 \cdot r$	$a_2 \cdot r$	$a_3 \cdot r$	…	$a_{n-1} \cdot r$
Expressed in Terms of *r* and the First Term	**numbers**	2	2(3)	2(9)	2(27)	…	
		$2(3^0)$	$2(3^1)$	$2(3^2)$	$2(3^3)$	…	
	symbols	$a_1 \cdot r^0$	$a_1 \cdot r^1$	$a_1 \cdot r^2$	$a_1 \cdot r^3$	…	$a_1 \cdot r^{n-1}$

The three entries in the last column of the table all describe the *n*th term of a geometric sequence. This leads us to the following formula for finding the *n*th term of a geometric sequence.

Key Concept — *n*th Term of a Geometric Sequence

The *n*th term a_n of a geometric sequence with first term a_1 and common ratio *r* is given by

$$a_n = a_1 \cdot r^{n-1},$$

where *n* is any positive integer.

Example 2 Find a Particular Term

Find the eighth term of a geometric sequence for which $a_1 = -3$ and $r = -2$.

$a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term

$a_8 = (-3) \cdot (-2)^{8-1}$ $n = 8, a_1 = -3, r = -2$

$a_8 = (-3) \cdot (-128)$ $(-2)^7 = -128$

$a_8 = 384$ Multiply.

The eighth term is 384.

Example 3 Write an Equation for the nth Term

Write an equation for the *n*th term of the geometric sequence 3, 12, 48, 192, … .

In this sequence, $a_1 = 3$ and $r = 4$. Use the *n*th term formula to write an equation.

$a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term

$a_n = 3 \cdot 4^{n-1}$ $a_1 = 3, r = 4$

An equation is $a_n = 3 \cdot 4^{n-1}$.

You can also use the formula for the *n*th term if you know the common ratio and one term of a geometric sequence, but not the first term.

Example 4 Find a Term Given the Fourth Term and the Ratio

Find the tenth term of a geometric sequence for which $a_4 = 108$ and $r = 3$.

First, find the value of a_1.

$a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term

$a_4 = a_1 \cdot 3^{4-1}$ $n = 4, r = 3$

$108 = 27a_1$ $a_4 = 108$

$4 = a_1$ Divide each side by 27.

Now find a_{10}.

$a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term

$a_{10} = 4 \cdot 3^{10-1}$ $n = 10, a_1 = 4, r = 3$

$a_{10} = 78{,}732$ Use a calculator.

The tenth term is 78,732.

www.algebra2.com/extra_examples

2 Teach

GEOMETRIC SEQUENCES

In-Class Examples Power Point®

1 Find the missing term in the geometric sequence: 324, 108, 36, 12, ____. **B**

A 972 **B** 4

C 0 **D** −12

Teaching Tip Discuss the fact that when a sequence has three consecutive terms that are decreasing (or increasing), it will continue to do so.

2 Find the sixth term of a geometric sequence for which $a_1 = -3$ and $r = -2$. $a_6 = 96$

3 Write an equation for the *n*th term of the geometric sequence 5, 10, 20, 40, … . $a_n = 5 \cdot 2^{n-1}$

Teaching Tip Encourage students to begin a geometric sequence problem by writing the known values for each of the variables *n*, *a*, and *r*.

4 Find the seventh term of a geometric sequence for which $a_3 = 96$ and $r = 2$. **1536**

Teaching Tip Emphasize the importance of writing every step of the calculations as an equation, so that each numeric value found during the process is clearly identified.

Teacher to Teacher

Holly K. Plunkett University H.S., Morgantown, WV

"I have my students investigate the problem presented at the beginning of this lesson using a CBL."

Standardized Test Practice

Ⓐ Ⓑ Ⓒ Ⓓ

Example 1 In discussing the Test-Taking Tip for Example 1, point out that a geometric sequence with a negative common ratio is neither increasing nor decreasing.

GEOMETRIC MEANS

In-Class Example

Power Point®

5 Find three geometric means between 3.12 and 49.92.
6.24, 12.48, 24.96 or −6.24, 12.48, −24.96

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 11.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

DAILY INTERVENTION

FIND THE ERROR

Help students see that if the first term is greater than 1, then a decreasing sequence must have a common ratio less than 1.

About the Exercises...

Organization by Objective
- **Geometric Sequences:** 13–42
- **Geometric Means:** 43–46

Odd/Even Assignments

Exercises 13–24, 27–36, and 39–46 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Alert! Exercise 47 requires the Internet or other reference materials.

Assignment Guide

Basic: 13, 15, 21–29 odd, 33–47 odd, 48–61

Average: 13–47 odd, 48–61

Advanced: 14–46 even, 47–58 (optional: 59–61)

All: Practice Quiz 1 (1–5)

GEOMETRIC MEANS In Lesson 11-1, you learned that missing terms between two nonsuccessive terms in an arithmetic sequence are called *arithmetic means*. Similarly, the missing terms(s) between two nonsuccessive terms of a geometric sequence are called **geometric means**. For example, 6, 18, and 54 are three geometric means between 2 and 162 in the sequence 2, 6, 18, 54, 162, … . You can use the common ratio to find the geometric means in a given sequence.

Study Tip

Alternate Method

You may prefer this method. The three means will be $2.25r$, $2.25r^2$, and $2.25r^3$. Then the common ratio is $r = \frac{576}{2.25r^3}$ or $r^4 = \frac{576}{2.25}$. Thus, $r = 4$.

Example 5 Find Geometric Means

Find three geometric means between 2.25 and 576.

Use the *n*th term formula to find the value of *r*. In the sequence 2.25, __?__, __?__, __?__, 576, a_1 is 2.25 and a_5 is 576.

$a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term

$a_5 = 2.25 \cdot r^{5-1}$ $n = 5$, $a_1 = 2.25$

$576 = 2.25r^4$ $a_5 = 576$

$256 = r^4$ Divide each side by 2.25.

$\pm 4 = r$ Take the fourth root of each side.

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of *r* to find three geometric means.

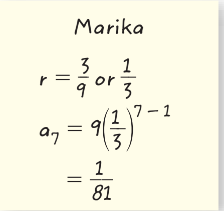
$r = 4$	$r = -4$
$a_2 = 2.25(4)$ or 9	$a_2 = 2.25(-4)$ or -9
$a_3 = 9(4)$ or 36	$a_3 = -9(-4)$ or 36
$a_4 = 36(4)$ or 144	$a_4 = 36(-4)$ or -144

The geometric means are 9, 36, and 144, or −9, 36, and −144.

Check for Understanding

Concept Check

1. **Decide** whether each sequence is *arithmetic* or *geometric*. Explain.
   a. 1, −2, 4, −8, …
   b. 1, −2, −5, −8, …
   1a. Geometric; the terms have a common ratio of −2.
   1b. Arithmetic; the terms have a common difference of −3.
2. **OPEN ENDED** Write a geometric sequence with a common ratio of $\frac{2}{3}$.
   2. Sample answer: 1, $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, …
3. **FIND THE ERROR** Marika and Lori are finding the seventh term of the geometric sequence 9, 3, 1, … .

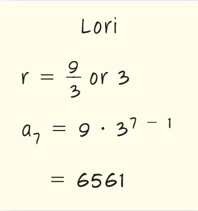


Marika

$r = \frac{3}{9}$ or $\frac{1}{3}$

$a_7 = 9\left(\frac{1}{3}\right)^{7-1}$

$= \frac{1}{81}$



Lori

$r = \frac{9}{3}$ or 3

$a_7 = 9 \cdot 3^{7-1}$

$= 6561$

Who is correct? Explain your reasoning.
Marika; Lori divided in the wrong order when finding *r*.

Guided Practice

Find the next two terms of each geometric sequence.

4. 20, 30, 45, … **67.5, 101.25**
5. $-\frac{1}{4}, \frac{1}{2}, -1, \ldots$ **2, −4**
6. Find the first five terms of the geometric sequence for which $a_1 = -2$ and $r = 3$.
   −2, −6, −18, −54, −162

DAILY INTERVENTION

Differentiated Instruction

Interpersonal Have students in small groups discuss any confusions they may have about the language, formulas, and definitions for arithmetic and geometric sequences and series. Suggest that they help each other organize their notes and thinking to make these topics clear.

GUIDED PRACTICE KEY

Exercises	Examples
4–6, 12	1
7, 8	2
9	3
10	4
11	5

7. Find a_9 for the geometric sequence 60, 30, 15, … . $\frac{15}{64}$

Find the indicated term of each geometric sequence.

8. $a_1 = 7, r = 2, n = 4$ **56**

9. $a_3 = 32, r = -0.5, n = 6$ **−4**

10. Write an equation for the nth term of the geometric sequence 4, 8, 16, … .

11. Find two geometric means between 1 and 27. **3, 9**

10. $a_n = 4 \cdot 2^{n-1}$

Standardized Test Practice

12. Find the missing term in the geometric sequence: $\frac{9}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{12}$, ____. **A**

Ⓐ $\frac{1}{36}$ Ⓑ $\frac{1}{20}$ Ⓒ $\frac{1}{6}$ Ⓓ $\frac{1}{3}$

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
13–24	1
25–30, 33–38, 47, 48	2
31, 32	4
39–42	3
43–46	5

Extra Practice

See page 852.

Find the next two terms of each geometric sequence.

13. 405, 135, 45, … **15, 5**

14. 81, 108, 144, … **192, 256**

15. 16, 24, 36, … **54, 81**

16. 162, 108, 72, … **48, 32**

★ 17. $\frac{5}{2}, \frac{5}{3}, \frac{10}{9}, \ldots$ $\frac{20}{27}, \frac{40}{81}$

★ 18. $\frac{1}{3}, \frac{5}{6}, \frac{25}{12}, \ldots$ $\frac{125}{24}, \frac{625}{48}$

★ 19. 1.25, −1.5, 1.8, … **−2.16, 2.592**

★ 20. 1.4, −3.5, 8.75, … **−21.875, 54.6875**

Find the first five terms of each geometric sequence described.

21. $a_1 = 2, r = -3$ **2, −6, 18, −54, 162**

22. $a_1 = 1, r = 4$ **1, 4, 16, 64, 256**

23. $a_1 = 243, r = \frac{1}{3}$ **243, 81, 27, 9, 3**

24. $a_1 = 576, r = -\frac{1}{2}$ **576, −288, 144, −72, 36**

25. Find a_7 if $a_n = 12\left(\frac{1}{2}\right)^{n-1}$. $\frac{3}{16}$

26. If $a_n = \frac{1}{3} \cdot 6^{n-1}$, what is a_6? **2592**

Find the indicated term of each geometric sequence.

27. $a_1 = \frac{1}{3}, r = 3, n = 8$ **729**

28. $a_1 = \frac{1}{64}, r = 4, n = 9$ **1024**

29. $a_1 = 16{,}807, r = \frac{3}{7}, n = 6$ **243**

30. $a_1 = 4096, r = \frac{1}{4}, n = 8$ $\frac{1}{4}$

★ 31. $a_4 = 16, r = 0.5, n = 8$ **1**

★ 32. $a_6 = 3, r = 2, n = 12$ **192**

33. a_9 for $\frac{1}{5}$, 1, 5, … **78,125**

34. a_7 for $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \ldots$ **2**

35. a_8 for 4, −12, 36, … **−8748**

36. a_6 for 540, 90, 15, … $\frac{5}{72}$

37. **ART** A one-ton ice sculpture is melting so that it loses one-fifth of its weight per hour. How much of the sculpture will be left after five hours? Write the answer in pounds. **655.36 lb**

38. **SALARIES** Geraldo's current salary is $40,000 per year. His annual pay raise is always a percent of his salary at the time. What would his salary be if he got four consecutive 4% increases? **$46,794.34**

Write an equation for the nth term of each geometric sequence.

39. 36, 12, 4, … $a_n = 36\left(\frac{1}{3}\right)^{n-1}$

40. 64, 16, 4, … $a_n = 64\left(\frac{1}{4}\right)^{n-1}$

41. −2, 10, −50, … $a_n = -2(-5)^{n-1}$

42. 4, −12, 36, … $a_n = 4(-3)^{n-1}$

Find the geometric means in each sequence. **46. 6, 12, 24, 48**

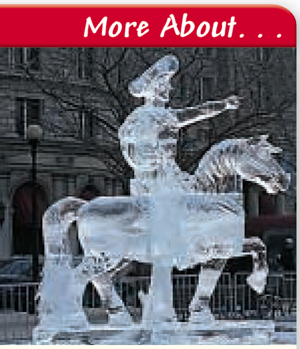
43. 9, __?__, __?__, __?__, 144

44. 4, __?__, __?__, __?__, 324

45. 32, __?__, __?__, __?__, __?__, 1

46. 3, __?__, __?__, __?__, __?__, 96

More About. . .



Art

The largest ever ice construction was an ice palace built for a carnival in St. Paul, Minnesota, in 1992. It contained 10.8 million pounds of ice.

Source: *The Guinness Book of Records*

43. ±18, 36, ±72
44. ±12, 36, ±108
45. 16, 8, 4, 2

www.algebra2.com/self_check_quiz

Study Guide and Intervention, p. 643 (shown) and p. 644

Geometric Sequences A **geometric sequence** is a sequence in which each term after the first is the product of the previous term and a constant called the **constant ratio**.

nth Term of a Geometric Sequence	$a_n = a_1 \cdot r^{n-1}$, where a_1 is the first term, r is the common ratio, and n is any positive integer

Example 1 **Find the next two terms of the geometric sequence 1200, 480, 192, … .**

Since $\frac{480}{1200} = 0.4$ and $\frac{192}{480} = 0.4$, the sequence has a common ratio of 0.4. The next two terms in the sequence are 192(0.4) = 76.8 and 76.8(0.4) = 30.72.

Example 2 **Write an equation for the nth term of the geometric sequence 3.6, 10.8, 32.4, … .**

In this sequence $a_1 = 3.6$ and $r = 3$. Use the nth term formula to write an equation.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term
$= 3.6 \cdot 3^{n-1}$ $a_1 = 3.6, r = 3$

An equation for the nth term is $a_n = 3.6 \cdot 3^{n-1}$.

Exercises

Find the next two terms of each geometric sequence.

1. 6, 12, 24, … **48, 96**
2. 180, 60, 20, … $\frac{20}{3}, \frac{20}{9}$
3. 2000, −1000, 500, … **−250, 125**
4. 0.8, −2.4, 7.2, … **−21.6, 64.8**
5. 80, 60, 45, … **33.75, 25.3125**
6. 3, 16.5, 90.75, … **499.125, 2745.1875**

Find the first five terms of each geometric sequence described.

7. $a_1 = \frac{1}{9}, r = 3$ $\frac{1}{9}, \frac{1}{3}$, 1, 3, 9
8. $a_1 = 240, r = -\frac{3}{4}$ 240, −180, 135, $-101\frac{1}{4}, 75\frac{15}{16}$
9. $a_1 = 10, r = \frac{5}{2}$ 10, 25, $62\frac{1}{2}, 156\frac{1}{4}, 390\frac{5}{8}$

Find the indicated term of each geometric sequence.

10. $a_1 = -10, r = 4, n = 2$ **−40**
11. $a_1 = -6, r = -\frac{1}{2}, n = 8$ $\frac{3}{64}$
12. $a_3 = 9, r = -3, n = 7$ **729**
13. $a_4 = 16, r = 2, n = 10$ **1024**
14. $a_4 = -54, r = -3, n = 6$ **−486**
15. $a_1 = 8, r = \frac{2}{3}, n = 5$ $\frac{128}{81}$

Write an equation for the nth term of each geometric sequence.

16. 500, 350, 245, … $500 \cdot 0.7^{n-1}$
17. 8, 32, 128, … $8 \cdot 4^{n-1}$
18. 11, −24.2, 53.24, … $11 \cdot (-2.2)^{n-1}$

Skills Practice, p. 645 and Practice, p. 646 (shown)

Find the next two terms of each geometric sequence.

1. −15, −30, −60, … **−120, −240**
2. 80, 40, 20, … **10, 5**
3. 90, 30, 10, … $\frac{10}{3}, \frac{10}{9}$
4. −1458, 486, −162, … **54, −18**
5. $\frac{1}{4}, \frac{3}{8}, \frac{9}{16}, \ldots$ $\frac{27}{32}, \frac{81}{64}$
6. 216, 144, 96, … 64, $\frac{128}{3}$

Find the first five terms of each geometric sequence described.

7. $a_1 = -1, r = -3$ **−1, 3, −9, 27, −81**
8. $a_1 = 7, r = -4$ **7, −28, 112, −448, 1792**
9. $a_1 = -\frac{1}{3}, r = 2$ $-\frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}$
10. $a_1 = 12, r = \frac{2}{3}$ 12, 8, $\frac{16}{3}, \frac{32}{9}, \frac{64}{27}$

Find the indicated term of each geometric sequence.

11. $a_1 = 5, r = 3, n = 6$ **1215**
12. $a_1 = 20, r = -3, n = 6$ **−4860**
13. $a_1 = -4, r = -2, n = 10$ **2048**
14. a_8 for $-\frac{1}{250}, -\frac{1}{50}, -\frac{1}{10}, \ldots$ $-\frac{625}{2}$
15. a_{12} for 96, 48, 24, … $\frac{3}{64}$
16. $a_1 = 8, r = \frac{1}{2}, n = 9$ $\frac{1}{32}$
17. $a_1 = -3125, r = -\frac{1}{5}, n = 9$ $-\frac{1}{125}$
18. $a_1 = 3, r = \frac{1}{10}, n = 8$ $\frac{3}{10{,}000{,}000}$

Write an equation for the nth term of each geometric sequence.

19. 1, 4, 16, … $a_n = (4)^{n-1}$
20. −1, −5, −25, … $a_n = -1(5)^{n-1}$
21. 1, $\frac{1}{2}, \frac{1}{4}, \ldots$ $a_n = \left(\frac{1}{2}\right)^{n-1}$
22. −3, −6, −12, … $a_n = -3(2)^{n-1}$
23. 7, −14, 28, … $a_n = 7(-2)^{n-1}$
24. −5, −30, −180, … $a_n = -5(6)^{n-1}$

Find the geometric means in each sequence.

25. 3, __?__, __?__, __?__, 768 **12, 48, 192**
26. 5, __?__, __?__, __?__, 1280 **±20, 80, ±320**
27. 144, __?__, __?__, __?__, 9 **±72, 36, ±18**
28. 37,500, __?__, __?__, __?__, __?__, −12 **−7500, 1500, −300, 60**

29. **BIOLOGY** A culture initially contains 200 bacteria. If the number of bacteria doubles every 2 hours, how many bacteria will be in the culture at the end of 12 hours? **12,800**

30. **LIGHT** If each foot of water in a lake screens out 60% of the light above, what percent of the light passes through 5 feet of water? **1.024%**

31. **INVESTING** Raul invests $1000 in a savings account that earns 5% interest compounded annually. How much money will he have in the account at the end of 5 years? **$1276.28**

Reading to Learn Mathematics, p. 647

ELL

Pre-Activity **How do geometric sequences apply to a bouncing ball?**

Read the introduction to Lesson 11-3 at the top of page 588 in your textbook.

Suppose that you drop a ball from a height of 4 feet, and that each time it falls, it bounces back to 74% of the height from which it fell. Describe how would you find the height of the third bounce. (Do not actually calculate the height of the bounce.)

Sample answer: Multiply 4 by 0.74 three times.

Reading the Lesson

1. Explain the difference between an arithmetic sequence and a geometric sequence.
Sample answer: In an arithmetic sequence, each term after the first is found by adding the common difference to the previous term. In a geometric sequence, each term after the first is found by multiplying the previous term by the common ratio.

2. Consider the formula $a_n = a_1 \cdot r^{n-1}$.
   a. What is this formula used to find? **a particular term of a geometric sequence**
   b. What do each of the following represent?
      a_n: **the nth term**
      a_1: **the first term**
      r: **the common ratio**
      n: **a positive integer that indicates which term you are finding**

3. a. In the sequence 5, 8, 11, 14, 17, 20, the numbers 8, 11, 14, and 17 are **arithmetic means** between 5 and 20.
   b. In the sequence 12, 4, $\frac{4}{3}, \frac{4}{9}, \frac{4}{27}$, the numbers 4, $\frac{4}{3}$, and $\frac{4}{9}$ are **geometric means** between 12 and $\frac{4}{27}$.

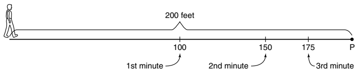
Helping You Remember

4. Suppose that your classmate Ricardo has trouble remembering the formula $a_n = a_1 \cdot r^{n-1}$ correctly. He thinks that the formula should be $a_n = a_1 \cdot r^n$. How would you explain to him that he should use r^{n-1} rather than r^n in the formula?
Sample answer: Each term after the first in a geometric sequence is found by multiplying the previous term by r. There are $n - 1$ terms before the nth term, so you would need to multiply by r a total of $n - 1$ times, not n times, to get the nth term.

Enrichment, p. 648

Half the Distance

Suppose you are 200 feet from a fixed point, P. Suppose that you are able to move to the halfway point in one minute, to the next halfway point one minute after that, and so on.



An interesting sequence results because according to the problem, you never actually reach the point P, although you do get arbitrarily close to it.

You can compute how long it will take to get within some specified small distance of the point. On a calculator, you enter the distance to be covered and then count the number of successive divisions by 2 necessary to get within the desired distance.

4 Assess

Open-Ended Assessment

Modeling With manipulatives or sketches, have students use various geometric elements (for example, numbers of sides and diagonals) to model problems involving arithmetic and geometric sequences.

Tips for New Teachers

Intervention Make sure that students understand the difference between arithmetic and geometric sequences by asking them to create a simple example of each one.

Getting Ready for Lesson 11-4

PREREQUISITE SKILL Students will find the sum of the first *n* terms of geometric series in Lesson 11-4. This will involve evaluating rational expressions for different values. Use Exercises 59–61 to determine your students' familiarity with evaluating rational expressions.

Assessment Options

Practice Quiz 1 The quiz provides students with a brief review of the concepts and skills in Lessons 11-1 through 11-3. Lesson numbers are given to the right of exercises or instruction lines so students can review concepts not yet mastered.

MEDICINE **For Exercises 47 and 48, use the following information.**
Iodine-131 is a radioactive element used to study the thyroid gland.

47. **RESEARCH** Use the Internet or other resource to find the *half-life* of Iodine-131, rounded to the nearest day. This is the amount of time it takes for half of a sample of Iodine-131 to decay into another element. **8 days**

48. How much of an 80-milligram sample of Iodine-131 would be left after 32 days? **5 mg**

CRITICAL THINKING **Determine whether each statement is *true* or *false*. If true, explain. If false, provide a counterexample.**

49. Every sequence is either arithmetic or geometric.

50. There is no sequence that is both arithmetic and geometric.

49. False; the sequence 1, 4, 9, 16, …, for example, is neither arithmetic nor geometric.

50. False, the sequence 1, 1, 1, 1, …, for example, is arithmetic ($d = 0$) and geometric ($r = 1$).

51. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

How do geometric sequences apply to a bouncing ball?

Include the following in your answer:
- the first five terms of the sequence of heights from which the ball falls, and
- any similarities or differences in the sequences for the heights the ball rebounds and the heights from which the ball falls.

Standardized Test Practice

52. Find the missing term in the geometric sequence: −5, 10, −20, 40, ___. **A**

Ⓐ −80 Ⓑ −35 Ⓒ 80 Ⓓ 100

53. What is the tenth term in the geometric sequence: 144, 72, 36, 18, …? **C**

Ⓐ 0 Ⓑ $\frac{9}{64}$ Ⓒ $\frac{9}{32}$ Ⓓ $\frac{9}{16}$

Maintain Your Skills

Mixed Review **Find S_n for each arithmetic series described.** *(Lesson 11-2)*

54. $a_1 = 11$, $a_n = 44$, $n = 23$ **632.5**

55. $a_1 = -5$, $d = 3$, $n = 14$ **203**

Find the arithmetic means in each sequence. *(Lesson 11-1)*

56. 15, __?__, __?__, 27 **19, 23**

57. −8, __?__, __?__, __?__, −24

57. −12, −16, −20

58. **GEOMETRY** Find the perimeter of a triangle with vertices at (2, 4), (−1, 3) and (1, −3). *(Lesson 8-1)* **$5\sqrt{2} + 3\sqrt{10}$ units**

Getting Ready for the Next Lesson **PREREQUISITE SKILL** **Evaluate each expression.** *(To review **expressions**, see Lesson 1-1.)*

59. $\frac{1 - 2^7}{1 - 2}$ **127**

60. $\frac{1 - \left(\frac{1}{2}\right)^6}{1 - \left(\frac{1}{2}\right)}$ **$\frac{63}{32}$**

61. $\frac{1 - \left(-\frac{1}{3}\right)^5}{1 - \left(-\frac{1}{3}\right)}$ **$\frac{61}{81}$**

Practice Quiz 1

Lessons 11-1 through 11-3

Find the indicated term of each arithmetic sequence. *(Lesson 11-1)*

1. $a_1 = 7$, $d = 3$, $n = 14$ **46**

2. $a_1 = 2$, $d = \frac{1}{2}$, $n = 8$ **$\frac{11}{2}$**

Find the sum of each arithmetic series described. *(Lesson 11-2)*

3. $a_1 = 5$, $a_n = 29$, $n = 11$ **187**

4. 6 + 12 + 18 + … + 96 **816**

5. Find a_7 for the geometric sequence 729, −243, 81, … . *(Lesson 11-3)* **1**

Answer

51. The heights of the bounces of a ball and the heights from which a bouncing ball falls each form geometric sequences. Answers should include the following.
- **3, 1.8, 1.08, 0.648, 0.3888**
- **The common ratios are the same, but the first terms are different. The sequence of heights from which the ball falls is the sequence of heights of the bounces with the term 3 inserted at the beginning.**

Graphing Calculator Investigation

A Preview of Lesson 11-4

Limits

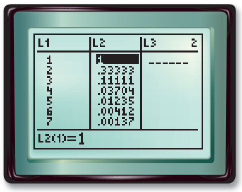
You may have noticed that in some geometric sequences, the later the term in the sequence, the closer the value is to 0. Another way to describe this is that as n increases, a_n approaches 0. The value that the terms of a sequence approach, in this case 0, is called the **limit** of the sequence. Other types of infinite sequences may also have limits. If the terms of a sequence do not approach a unique value, we say that the limit of the sequence does not exist.

Find the limit of the geometric sequence 1, $\frac{1}{3}$, $\frac{1}{9}$, … .

Step 1 *Enter the sequence.*

- The formula for this sequence is $a_n = \left(\frac{1}{3}\right)^{n-1}$.
- Position the cursor on L1 in the STAT EDIT Edit … screen and enter the formula seq(N,N,1,10,1). This generates the values 1, 2, …, 10 of the index N.
- Position the cursor on L2 and enter the formula seq((1/3)^(N-1),N,1,10,1). This generates the first ten terms of the sequence.

KEYSTROKES: *Review sequences in the Graphing Calculator Investigation on page 585.*



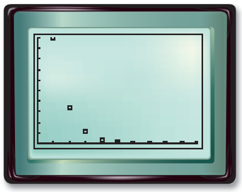
```
L1      L2      L3     2
  1     1       ------
  2     .33333
  3     .11111
  4     .03704
  5     .01235
  6     .00412
  7     .00137
L2(1)=1
```

Notice that as n increases, the terms of the given sequence get closer and closer to 0. If you scroll down, you can see that for $n \geq 8$ the terms are so close to 0 that the calculator expresses them in scientific notation. This suggests that the limit of the sequence is 0.

Step 2 *Graph the sequence.*

- Use a STAT PLOT to graph the sequence. Use L1 as the Xlist and L2 as the Ylist.

KEYSTROKES: *Review STAT PLOTs on page 87.*



[0, 10] scl: 1 by [0, 1] scl: 0.1

The graph also shows that, as n increases, the terms approach 0. In fact, for $n \geq 6$, the marks appear to lie on the horizontal axis. This strongly suggests that the limit of the sequence is 0.

Exercises

Use a graphing calculator to find the limit, if it exists, of each sequence.

1. $a_n = \left(\frac{1}{2}\right)^n$ **0**

2. $a_n = \left(-\frac{1}{2}\right)^n$ **0**

3. $a_n = 4^n$ **does not exist**

4. $a_n = \frac{1}{n^2}$ **0**

5. $a_n = \frac{2^n}{2^n + 1}$ **1**

6. $a_n = \frac{n^2}{n + 1}$ **does not exist**

www.algebra2.com/other_calculator_keystrokes

Graphing Calculator Investigation

A Preview of Lesson 11-4

Getting Started

Entering Sequences To enter the formula seq (N,N,1,10,1) in Step 1, use the keystrokes 2nd [LIST] ▶ 5 ALPHA [N] , ALPHA [N] , 1 , 10 , 1). Follow a similar procedure to enter the formula for L2.

Graphing Sequences Stat plots for sequences are graphed in the same way as any other stat plot. It is essential that lists L1 and L2 contain the same number of elements.

Graphing Window The x-axis settings are determined by the values in L1. The y-axis settings are determined by the values in L2.

Teach

Ask: Does the sequence 3, 9, 27, 81, … have a limit? **no** Does 0.1, 0.01, 0.001, 0.0001, …? **yes**

Assess

Ask the students:

- Describe the graph in the example in terms of asymptotes. **The graph has the x-axis as an asymptote.**
- Does every decreasing geometric sequence have a limit? Explain. **No; A sequence such as −2, −4, −8, −16, … is decreasing and has no limit.**

11-4 Lesson Notes

1 Focus

5-Minute Check Transparency 11-4 Use as a quiz or review of Lesson 11-3.

Mathematical Background notes are available for this lesson on p. 576D.

How is e-mailing a joke like a geometric series?

Ask students:

- How many people have read your joke at the end of Monday? **3** at the end of Tuesday? **12** at the end of Wednesday? **39** at the end of Thursday? **120**

11-4 Geometric Series

What You'll Learn

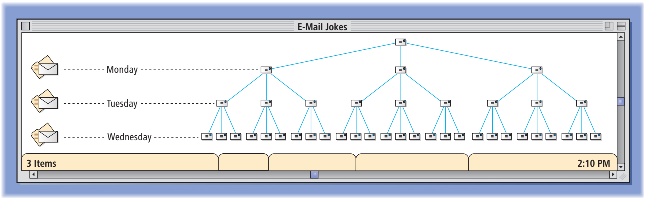
- Find sums of geometric series.
- Find specific terms of geometric series.

Vocabulary

- geometric series

How is e-mailing a joke like a geometric series?

Suppose you e-mail a joke to three friends on Monday. Each of those friends sends the joke on to three of their friends on Tuesday. Each person who receives the joke on Tuesday sends it to three more people on Wednesday, and so on.



GEOMETRIC SERIES Notice that every day, the number of people who read your joke is three times the number that read it the day before. By Sunday, the number of people, including yourself, who have read the joke is 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 or 3280!

The numbers 1, 3, 9, 27, 81, 243, 729, and 2187 form a geometric sequence in which $a_1 = 1$ and $r = 3$. Since 1, 3, 9, 27, 81, 243, 729, 2187 is a geometric sequence, 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 is called a **geometric series**. Below are some more examples of geometric sequences and their corresponding geometric series.

Geometric Sequences	Geometric Series
1, 2, 4, 8, 16	$1 + 2 + 4 + 8 + 16$
4, −12, 36	$4 + (-12) + 36$
$5, 1, \frac{1}{5}, \frac{1}{25}$	$5 + 1 + \frac{1}{5} + \frac{1}{25}$

To develop a formula for the sum of a geometric series, consider the series given in the e-mail situation above.

$$\begin{aligned} S_8 &= 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 \\ (-)\ 3S_8 &= \phantom{1 + {}} 3 + 9 + 27 + 81 + 243 + 729 + 2187 + 6561 \\ \hline (1 - 3)S_8 &= 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 - 6561 \end{aligned}$$

first term in series

$$S_8 = \frac{1 - 6561}{1 - 3} \text{ or } 3280$$

last term in series multiplied by common ratio; in this case, a_9

common ratio

Study Tip

Terms of Geometric Sequences
Remember that a_9 can also be written as a_1r^8.

The expression for S_8 can be written as $S_8 = \frac{a_1 - a_1r^8}{1 - r}$. A rational expression like this can be used to find the sum of any geometric series.

Resource Manager

Workbook and Reproducible Masters

Chapter 11 Resource Masters
- Study Guide and Intervention, pp. 649–650
- Skills Practice, p. 651
- Practice, p. 652
- Reading to Learn Mathematics, p. 653
- Enrichment, p. 654
- Assessment, pp. 693, 695

School-to-Career Masters, p. 22

Transparencies

5-Minute Check Transparency 11-4
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lesson 21
Interactive Chalkboard

Key Concept — Sum of a Geometric Series

The sum S_n of the first n terms of a geometric series is given by

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \text{ or } S_n = \frac{a_1(1 - r^n)}{1 - r}, \text{ where } r \neq 1.$$

You cannot use the formula for the sum with a geometric series for which $r = 1$ because division by 0 would result. In a geometric series with $r = 1$, the terms are constant. For example, $4 + 4 + 4 + \ldots + 4$ is such a series. In general, the sum of n terms of a geometric series with $r = 1$ is $n \cdot a^1$.

Example 1 Find the Sum of the First n Terms

GENEALOGY **In the book *Roots*, author Alex Haley traced his family history back many generations to the time one of his ancestors was brought to America from Africa. If you could trace your family back for 15 generations, starting with your parents, how many ancestors would there be?**

Counting your two parents, four grandparents, eight great-grandparents, and so on gives you a geometric series with $a_1 = 2$, $r = 2$, and $n = 15$.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \qquad \text{Sum formula}$$

$$S_{15} = \frac{2(1 - 2^{15})}{1 - 2} \qquad n = 15,\ a_1 = 2,\ r = 2$$

$$S_{15} = 65{,}534 \qquad \text{Use a calculator.}$$

Going back 15 generations, you have 65,534 ancestors.

More About. . .

Genealogy

When he died in 1992, Samuel Must of Fryburg, Pennsylvania, had a record 824 living descendants.

Source: *The Guinness Book of Records*

As with arithmetic series, you can use sigma notation to represent geometric series.

Example 2 Evaluate a Sum Written in Sigma Notation

Evaluate $\sum_{n=1}^{6} 5 \cdot 2^{n-1}$.

Method 1

Find the terms by replacing n with 1, 2, 3, 4, 5, and 6. Then add.

$$\begin{aligned}\sum_{n=1}^{6} 5 \cdot 2^{n-1} &= 5(2^{1-1}) + 5(2^{2-1}) \\ &\quad + 5(2^{3-1}) + 5(2^{4-1}) \\ &\quad + 5(2^{5-1}) + 5(2^{6-1}) \\ &= 5(1) + 5(2) + 5(4) + 5(8) \\ &\quad + 5(16) + 5(32) \\ &= 5 + 10 + 20 + 40 + 80 \\ &\quad + 160 \\ &= 315\end{aligned}$$

The sum of the series is 315.

Method 2

Since the sum is a geometric series, you can use the formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$

$$S_6 = \frac{5(1 - 2^6)}{1 - 2} \qquad n = 6,\ a_1 = 5,\ r = 2$$

$$S_6 = \frac{5(-63)}{-1} \qquad 2^6 = 64$$

$$S_6 = 315 \qquad \text{Simplify.}$$

How can you find the sum of a geometric series if you know the first and last terms and the common ratio, but not the number of terms? Remember the formula for the nth term of a geometric sequence or series, $a_n = a_1 \cdot r^{n-1}$. You can use this formula to find an expression involving r^n.

$$a_n = a_1 \cdot r^{n-1} \qquad \text{Formula for } n\text{th term}$$

$$a_n \cdot r = a_1 \cdot r^{n-1} \cdot r \qquad \text{Multiply each side by } r.$$

$$a_n \cdot r = a_1 \cdot r^n \qquad r^{n-1} \cdot r^1 = r^{n-1+1} \text{ or } r^n$$

2 Teach

GEOMETRIC SERIES

In-Class Examples Power Point®

Teaching Tip Ask students to explain the difference between counting direct ancestors, as in Example 1, and counting living descendants. Point out that this example counts only direct biological parents, not taking into consideration step-parents, adoptive parents, aunts, uncles, and so on. Discuss how the counting process might change if this assumption were not made.

1 **GENEALOGY** Use the information in Example 2. How many direct ancestors would a person have after 8 generations? **510**

Teaching Tip Review the basic ideas by asking students to explain the difference between a sequence and a series. Ask them to read Example 2 aloud to be sure they can interpret the sigma notation correctly.

2 Evaluate $\sum_{n=1}^{12} 3 \cdot 2^{n-1}$. **12,285**

In-Class Example

PowerPoint®

3 Find the sum of a geometric series for which $a_1 = 7776$, $a_n = 6$, and $r = -\frac{1}{6}$. **6666**

SPECIFIC TERMS

In-Class Example

PowerPoint®

4 Find a_1 in a geometric series for which $S_8 = 765$ and $r = 2$. **3**

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 11.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective

- **Geometric Series:** 15–40, 47
- **Specific Terms:** 41–46

Odd/Even Assignments

Exercises 15–46 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Alert! A graphing calculator is needed for Exercises 52–54.

Assignment Guide

Basic: 15–25 odd, 29–37 odd, 41, 43, 47–51, 55–67

Average: 15–47 odd, 48–51, 55–67 (optional: 52–54)

Advanced: 16–48 even, 49–61 (optional: 62–67)

Now substitute $a_n \cdot r$ for $a_1 \cdot r^n$ in the formula for the sum of a geometric series. The result is $S_n = \frac{a_1 - a_n r}{1 - r}$.

Example 3 Use the Alternate Formula for a Sum

Find the sum of a geometric series for which $a_1 = 15{,}625$, $a_n = -5$, and $r = -\frac{1}{5}$.

Since you do not know the value of n, use the formula derived above.

$S_n = \frac{a_1 - a_n r}{1 - r}$ Alternate sum formula

$= \frac{15{,}625 - (-5)\left(-\frac{1}{5}\right)}{1 - \left(-\frac{1}{5}\right)}$ $a_1 = 15{,}625$, $a_n = -5$, $r = -\frac{1}{5}$

$= \frac{15{,}624}{\frac{6}{5}}$ or 13,020 Simplify.

SPECIFIC TERMS You can use the formula for the sum of a geometric series to help find a particular term of the series.

Example 4 Find the First Term of a Series

Find a_1 in a geometric series for which $S_8 = 39{,}360$ and $r = 3$.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$39{,}360 = \frac{a_1(1 - 3^8)}{1 - 3}$ $S_8 = 39{,}360$; $r = 3$; $n = 8$

$39{,}360 = \frac{-6560a_1}{-2}$ Subtract.

$39{,}360 = 3280a_1$ Divide.

$12 = a_1$ Divide each side by 3280.

The first term of the series is 12.

Check for Understanding

Concept Check

1. **OPEN ENDED** Write a geometric series for which $r = \frac{1}{2}$ and $n = 4$.

1. Sample answer: $4 + 2 + 1 + \frac{1}{2}$

2. **Explain,** using geometric series, why the polynomial $1 + x + x^2 + x^3$ can be written as $\frac{x^4 - 1}{x - 1}$, assuming $x \neq 1$. **See margin.**

3. **Explain** how to write the series 2 + 12 + 72 + 432 + 2592 using sigma notation. **See pp. 629A–629F.**

Guided Practice

GUIDED PRACTICE KEY	
Exercises	**Examples**
4, 5	3
6–9, 14	1
10, 11	2
12, 13	4

Find S_n for each geometric series described.

4. $a_1 = 12$, $a_5 = 972$, $r = -3$ **732**

5. $a_1 = 3$, $a_n = 46{,}875$, $r = -5$ **39,063**

6. $a_1 = 5$, $r = 2$, $n = 14$ **81,915**

7. $a_1 = 243$, $r = -\frac{2}{3}$, $n = 5$ **165**

Find the sum of each geometric series.

8. 54 + 36 + 24 + 16 + … to 6 terms

8. $\frac{1330}{9}$

9. 3 − 6 + 12 − … to 7 terms **129**

10. $\sum_{n=1}^{5} \frac{1}{4} \cdot 2^{n-1}$ $\frac{31}{4}$

11. $\sum_{n=1}^{7} 81\left(\frac{1}{3}\right)^{n-1}$ $\frac{1093}{9}$

DAILY INTERVENTION

Differentiated Instruction

Naturalist Have students research how biologists and ecologists use geometric series in their work to count and predict the population changes for various organisms.

Find the indicated term for each geometric series described.

12. $S_n = \frac{381}{64}$, $r = \frac{1}{2}$, $n = 7$; a_1 **3**

13. $S_n = 33$, $a_n = 48$, $r = -2$; a_1 **3**

Application

14. WEATHER Heavy rain caused a river to rise. The river rose three inches the first day, and each additional day it rose twice as much as the previous day. How much did the river rise in five days? **93 in. or 7 ft 9 in.**

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
15–34, 47	1, 3
35–40	2
41–46	4

Extra Practice

See page 852.

Find S_n for each geometric series described.

15. $a_1 = 2$, $a_6 = 486$, $r = 3$ **728**

16. $a_1 = 3$, $a_8 = 384$, $r = 2$ **765**

17. $a_1 = 1296$, $a_n = 1$, $r = -\frac{1}{6}$ **1111**

18. $a_1 = 343$, $a_n = -1$, $r = -\frac{1}{7}$ **300**

19. $a_1 = 4$, $r = -3$, $n = 5$ **244**

20. $a_1 = 5$, $r = 3$, $n = 12$ **1,328,600**

21. $a_1 = 2401$, $r = -\frac{1}{7}$, $n = 5$ **2101**

22. $a_1 = 625$, $r = \frac{3}{5}$, $n = 5$ **1441**

23. $a_1 = 162$, $r = \frac{1}{3}$, $n = 6$ $\mathbf{\frac{728}{3}}$

24. $a_1 = 80$, $r = -\frac{1}{2}$, $n = 7$ $\mathbf{\frac{215}{4}}$

25. $a_1 = 625$, $r = 0.4$, $n = 8$ **1040.984**

26. $a_1 = 4$, $r = 0.5$, $n = 8$ **7.96875**

★ **27.** $a_2 = -36$, $a_5 = 972$, $n = 7$ **6564**

★ **28.** $a_3 = -36$, $a_6 = -972$, $n = 10$ **−118,096**

29. HEALTH Contagious diseases can spread very quickly. Suppose five people are ill during the first week of an epidemic and that each person who is ill spreads the disease to four people by the end of the next week. By the end of the tenth week of the epidemic, how many people have been affected by the illness? **1,747,625**

30. LEGENDS There is a legend of a king who wanted to reward a boy for a good deed. The king gave the boy a choice. He could have $1,000,000 at once, or he could be rewarded daily for a 30-day month, with one penny on the first day, two pennies on the second day, and so on, receiving twice as many pennies each day as the previous day. How much would the second option be worth? **$10,737,418.23**

More About. . .

Legends

Some of the best-known legends involving a king are the Arthurian legends. According to legend, King Arthur reigned over Britain before the Saxon conquest. Camelot was the most famous castle in the medieval legends of King Arthur.

Find the sum of each geometric series.

31. $4096 - 512 + 64 - \ldots$ to 5 terms **3641**

32. $7 + 21 + 63 + \ldots$ to 10 terms **206,668**

33. $\frac{1}{16} + \frac{1}{4} + 1 + \ldots$ to 7 terms $\mathbf{\frac{5461}{16}}$

34. $\frac{1}{9} - \frac{1}{3} + 1 - \ldots$ to 6 terms $\mathbf{-\frac{182}{9}}$

35. $\sum_{n=1}^{9} 5 \cdot 2^{n-1}$ **2555**

36. $\sum_{n=1}^{6} 2(-3)^{n-1}$ **−364**

37. $\sum_{n=1}^{7} 144\left(-\frac{1}{2}\right)^{n-1}$ $\mathbf{\frac{387}{4}}$

38. $\sum_{n=1}^{8} 64\left(\frac{3}{4}\right)^{n-1}$ $\mathbf{\frac{58,975}{256}}$

★ **39.** $\sum_{n=1}^{20} 3 \cdot 2^{n-1}$ **3,145,725**

★ **40.** $\sum_{n=1}^{16} 4 \cdot 3^{n-1}$ **86,093,440**

Find the indicated term for each geometric series described.

41. $S_n = 165$, $a_n = 48$, $r = -\frac{2}{3}$; a_1 **243**

42. $S_n = 688$, $a_n = 16$, $r = -\frac{1}{2}$; a_1 **1024**

43. $S_n = -364$, $r = -3$, $n = 6$; a_1 **2**

44. $S_n = 1530$, $r = 2$, $n = 8$; a_1 **6**

★ **45.** $S_n = 315$, $r = 0.5$, $n = 6$; a_2 **80**

★ **46.** $S_n = 249.92$, $r = 0.2$, $n = 5$, a_3 **8**

47. LANDSCAPING Rob is helping his dad install a fence. He is using a sledgehammer to drive the pointed fence posts into the ground. On his first swing, he drives a post five inches into the ground. Since the soil is denser the deeper he drives, on each swing after the first, he can only drive the post 30% as far into the ground as he did on the previous swing. How far has he driven the post into the ground after five swings? **about 7.13 in.**

www.algebra2.com/self_check_quiz

Answer

2. The polynomial is a geometric series with first term 1, common ratio x, and 4 terms. The sum is $\frac{1(1 - x^4)}{1 - x} = \frac{x^4 - 1}{x - 1}$.

Enrichment, p. 654

Annuities

An annuity is a fixed amount of money payable at given intervals. For example, suppose you wanted to set up a trust fund so that $30,000 could be withdrawn each year for 14 years before the money ran out. Assume the money can be invested at 9%.

You must find the amount of money that needs to be invested. Call this amount A. After the third payment, the amount left is

$$1.09[1.09A - 30{,}000(1 + 1.09)] - 30{,}000 = 1.09^2A - 30{,}000(1 + 1.09 + 1.09^2).$$

The results are summarized in the table below.

Payment Number	Number of Dollars Left After Payment
1	$A - 30{,}000$
2	$1.09A - 30{,}000(1 + 1.09)$
3	$1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$

1. Use the pattern shown in the table to find the number of dollars left after the fourth payment. $\mathbf{1.09^3A - 30{,}000(1 + 1.09 + 1.09^2 + 1.09^3)}$

Study Guide and Intervention, p. 649 (shown) and p. 650

Geometric Series A **geometric series** is the indicated sum of consecutive terms of a geometric sequence.

Sum of a Geometric Series	The sum S_n of the first n terms of a geometric series is given by $S_n = \frac{a_1(1 - r^n)}{1 - r}$ or $S_n = \frac{a_1 - a_1r^n}{1 - r}$, where $r \neq 1$.

Example 1 **Find the sum of the first four terms of the geometric sequence for which $a_1 = 120$ and $r = \frac{1}{3}$.**

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$S_4 = \frac{120\left(1 - \left(\frac{1}{3}\right)^4\right)}{1 - \frac{1}{3}}$ $n = 4$, $a_1 = 120$, $r = \frac{1}{3}$

≈ 177.78 Use a calculator.

The sum of the series is 177.78.

Example 2 **Find the sum of the geometric series $\sum_{j=1}^{7} 4 \cdot 3^{j-2}$.**

Since the sum is a geometric series, you can use the sum formula.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$S_7 = \frac{\frac{4}{3}(1 - 3^7)}{1 - 3}$ $n = 7$, $a_1 = \frac{4}{3}$, $r = 3$

≈ 1457.33 Use a calculator.

The sum of the series is 1457.33.

Exercises

Find S_n for each geometric series described.

1. $a_1 = 2$, $a_n = 486$, $r = 3$ **728**

2. $a_1 = 1200$, $a_n = 75$, $r = \frac{1}{2}$ **2325**

3. $a_1 = \frac{1}{25}$, $a_n = 125$, $r = 5$ **156.24**

4. $a_1 = 3$, $r = \frac{1}{3}$, $n = 4$ **4.44**

5. $a_1 = 2$, $r = 6$, $n = 4$ **518**

6. $a_1 = 2$, $r = 4$, $n = 6$ **2730**

7. $a_1 = 100$, $r = -\frac{1}{2}$, $n = 5$ **68.75**

8. $a_3 = 20$, $a_6 = 160$, $n= 8$ **1275**

9. $a_4 = 16$, $a_7 = 1024$, $n = 10$ **87,381.25**

Find the sum of each geometric series.

10. $6 + 18 + 54 + \ldots$ to 6 terms **2184**

11. $\frac{1}{4} + \frac{1}{2} + 1 + \ldots$ to 10 terms **255.75**

12. $\sum_{j=4}^{8} 2^j$ **496**

13. $\sum_{k=1}^{7} 3 \cdot 2^{k-1}$ **381**

Skills Practice, p. 651 and Practice, p. 652 (shown)

Find S_n for each geometric series described.

1. $a_1 = 2$, $a_6 = 64$, $r = 2$ **126**

2. $a_1 = 160$, $a_6 = 5$, $r = \frac{1}{2}$ **315**

3. $a_1 = -3$, $a_n = -192$, $r = -2$ **−129**

4. $a_1 = -81$, $a_n = -16$, $r = -\frac{2}{3}$ **−55**

5. $a_1 = -3$, $a_n = 3072$, $r = -4$ **2457**

6. $a_1 = 54$, $a_6 = \frac{2}{9}$, $r = \frac{1}{3}$ $\mathbf{\frac{728}{9}}$

7. $a_1 = 5$, $r = 3$, $n = 9$ **49,205**

8. $a_1 = -6$, $r = -1$, $n = 21$ **−6**

9. $a_1 = -6$, $r = -3$, $n = 7$ **−3282**

10. $a_1 = -9$, $r = \frac{2}{3}$, $n = 4$ $\mathbf{-\frac{65}{3}}$

11. $a_1 = \frac{1}{3}$, $r = 3$, $n = 10$ $\mathbf{\frac{29,524}{3}}$

12. $a_1 = 16$, $r = -1.5$, $n = 6$ **−66.5**

Find the sum of each geometric series.

13. $162 + 54 + 18 + \ldots$ to 6 terms $\mathbf{\frac{728}{3}}$

14. $2 + 4 + 8 + \ldots$ to 8 terms **510**

15. $64 - 96 + 144 - \ldots$ to 7 terms **463**

16. $\frac{1}{9} - \frac{1}{3} + 1 - \ldots$ to 6 terms $\mathbf{-\frac{182}{9}}$

17. $\sum_{n=1}^{8} (-3)^{n-1}$ **−1640**

18. $\sum_{n=1}^{9} 5(-2)^{n-1}$ **855**

19. $\sum_{n=1}^{5} -1(4)^{n-1}$ **−341**

20. $\sum_{n=1}^{6} \left(\frac{1}{2}\right)^{n-1}$ $\mathbf{\frac{63}{32}}$

21. $\sum_{n=1}^{10} 2560\left(\frac{1}{2}\right)^{n-1}$ **5115**

22. $\sum_{n=1}^{4} 9\left(\frac{2}{3}\right)^{n-1}$ $\mathbf{\frac{65}{3}}$

Find the indicated term for each geometric series described.

23. $S_n = 1023$, $a_n = 768$, $r = 4$; a_1 **3**

24. $S_n = 10{,}160$, $a_n = 5120$, $r = 2$; a_1 **80**

25. $S_n = -1365$, $n = 12$, $r = -2$; a_1 **1**

26. $S_n = 665$, $n = 6$, $r = 1.5$; a_1 **32**

27. CONSTRUCTION A pile driver drives a post 27 inches into the ground on its first hit. Each additional hit drives the post $\frac{2}{3}$ the distance of the prior hit. Find the total distance the post has been driven after 5 hits. $\mathbf{70\frac{1}{3}}$ **in.**

28. COMMUNICATIONS Hugh Moore e-mails a joke to 5 friends on Sunday morning. Each of these friends e-mails the joke to 5 of her or his friends on Monday morning, and so on. Assuming no duplication, how many people will have heard the joke by the end of Saturday, not including Hugh? **97,655 people**

Reading to Learn Mathematics, p. 653

ELL

Pre-Activity **How is e-mailing a joke like a geometric series?**

Read the introduction to Lesson 11-4 at the top of page 594 in your textbook.

- Suppose that you e-mail the joke on Monday to five friends, rather than three, and that each of those friends e-mails it to five friends on Tuesday, and so on. Write a sum that shows that total number of people, including yourself, who will have read the joke by Thursday. (Write out the sum using plus signs rather than sigma notation. Do not actually find the sum.) **1 + 5 + 25 + 125**
- Use exponents to rewrite the sum you found above. (Use an exponent in each term, and use the same base for all terms.) $\mathbf{5^0 + 5^1 + 5^2 + 5^3}$

Reading the Lesson

1. Consider the formula $S_n = \frac{a_1(1 - r^n)}{1 - r}$.

a. What is this formula used to find? **the sum of the first *n* terms of a geometric series**

b. What do each of the following represent?

S_n: **the sum of the first *n* terms**

a_1: **the first term**

r: **the common ratio**

c. Suppose that you want to use the formula to evaluate $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27}$. Indicate the values you would substitute into the formula in order to find S_n. (Do not actually calculate the sum.)

$n =$ **5** $a_1 =$ **3** $r =$ $\mathbf{-\frac{1}{3}}$ $r^n =$ $\mathbf{\left(-\frac{1}{3}\right)^5}$ **or** $\mathbf{-\frac{1}{243}}$

d. Suppose that you want to use the formula to evaluate the sum $\sum_{n=1}^{6} 8(-2)^{n-1}$. Indicate the values you would substitute into the formula in order to find S_n. (Do not actually calculate the sum.)

$n =$ **6** $a_1 =$ **8** $r =$ **−2** $r^n =$ $\mathbf{(-2)^6}$ **or 64**

Helping You Remember

2. This lesson includes three formulas for the sum of the first n terms of a geometric series. All of these formulas have the same denominator and have the restriction $r \neq 1$. How can this restriction help you to remember the denominator in the formulas? **Sample answer: If $r = 1$, then $r - 1 = 0$. Because division by 0 is undefined, a formula with $r - 1$ in the denominator will not apply when $r = 1$.**

4 Assess

Open-Ended Assessment

Writing Have students make a chart that compares and contrasts arithmetic and geometric sequences and series, explaining what the variables represent in each formula.

Tips for New Teachers

Intervention Make sure that students can read the notation used in the various formulas and that they understand what each variable and subscript means.

Getting Ready for Lesson 11-5

PREREQUISITE SKILL Students will find the sum of infinite geometric series in Lesson 11-5. This will involve their evaluating rational expressions for different values. Use Exercises 62–67 to determine your students' familiarity with evaluating rational expressions for given values.

Assessment Options

Quiz (Lessons 11-3 and 11-4) is available on p. 693 of the *Chapter 11 Resource Masters*.

Mid-Chapter Test (Lessons 11-1 through 11-4) is available on p. 695 of the *Chapter 11 Resource Masters*.

Answers

49. If the number of people that each person sends the joke to is constant, then the total number of people who have seen the joke is the sum of a geometric series. Answers should include the following.
    - The common ratio would change from 3 to 4.
    - Increase the number of days that the joke circulates so that it is inconvenient to find and add all the terms of the series.

48. If the first term and common ratio of a geometric series are integers, then all the terms of the series are integers. Therefore, the sum of the series is an integer.

48. CRITICAL THINKING If a_1 and r are integers, explain why the value of $\frac{a_1 - a_1r^n}{1 - r}$ must also be an integer.

49. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. **See margin.**

How is e-mailing a joke like a geometric series?

Include the following in your answer:

- how the related geometric series would change if each person e-mailed the joke on to four people instead of three, and
- how the situation could be changed to make it better to use a formula than to add terms.

Standardized Test Practice

50. The first term of a geometric series is −1, and the common ratio is −3. How many terms are in the series if its sum is 182? **A**

Ⓐ 6 Ⓑ 7 Ⓒ 8 Ⓓ 9

51. What is the first term in a geometric series with ten terms, a common ratio of 0.5, and a sum of 511.5? **C**

Ⓐ 64 Ⓑ 128 Ⓒ 256 Ⓓ 512

Graphing Calculator

Use a graphing calculator to find the sum of each geometric series.

52. $\sum_{n=1}^{20} 3(-2)^{n-1}$ **−1,048,575**

53. $\sum_{n=1}^{15} 2\left(\frac{1}{2}\right)^{n-1}$ **3.99987793**

54. $\sum_{n=1}^{10} 5(0.2)^{n-1}$ **6.24999936**

Maintain Your Skills

Mixed Review

55. $\pm\frac{1}{4}, \frac{3}{4}, \pm 9$

56. $-3, -\frac{9}{2}, -\frac{27}{4}, -\frac{81}{8}$

Find the geometric means in each sequence. *(Lesson 11-3)*

55. $\frac{1}{24}$, ___?___, ___?___, ___?___, 54

56. −2, ___?___, ___?___, ___?___, ___?___, $-\frac{243}{16}$

Find the sum of each arithmetic series. *(Lesson 11-2)*

57. 50 + 44 + 38 + … + 8 **232**

58. $\sum_{n=1}^{12} (2n + 3)$ **192**

ENTERTAINMENT **For Exercises 59–61, use the table that shows the number of drive-in movie screens in the United States for 1995–2000.** *(Lesson 2-5)*

Year	1995	1996	1997	1998	1999	2000
Screens	848	826	815	750	737	637

Source: National Association of Theatre Owners

59. See margin.

60. Sample answer using (1, 826) and (3, 750): $y = -38x + 864$

59. Draw a scatter plot, in which x is the number of years since 1995.

60. Find a prediction equation.

61. Predict the number of screens in 2010. **Sample answer: 274**

Online Research **Data Update** For the latest statistics on the movie industry, visit: www.algebra2.com/data_update

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Evaluate $\frac{a}{1-b}$ for the given values of a and b.**

*(To review **evaluating expressions**, see Lesson 1-1.)*

62. $a = 1, b = \frac{1}{2}$ **2**

63. $a = 3, b = -\frac{1}{2}$ **2**

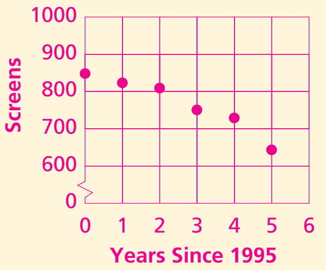
64. $a = \frac{1}{3}, b = -\frac{1}{3}$ $\frac{1}{4}$

65. $a = \frac{1}{2}, b = \frac{1}{4}$ $\frac{2}{3}$

66. $a = -1, b = 0.5$ **−2**

67. $a = 0.9, b = -0.5$ **0.6**

59. **Drive-In Movie Screens**



11-5 Infinite Geometric Series

What You'll Learn

- Find the sum of an infinite geometric series.
- Write repeating decimals as fractions.

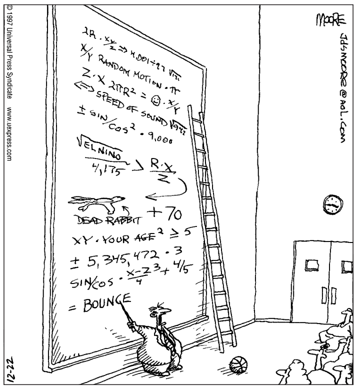
Vocabulary

- infinite geometric series
- partial sum

How does an infinite geometric series apply to a bouncing ball?

Refer to the beginning of Lesson 11-3. Suppose you wrote a geometric series to find the sum of the heights of the rebounds of the ball. The series would have no last term because theoretically there is no last bounce of the ball. For every rebound of the ball, there is another rebound, 60% as high. Such a geometric series is called an **infinite geometric series**.

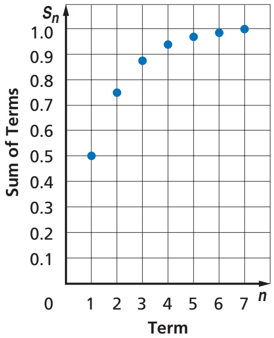
In the Bleachers By Steve Moore



"And that, ladies and gentlemen, is the way the ball bounces."

INFINITE GEOMETRIC SERIES Consider the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$. You have already learned how to find the sum S_n of the first n terms of a geometric series. For an infinite series, S_n is called a **partial sum** of the series. The table and graph show some values of S_n.

n	S_n
1	$\frac{1}{2}$ or 0.5
2	$\frac{3}{4}$ or 0.75
3	$\frac{7}{8}$ or 0.875
4	$\frac{15}{16}$ or 0.9375
5	$\frac{31}{32}$ or 0.96875
6	$\frac{63}{64}$ or 0.984375
7	$\frac{127}{128}$ or 0.9921875



Notice that as n increases, the partial sums level off and approach a limit of 1. This leveling-off behavior is characteristic of infinite geometric series for which $|r| < 1$.

Study Tip

Absolute Value
Recall that $|r| < 1$ means $-1 < r < 1$.

11-5 Lesson Notes

1 Focus

5-Minute Check Transparency 11-5 Use as a quiz or review of Lesson 11-4.

Mathematical Background notes are available for this lesson on p. 576D.

Building on Prior Knowledge

In Lesson 11-4, students worked with geometric series that had a specific number of terms. In this lesson, students extend these skills to finding the sum of an infinite geometric series.

How does an infinite geometric series apply to a bouncing ball?

Ask students:

- Why might someone find this cartoon amusing? **Answers will vary.**
- What is the difference between what is happening theoretically and what really happens with the ball? **Answers will vary.**

Resource Manager

Workbook and Reproducible Masters

Chapter 11 Resource Masters

- Study Guide and Intervention, pp. 655–656
- Skills Practice, p. 657
- Practice, p. 658
- Reading to Learn Mathematics, p. 659
- Enrichment, p. 660

Transparencies

5-Minute Check Transparency 11-5
Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

INFINITE GEOMETRIC SERIES

In-Class Example

Power Point®

1 Find the sum of each infinite geometric series, if it exists.

a. $-\frac{4}{3} + 4 - 12 + 36 - 108 + \ldots$ no sum

b. $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \ldots$ 2

Teaching Tip To help students understand when an infinite geometric series has a sum, lead students to make a generalization about the size of a product of a number and a fraction between −1 and 1. **The absolute value of such a product will always be less than the absolute value of the original number.**

Let's look at the formula for the sum of a finite geometric series and use it to find a formula for the sum of an infinite geometric series.

$$S_n = \frac{a_1 - a_1r^n}{1 - r} \qquad \text{Sum of first } n \text{ terms}$$

$$= \frac{a_1}{1 - r} - \frac{a_1r^n}{1 - r} \qquad \text{Write the fraction as a difference of fractions.}$$

If $-1 < r < 1$, the value of r^n will approach 0 as n increases. Therefore, the partial sums of an infinite geometric series will approach $\frac{a_1}{1 - r} - \frac{a_1(0)}{1 - r}$ or $\frac{a_1}{1 - r}$. This expression gives the sum of an infinite geometric series.

Key Concept — Sum of an Infinite Geometric Series

The sum S of an infinite geometric series with $-1 < r < 1$ is given by

$$S = \frac{a_1}{1 - r}.$$

Study Tip

Formula for Sum if $-1 < r < 1$

To convince yourself of this formula, make a table of the first ten partial sums of the geometric series with $r = \frac{1}{2}$ and $a_1 = 100$.

Term Number	Term	Partial Sum
1	100	100
2	50	150
3	25	175
⋮	⋮	⋮
10		

Complete the table and compare the sum that the series is approaching to that obtained by using the formula.

An infinite geometric series for which $|r| \geq 1$ does not have a sum. Consider the series $1 + 3 + 9 + 27 + 81 + \ldots$. In this series, $a_1 = 1$ and $r = 3$. The table shows some of the partial sums of this series. As n increases, S_n rapidly increases and has no limit. That is, the partial sums do not approach a particular value.

n	S_n
5	121
10	29,524
15	7,174,453
20	1,743,392,200

Example 1 Sum of an Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

a. $\frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \ldots$

First, find the value of r to determine if the sum exists.

$a_1 = \frac{1}{2}$ and $a_2 = \frac{3}{8}$, so $r = \frac{\frac{3}{8}}{\frac{1}{2}}$ or $\frac{3}{4}$. Since $\left|\frac{3}{4}\right| < 1$, the sum exists.

Now use the formula for the sum of an infinite geometric series.

$$S = \frac{a_1}{1 - r} \qquad \text{Sum formula}$$

$$= \frac{\frac{1}{2}}{1 - \frac{3}{4}} \qquad a_1 = \frac{1}{2}, r + \frac{3}{4}$$

$$= \frac{\frac{1}{2}}{\frac{1}{4}} \text{ or } 2 \qquad \text{Simplify.}$$

The sum of the series is 2.

b. $1 - 2 + 4 - 8 + \ldots$

$a_1 = 1$ and $a_2 = -2$, so $r = \frac{-2}{1}$ or -2. Since $|-2| \geq 1$, the sum does not exist.

DAILY INTERVENTION

Unlocking Misconceptions

Absolute Value Make sure students can explain why $|r| < 1$ can also be written as $-1 < r < 1$. Graphing this inequality on a number line may help students understand what is meant by these two different mathematical notations.

In Lessons 11-2 and 11-4, we used sigma notation to represent finite series. You can also use sigma notation to represent infinite series. An *infinity symbol* ∞ is placed above the Σ to indicate that a series is infinite.

Example 2 Infinite Series in Sigma Notation

Evaluate $\sum_{n=1}^{\infty} 24\left(-\frac{1}{5}\right)^{n-1}$.

In this infinite geometric series, $a_1 = 24$ and $r = -\frac{1}{5}$.

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{24}{1 - \left(-\frac{1}{5}\right)}$ $a_1 = 24, r = -\frac{1}{5}$

$= \frac{24}{\frac{6}{5}}$ or 20 Simplify.

Thus, $\sum_{n=1}^{\infty} 24\left(-\frac{1}{5}\right)^{n-1} = 20$.

REPEATING DECIMALS The formula for the sum of an infinite geometric series can be used to write a repeating decimal as a fraction. Remember that decimals with bar notation such as $0.\overline{2}$ and $0.\overline{47}$ represent 0.222222… and 0.474747…, respectively. Each of these expressions can be written as an infinite geometric series.

Example 3 Write a Repeating Decimal as a Fraction

Write $0.\overline{39}$ as a fraction.

Method 1

Write the repeating decimal as a sum.

$0.\overline{39} = 0.393939\ldots$

$= 0.39 + 0.0039 + 0.000039 + \ldots$

$= \frac{39}{100} + \frac{39}{10{,}000} + \frac{39}{1{,}000{,}000} + \ldots$

In this series, $a_1 = \frac{39}{100}$ and $r = \frac{1}{100}$.

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{\frac{39}{100}}{1 - \frac{1}{100}}$ $a_1 = \frac{39}{100}, r = \frac{1}{100}$

$= \frac{\frac{39}{100}}{\frac{99}{100}}$ Subtract.

$= \frac{39}{99}$ or $\frac{13}{33}$ Simplify.

Thus, $0.\overline{39} = \frac{13}{33}$.

Method 2

$S = 0.\overline{39}$ Label the given decimal.

$S = 0.393939\ldots$ Repeating decimal

$100S = 39.393939\ldots$ Multiply each side by 100.

$99S = 39$ Subtract the second equation from the third.

$S = \frac{39}{99}$ or $\frac{13}{33}$ Divide each side by 99.

www.algebra2.com/extra_examples

In-Class Example Power Point®

2 Evaluate $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n-1}$. **10**

Teaching Tip Ask students to write a few terms of the series in Example 2 to make sure they know how to read the notation.

REPEATING DECIMALS

In-Class Example Power Point®

3 Write $0.\overline{25}$ as a fraction. $\frac{25}{99}$

DAILY INTERVENTION

Differentiated Instruction

Logical Have students research and read about the famous mathematical puzzle called Zeno's paradox. Have them discuss this story of the tortoise's race in terms of the content of this lesson.

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 11.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

DAILY INTERVENTION

FIND THE ERROR

Help students realize that, although $\frac{a_1}{1-r}$ may have a value, that value represents the sum of an infinite geometric series *only* when $|r| < 1$.

About the Exercises…

Organization by Objective

- **Infinite Geometric Series:** 14–39
- **Repeating Decimals:** 40–47

Odd/Even Assignments

Exercises 14–31 and 36–47 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 15–33 odd, 34, 35, 41–45 odd, 48–75

Average: 15–33 odd, 34, 35–47 odd, 48–75

Advanced: 14–32 even, 33, 34–48 even, 49–69 (optional: 70–75)

Check for Understanding

Concept Check

1. **OPEN ENDED** Write the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ using sigma notation.

1. Sample answer: $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$

2. **Explain** why $0.999999\ldots = 1$. **See margin.**

3. **FIND THE ERROR** Miguel and Beth are discussing the series $-\frac{1}{3} + \frac{4}{9} - \frac{16}{27} + \ldots$. Miguel says that the sum of the series is $-\frac{1}{7}$. Beth says that the series does not have a sum. Who is correct? Explain your reasoning. **Beth; see margin for explanation.**

Miguel

$$S = \frac{-\frac{1}{3}}{1-\left(-\frac{4}{3}\right)}$$

$$= -\frac{1}{7}$$

Guided Practice

GUIDED PRACTICE KEY

Exercises	Examples
4–8, 13	1
9	2
10–12	3

Find the sum of each infinite geometric series, if it exists.

4. $a_1 = 36, r = \frac{2}{3}$ **108**

5. $a_1 = 18, r = -1.5$ **does not exist**

6. $16 + 24 + 36 + \ldots$ **does not exist**

7. $\frac{1}{4} + \frac{1}{6} + \frac{2}{18} + \ldots$ $\mathbf{\frac{3}{4}}$

8. $6 - 2.4 + 0.96 - \ldots$ $\mathbf{\frac{30}{7}}$

9. $\sum_{n=1}^{\infty} 40\left(\frac{3}{5}\right)^{n-1}$ **100**

Write each repeating decimal as a fraction.

10. $0.\overline{5}$ $\mathbf{\frac{5}{9}}$

11. $0.\overline{73}$ $\mathbf{\frac{73}{99}}$

12. $0.\overline{175}$ $\mathbf{\frac{175}{999}}$

Application

13. **CLOCKS** Jasmine's old grandfather clock is broken. When she tries to set the pendulum in motion by holding it against the side of the clock and letting it go, it first swings 24 centimeters to the other side, then 18 centimeters back, then 13.5 centimeters, and so on. What is the total distance that the pendulum swings? **96 cm**

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
14–27, 32–39	1
28–31	2
40–47	3

Extra Practice

See page 852.

Find the sum of each infinite geometric series, if it exists.

14. $a_1 = 4, r = \frac{5}{7}$ **14**

15. $a_1 = 14, r = \frac{7}{3}$

16. $a_1 = 12, r = -0.6$ **7.5**

17. $a_1 = 18, r = 0.6$ **45**

18. $16 + 12 + 9 + \ldots$ **64**

19. $-8 - 4 - 2 - \ldots$ **−16**

20. $12 - 18 + 24 - \ldots$

21. $18 - 12 + 8 - \ldots$ $\mathbf{\frac{54}{5}}$

22. $1 + \frac{2}{3} + \frac{4}{9} + \ldots$ **3**

23. $\frac{5}{3} + \frac{25}{3} + \frac{125}{3} + \ldots$

24. $\frac{5}{3} - \frac{10}{9} + \frac{20}{27} - \ldots$ **1**

25. $\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \ldots$ **1**

26. $3 + 1.8 + 1.08 + \ldots$ **7.5**

27. $1 - 0.5 + 0.25 - \ldots$ $\mathbf{\frac{2}{3}}$

28. $\sum_{n=1}^{\infty} 48\left(\frac{2}{3}\right)^{n-1}$ **144**

29. $\sum_{n=1}^{\infty}\left(\frac{3}{8}\right)\left(\frac{3}{4}\right)^{n-1}$ $\mathbf{\frac{3}{2}}$

30. $\sum_{n=1}^{\infty} 3(0.5)^{n-1}$ **6**

31. $\sum_{n=1}^{\infty}(1.5)(0.25)^{n-1}$ **2**

15. does not exist
20. does not exist
23. does not exist

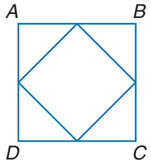
32. **CHILD'S PLAY** Kimimela's little sister likes to swing at the playground. Yesterday, Kimimela pulled the swing back and let it go. The swing traveled a distance of 9 feet before heading back the other way. Each swing afterward was only 70% as long as the previous one. Find the total distance the swing traveled. **30 ft**

Answers

2. $0.999999\ldots$ can be written as the infinite geometric series $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots$. The first term of this series is $\frac{9}{10}$ and the common ratio is $\frac{1}{10}$, so the sum is $\frac{\frac{9}{10}}{1-\frac{1}{10}}$ or 1.

3. The common ratio for the infinite geometric series is $-\frac{4}{3}$. Since $\left|-\frac{4}{3}\right| \geq 1$, the series does not have a sum and the formula $S = \frac{a_1}{1-r}$ does not apply.

GEOMETRY **For Exercises 33 and 34, refer to square *ABCD*, which has a perimeter of 40 centimeters.**
If the midpoints of the sides are connected, a smaller square results. Suppose the process of connecting midpoints of sides and drawing new squares is continued indefinitely.



33. Write an infinite geometric series to represent the sum of the perimeters of all of the squares. **$40 + 20\sqrt{2} + 20 + \ldots$**

34. Find the sum of the perimeters of all of the squares. **$80 + 40\sqrt{2}$ or about 136.6 cm**

35. AVIATION A hot-air balloon rises 90 feet in its first minute of flight. In each succeeding minute, it rises only 90% as far as it did during the preceding minute. What is the final height of the balloon? **900 ft**

More About. . .

Aviation

The largest hot-air balloon ever flown had a capacity of 2.6 million cubic feet.

Source: *The Guinness Book of Records*

★ **36.** The sum of an infinite geometric series is 81, and its common ratio is $\frac{2}{3}$. Find the first three terms of the series. **27, 18, 12**

★ **37.** The sum of an infinite geometric series is 125, and the value of r is 0.4. Find the first three terms of the series. **75, 30, 12**

★ **38.** The common ratio of an infinite geometric series is $\frac{11}{16}$, and its sum is $76\frac{4}{5}$. Find the first four terms of the series. **$24, 16\frac{1}{2}, 11\frac{11}{32}, 7\frac{409}{512}$**

★ **39.** The first term of an infinite geometric series is -8, and its sum is $-13\frac{1}{3}$. Find the first four terms of the series. **$-8, -3\frac{1}{5}, -1\frac{7}{25}, -\frac{64}{125}$**

Write each repeating decimal as a fraction.

40. $0.\overline{7}$ **$\frac{7}{9}$**
41. $0.\overline{1}$ **$\frac{1}{9}$**
42. $0.\overline{36}$ **$\frac{4}{11}$**
43. $0.\overline{82}$ **$\frac{82}{99}$**
44. $0.\overline{246}$ **$\frac{82}{333}$**
45. $0.\overline{427}$ **$\frac{427}{999}$**
★ **46.** $0.\overline{45}$ **$\frac{5}{11}$**
★ **47.** $0.2\overline{31}$ **$\frac{229}{990}$**

48. CRITICAL THINKING Derive the formula for the sum of an infinite geometric series by using the technique in Lessons 11-2 and 11-4. That is, write an equation for the sum S of a general infinite geometric series, multiply each side of the equation by r, and subtract equations. **See pp. 629A–629F.**

49. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. **See pp. 629A–629F.**

How does an infinite geometric series apply to a bouncing ball?

Include the following in your answer:

- some formulas you might expect to see on the chalkboard if the character in the comic strip really was discussing a bouncing ball, and
- an explanation of how to find the total distance traveled, both up and down, by the bouncing ball described at the beginning of Lesson 11-3.

Standardized Test Practice

50. What is the sum of an infinite geometric series with a first term of 6 and a common ratio of $\frac{1}{2}$? **D**

Ⓐ 3 Ⓑ 4 Ⓒ 9 Ⓓ 12

51. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots =$ **C**

Ⓐ $\frac{3}{2}$ Ⓑ $\frac{80}{27}$ Ⓒ 3 Ⓓ does not exist

www.algebra2.com/self_check_quiz

Study Guide and Intervention, p. 655 (shown) and p. 656

Infinite Geometric Series A geometric series that does not end is called an **infinite geometric series**. Some infinite geometric series have sums, but others do not because the **partial sums** increase without approaching a limiting value.

Sum of an Infinite Geometric Series	$S = \frac{a_1}{1 - r}$ for $-1 < r < 1$. If $\lvert r \rvert \geq 1$, the infinite geometric series does not have a sum.

Example **Find the sum of each infinite geometric series, if it exists.**

a. 75 + 15 + 3 + ...

First, find the value of r to determine if the sum exists. $a_1 = 75$ and $a_2 = 15$, so $r = \frac{15}{75}$ or $\frac{1}{5}$. Since $\left|\frac{1}{5}\right| < 1$, the sum exists. Now use the formula for the sum of an infinite geometric series.

$S = \frac{a_1}{1 - r}$ Sum formula
$= \frac{75}{1 - \frac{1}{5}}$ $a_1 = 75, r = \frac{1}{5}$
$= \frac{75}{\frac{4}{5}}$ or 93.75 Simplify.

The sum of the series is 93.75.

b. $\sum_{n=1}^{\infty} 48\left(-\frac{1}{3}\right)^{n-1}$

In this infinite geometric series, $a_1 = 48$ and $r = -\frac{1}{3}$.

$S = \frac{a_1}{1 - r}$ Sum formula
$= \frac{48}{1 - \left(-\frac{1}{3}\right)}$ $a_1 = 48, r = -\frac{1}{3}$
$= \frac{48}{\frac{4}{3}}$ or 36 Simplify.

Thus $\sum_{n=1}^{\infty} 48\left(-\frac{1}{3}\right)^{n-1} = 36$.

Exercises

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = -7, r = \frac{5}{8}$ **$-18\frac{2}{3}$**
2. $1 + \frac{5}{4} + \frac{25}{16} + \ldots$ **does not exist**
3. $a_1 = 4, r = \frac{1}{2}$ **8**
4. $\frac{2}{9} + \frac{5}{27} + \frac{25}{162} + \ldots$ **$1\frac{1}{3}$**
5. $15 + 10 + 6\frac{2}{3} + \ldots$ **45**
6. $18 - 9 + 4\frac{1}{2} - 2\frac{1}{4} + \ldots$ **12**
7. $\frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \ldots$ **$\frac{1}{5}$**
8. $1000 + 800 + 640 + \ldots$ **5000**
9. $6 - 12 + 24 - 48 + \ldots$ **does not exist**
10. $\sum_{n=1}^{\infty} 50\left(\frac{4}{5}\right)^{n-1}$ **250**
11. $\sum_{k=1}^{\infty} 22\left(-\frac{1}{2}\right)^{k-1}$ **$14\frac{2}{3}$**
12. $\sum_{s=1}^{\infty} 24\left(\frac{7}{12}\right)^{s-1}$ **$57\frac{3}{5}$**

Skills Practice, p. 657 and Practice, p. 658 (shown)

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 35, r = \frac{2}{7}$ **49**
2. $a_1 = 26, r = \frac{1}{2}$ **52**
3. $a_1 = 98, r = -\frac{3}{4}$ **56**
4. $a_1 = 42, r = \frac{6}{5}$ **does not exist**
5. $a_1 = 112, r = -\frac{3}{5}$ **70**
6. $a_1 = 500, r = \frac{1}{5}$ **625**
7. $a_1 = 135, r = -\frac{1}{2}$ **90**
8. $18 - 6 + 2 - \ldots$ **$\frac{27}{2}$**
9. $2 + 6 + 18 + \ldots$ **does not exist**
10. $6 + 4 + \frac{8}{3} + \ldots$ **18**
11. $\frac{4}{25} + \frac{2}{5} + 1 + \ldots$ **does not exist**
12. $10 + 1 + 0.1 + \ldots$ **$\frac{100}{9}$**
13. $100 + 20 + 4 + \ldots$ **125**
14. $-270 + 135 - 67.5 + \ldots$ **-180**
15. $0.5 + 0.25 + 0.125 + \ldots$ **1**
16. $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$ **$\frac{7}{9}$**
17. $0.8 + 0.08 + 0.008 + \ldots$ **$\frac{8}{9}$**
18. $\frac{1}{12} - \frac{1}{6} + \frac{1}{3} - \ldots$ **does not exist**
19. $3 + \frac{9}{7} + \frac{27}{49} + \ldots$ **$\frac{21}{4}$**
20. $0.3 - 0.003 + 0.00003 - \ldots$ **$\frac{30}{101}$**
21. $0.06 + 0.006 + 0.0006 + \ldots$ **$\frac{1}{15}$**
22. $\frac{2}{3} - 2 + 6 - \ldots$ **does not exist**
23. $\sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^{n-1}$ **4**
24. $\sum_{n=1}^{\infty} \frac{2}{3}\left(-\frac{3}{4}\right)^{n-1}$ **$\frac{8}{21}$**
25. $\sum_{n=1}^{\infty} 18\left(\frac{2}{3}\right)^{n-1}$ **54**
26. $\sum_{n=1}^{\infty} 5(-0.1)^{n-1}$ **$\frac{50}{11}$**

Write each repeating decimal as a fraction.

27. $0.\overline{6}$ **$\frac{2}{3}$**
28. $0.\overline{09}$ **$\frac{1}{11}$**
29. $0.\overline{43}$ **$\frac{43}{99}$**
30. $0.\overline{27}$ **$\frac{3}{11}$**
31. $0.\overline{243}$ **$\frac{9}{37}$**
32. $0.\overline{84}$ **$\frac{28}{33}$**
33. $0.\overline{990}$ **$\frac{110}{111}$**
34. $0.\overline{150}$ **$\frac{50}{333}$**

35. PENDULUMS On its first swing, a pendulum travels 8 feet. On each successive swing, the pendulum travels $\frac{4}{5}$ the distance of its previous swing. What is the total distance traveled by the pendulum when it stops swinging? **40 ft**

36. ELASTICITY A ball dropped from a height of 10 feet bounces back $\frac{9}{10}$ of that distance. With each successive bounce, the ball continues to reach $\frac{9}{10}$ of its previous height. What is the total vertical distance (both up and down) traveled by the ball when it stops bouncing? (*Hint*: Add the total distance the ball falls to the total distance it rises.) **190 ft**

Reading to Learn Mathematics, p. 659 ELL

Pre-Activity **How does an infinite geometric series apply to a bouncing ball?**

Read the introduction to Lesson 11-5 at the top of page 599 in your textbook.

Note the following powers of 0.6: $0.6^1 = 0.6$; $0.6^2 = 0.36$; $0.6^3 = 0.216$; $0.6^4 = 0.1296$; $0.6^5 = 0.07776$; $0.6^6 = 0.046656$; $0.6^7 = 0.0279936$. If a ball is dropped from a height of 10 feet and bounces back to 60% of its previous height on each bounce, after how many bounces will it bounce back to a height of less than 1 foot? **5 bounces**

Reading the Lesson

1. Consider the formula $S = \frac{a_1}{1 - r}$.

a. What is the formula used to find? **the sum of an infinite geometric series**

b. What do each of the following represent?

S: **the sum**
a_1: **the first term**
r: **the common ratio**

c. For what values of r does an infinite geometric sequence have a sum? **$-1 < r < 1$**

d. Rewrite your answer for part d as an absolute value inequality. **$|r| < 1$**

2. For each of the following geometric series, give the values of a_1 and r. Then state whether the sum of the series exists. (Do not actually find the sum.)

a. $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots$ $a_1 =$ **$\frac{2}{3}$** $r =$ **$\frac{1}{3}$**
Does the sum exist? **yes**

b. $2 - 1 + \frac{1}{2} - \frac{1}{4} + \ldots$ $a_1 =$ **2** $r =$ **$-\frac{1}{2}$**
Does the sum exist? **yes**

c. $\sum_{i=1}^{\infty} 3^i$ $a_1 =$ **3** $r =$ **3**
Does the sum exist? **no**

Helping You Remember

3. One good way to remember something is to relate it to something you already know. How can you use the formula $S_n = \frac{a_1(1 - r^n)}{1 - r}$ that you learned in Lesson 11-4 for finding the sum of a geometric series to help you remember the formula for finding the sum of an infinite geometric series? **Sample answer: If $-1 < r < 1$, then as n gets large, r^n approaches 0, so $1 - r^n$ approaches 1. Therefore, S_n approaches $\frac{a_1 \cdot 1}{1 - r}$, or $\frac{a_1}{1 - r}$.**

Enrichment, p. 660

Convergence and Divergence

Convergence and divergence are terms that relate to the existence of a sum of an infinite series. If a sum exists, the series is convergent. If not, the series is divergent. Consider the series $12 + 3 + \frac{3}{4} + \frac{3}{16} + \ldots$. This is a geometric series with $r = \frac{1}{4}$. The sum is given by the formula $S = \frac{a_1}{1 - r}$. Thus, the sum is $12 \div \frac{3}{4}$ or 16. This series is convergent since a sum exists. Notice that the first two terms have a sum of 15. As more terms are added, the sum comes closer (or converges) to 16.

Recall that a geometric series has a sum if and only if $-1 < r < 1$. Thus, a geometric series is convergent if r is between -1 and 1, and divergent if r has another value. An infinite arithmetic series cannot have a sum unless all of the terms are equal to zero.

Example **Determine whether each series is convergent or divergent.**

a. 2 + 5 + 8 + 11 + ... divergent

4 Assess

Open-Ended Assessment

Writing Have students write their own examples of an infinite geometric series—one that has a sum and one that does not. Have them also write an example of a repeating decimal and then express it as a fraction.

Tips for New Teachers

Intervention Make sure that students can read the notation used in the various formulas and that they understand what each variable and subscript means.

Getting Ready for Lesson 11-6

PREREQUISITE SKILL Students will use recursive formulas in Lesson 11-6. This will involve their evaluating functions for given values. Use Exercises 70–75 to determine your students' familiarity with evaluating functions for given values.

Maintain Your Skills

Mixed Review

Find S_n for each geometric series described. *(Lesson 11-4)*

52. $a_1 = 1, a_6 = -243, r = -3$ **-182**

53. $a_1 = 72, r = \frac{1}{3}, n = 7$ **$\frac{8744}{81}$**

54. **PHYSICS** A vacuum pump removes 20% of the air from a container with each stroke of its piston. What percent of the original air remains after five strokes of the piston? *(Lesson 11-3)* **32.768%**

Solve each equation or inequality. Check your solution. *(Lesson 10-1)*

55. $6^x = 216$ **3**

56. $2^{2x} = \frac{1}{8}$ **$-\frac{3}{2}$**

57. $3^{x-2} \geq 27$ **$x \geq 5$**

Simplify each expression. *(Lesson 9-2)*

58. $\frac{-2}{ab} + \frac{5}{a^2}$ **$\frac{-2a + 5b}{a^2b}$**

59. $\frac{1}{x-3} - \frac{2}{x+1}$

60. $\frac{1}{x^2 + 6x + 8} + \frac{3}{x+4}$

59. $\frac{-x + 7}{(x - 3)(x + 1)}$

60. $\frac{3x + 7}{(x + 4)(x + 2)}$

Write an equation for the circle that satisfies each set of conditions. *(Lesson 8-3)*

61. center (2, 4), radius 6 **$(x - 2)^2 + (y - 4)^2 = 36$**

62. endpoints of a diameter at (7, 3) and (−1, −5) **$(x - 3)^2 + (y + 1)^2 = 32$**

Find all the zeros of each function. *(Lesson 7-5)*

63. $f(x) = 8x^3 - 36x^2 + 22x + 21$

64. $g(x) = 12x^4 + 4x^3 - 3x^2 - x$

63. $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}$

64. $-\frac{1}{2}, -\frac{1}{3}, 0, \frac{1}{2}$

Write a quadratic equation with the given roots. Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers. *(Lesson 6-3)*

65. 6, −6 **$x^2 - 36 = 0$**

66. −2, −7 **$x^2 + 9x + 14 = 0$**

67. 6, 4 **$x^2 - 10x + 24 = 0$**

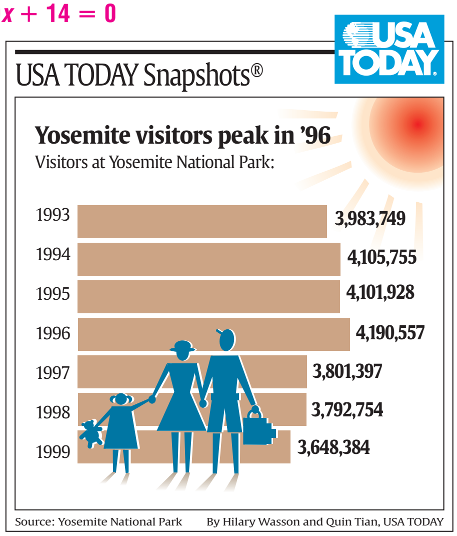
RECREATION **For Exercises 68 and 69, refer to the graph at the right.** *(Lesson 2-3)*

68. Find the average rate of change of the number of visitors to Yosemite National Park from 1996 to 1999.

68. about −180,724 visitors per year

69. Was the number of visitors increasing or decreasing from 1996 to 1999? **The number of visitors was decreasing.**

USA TODAY Snapshots®



Source: Yosemite National Park By Hilary Wasson and Quin Tian, USA TODAY

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each function value.**
*(To review **evaluating functions**, see Lesson 2-1.)*

70. $f(x) = 2x, f(1)$ **2**

71. $g(x) = 3x - 3, g(2)$ **3**

72. $h(x) = -2x + 2, h(0)$ **2**

73. $f(x) = 3x - 1, f\left(\frac{1}{2}\right)$ **$\frac{1}{2}$**

74. $g(x) = x^2, g(2)$ **4**

75. $h(x) = 2x^2 - 4, h(0)$ **-4**

Online Lesson Plans

USA TODAY Education's Online site offers resources and interactive features connected to each day's newspaper. *Experience TODAY*, USA TODAY's daily lesson plan, is available on the site and delivered daily to subscribers. This plan provides instruction for integrating USA TODAY graphics and key editorial features into your mathematics classroom. Log on to www.education.usatoday.com.

Spreadsheet Investigation

A Preview of Lesson 11-6

Amortizing Loans

When a payment is made on a loan, part of the payment is used to cover the interest that has accumulated since the last payment. The rest is used to reduce the *principal*, or original amount of the loan. This process is called *amortization*. You can use a spreadsheet to analyze the payments, interest, and balance on a loan. A table that shows this kind of information is called an *amortization schedule*.

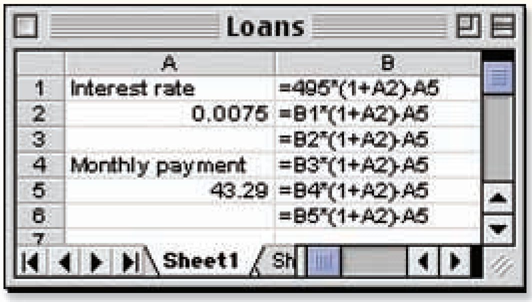
Example

Marisela just bought a new sofa for $495. The store is letting her make monthly payments of $43.29 at an interest rate of 9% for one year. How much will she still owe after six months?

Every month, the interest on the remaining balance will be $\frac{9\%}{12}$ or 0.75%. You can find the balance after a payment by multiplying the balance after the previous payment by 1 + 0.0075 or 1.0075 and then subtracting 43.29.

In a spreadsheet, use the column of numbers for the number of payments and use column B for the balance. Enter the interest rate and monthly payment in cells in column A so that they can be easily updated if the information changes.

The spreadsheet at the right shows the formulas for the balances after each of the first six payments. After six months, Marisela still owes $253.04.



Loans

	A	B
1	Interest rate	=495*(1+A2)-A5
2	0.0075	=B1*(1+A2)-A5
3		=B2*(1+A2)-A5
4	Monthly payment	=B3*(1+A2)-A5
5	43.29	=B4*(1+A2)-A5
6		=B5*(1+A2)-A5
7		

Exercises

1. Let b_n be the balance left on Marisela's loan after n months. Write an equation relating b_n and b_{n+1}. **$b_{n+1} = 1.0075b_n - 43.29$**
2. Payments at the beginning of a loan go more toward interest than payments at the end. What percent of Marisela's loan remains to be paid after half a year? **about 51%**
3. Extend the spreadsheet to the whole year. What is the balance after 12 payments? Why is it not 0? **About −$0.02; the balance is not exactly 0 due to rounding.**
4. Suppose Marisela decides to pay $50 every month. How long would it take her to pay off the loan? **11 months**
5. Suppose that, based on how much she can afford, Marisela will pay a variable amount each month in addition to the $43.29. Explain how the flexibility of a spreadsheet can be used to adapt to this situation. **See margin.**
6. Jamie has a three-year, $12,000 car loan. The annual interest rate is 6%, and his monthly payment is $365.06. After twelve months, he receives an inheritance which he wants to use to pay off the loan. How much does he owe at that point? **$8236.91**

Spreadsheet Investigation

A Preview of Lesson 11-6

Getting Started

Objective To discover how a spreadsheet can express a relationship in which the calculation of the value of the next term involves using the value of the previous term.

Teach

Ask students why the balance is multiplied by 1.0075 rather than 0.75%. **to find the balance plus the interest**

Assess

In **Exercises 1–5**, students should

- be able to relate the list of payments to the series and sequences they have been studying.
- be able to use a spreadsheet to make a flexible table of payments.

Study Notebook

You may wish to have students summarize this activity and what they learned from it.

Answer

5. Changing the monthly payment only requires editing the amount subtracted in the formula in each cell.

11-6 Lesson Notes

1 Focus

5-Minute Check Transparency 11-6 Use as a quiz or review of Lesson 11-5.

Mathematical Background notes are available for this lesson on p. 576D.

How is the Fibonacci sequence illustrated in nature?

Ask students:

- What number follows 5 in the Fibonacci sequence? **8**
- Is the Fibonacci sequence an arithmetic sequence? **no** Is it a geometric sequence? **no** Explain. **There is no common difference and no common ratio.**

11-6 Recursion and Special Sequences

What You'll Learn

- Recognize and use special sequences.
- Iterate functions.

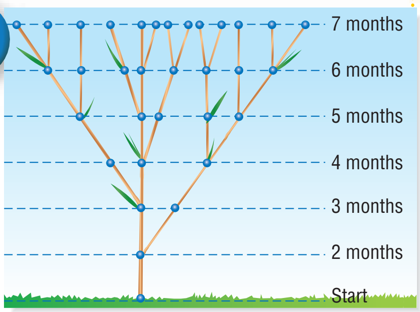
Vocabulary

- Fibonacci sequence
- recursive formula
- iteration

How is the Fibonacci sequence illustrated in nature?

A shoot on a sneezewort plant must grow for two months before it is strong enough to put out another shoot. After that, it puts out at least one shoot every month.

Month	1	2	3	4	5
Shoots	1	1	2	3	5



SPECIAL SEQUENCES Notice that the sequence 1, 1, 2, 3, 5, 8, 13, … has a pattern. Each term in the sequence is the sum of the two previous terms. For example, 8 = 3 + 5 and 13 = 5 + 8. This sequence is called the **Fibonacci sequence**, and it is found in many places in nature.

first term	a_1		1
second term	a_2		1
third term	a_3	$a_1 + a_2$	$1 + 1 = 2$
fourth term	a_4	$a_2 + a_3$	$1 + 2 = 3$
fifth term	a_5	$a_3 + a_4$	$2 + 3 = 5$
⋮	⋮	⋮	
nth term	a_n	$a_{n-2} + a_{n-1}$	

Study Tip

Reading Math

A recursive formula is often called a *recursive relation* or a *recurrence relation*.

The formula $a_n = a_{n-2} + a_{n-1}$ is an example of a **recursive formula**. This means that each term is formulated from one or more previous terms. To be able to use a recursive formula, you must be given the value(s) of the first term(s) so that you can start the sequence and then use the formula to generate the rest of the terms.

TEACHING TIP

$a_n = a_1 + (n - 1)d$ and $a_n = a_1r^{n-1}$ are not recursive. These sequences are determined by the number of the term n rather than by the preceding term.

Example 1 Use a Recursive Formula

Find the first five terms of the sequence in which $a_1 = 4$ and $a_{n+1} = 3a_n - 2$, $n \geq 1$.

$a_{n+1} = 3a_n - 2$ Recursive formula

$a_{1+1} = 3a_1 - 2$ $n = 1$

$a_2 = 3(4) - 2$ or 10 $a_1 = 4$

$a_{2+1} = 3a_2 - 2$ $n = 2$

$a_3 = 3(10) - 2$ or 28 $a_2 = 10$

$a_{3+1} = 3a_3 - 2$ $n = 3$

$a_4 = 3(28) - 2$ or 82 $a_3 = 28$

$a_4 + 1 = 3a_4 - 2$ $n = 4$

$a_5 = 3(82) - 2$ or 244 $a_4 = 82$

The first five terms of the sequence are 4, 10, 28, 82, and 244.

Resource Manager

Workbook and Reproducible Masters

Chapter 11 Resource Masters

- Study Guide and Intervention, pp. 661–662
- Skills Practice, p. 663
- Practice, p. 664
- Reading to Learn Mathematics, p. 665
- Enrichment, p. 666
- Assessment, p. 694

Graphing Calculator and Spreadsheet Masters, p. 47

Teaching Algebra With Manipulatives Masters, pp. 285, 286–287

Transparencies

5-Minute Check Transparency 11-6
Real-World Transparency 11
Answer Key Transparencies

Technology

Interactive Chalkboard

Example 2 Find and Use a Recursive Formula

GARDENING **Mr. Yazaki discovered that there were 225 dandelions in his garden on the first Saturday of spring. He had time to pull out 100, but by the next Saturday, there were twice as many as he had left. Each Saturday in spring, he removed 100 dandelions, only to find that the number of remaining dandelions had doubled by the following Saturday.**

a. Write a recursive formula for the number of dandelions Mr. Yazaki finds in his garden each Saturday.

Let d_n represent the number of dandelions at the beginning of the nth Saturday. Mr. Yazaki will pull 100 of these out of his garden, leaving $d_n - 100$. The number d_{n+1} of dandelions the next Saturday will be twice this number. So, $d_{n+1} = 2(d_n - 100)$ or $2d_n - 200$.

b. Find the number of dandelions Mr. Yazaki would find on the fifth Saturday.

On the first Saturday, there were 225 dandelions, so $d_1 = 225$.

$d_{n+1} = 2d_n - 200$ Recursive formula

$d_{1+1} = 2d_1 - 200$ $n = 1$
$d_2 = 2(225) - 200$ or 250

$d_{2+1} = 2d_2 - 200$ $n = 2$
$d_3 = 2(250) - 200$ or 300

$d_{3+1} = 2d_3 - 200$ $n = 3$
$d_4 = 2(300) - 200$ or 400

$d_{4+1} = 2\,d_4 - 200$ $n = 4$
$d_5 = 2(400) - 200$ or 600

On the fifth Saturday, there would be 600 dandelions in Mr. Yazaki's garden.

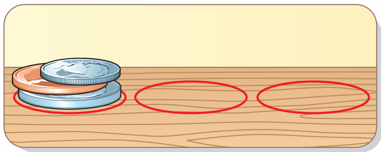
You can use sequences to analyze some games.

Algebra Activity

Special Sequences

The object of the *Towers of Hanoi* game is to move a stack of n coins from one position to another in the fewest number a_n of moves with these rules.

- You may only move one coin at a time.
- A coin must be placed on top of another coin, not underneath.
- A smaller coin may be placed on top of a larger coin, but not vice versa. For example, a penny may not be placed on top of a dime.



Model and Analyze

1. Draw three circles on a sheet of paper, as shown above. Place a penny on the first circle. What is the least number of moves required to get the penny to the second circle? **1**
2. Place a nickel and a penny on the first circle, with the penny on top. What is the least number of moves that you can make to get the stack to another circle? (Remember, a nickel cannot be placed on top of a penny.) **3**
3. Place a nickel, penny, and dime on the first circle. What is the least number of moves that you can take to get the stack to another circle? **7**

Make a Conjecture

4. Place a quarter, nickel, penny, and dime on the first circle. Experiment to find the least number of moves needed to get the stack to another circle. Make a conjecture about a formula for the minimum number a_n of moves required to move a stack of n coins. **15; $a_n = 2^n - 1$**

2 Teach

SPECIAL SEQUENCES

In-Class Examples

Power Point®

1 Find the first five terms of the sequence in which $a_1 = 5$ and $a_{n+1} = 2a_n + 7$, $n \geq 1$. **5, 17, 41, 89, 185**

Teaching Tip Make sure students understand that you use the value of one term to find the value of the next term.

2 **BIOLOGY** Dr. Elliot is growing cells in lab dishes. She starts with 108 cells Monday morning and then removes 20 of these for her experiment. By Tuesday the remaining cells have multiplied by 1.5. She again removes 20. This pattern repeats each day in the week.

a. Write a recursive formula for the number of cells Dr. Elliot finds each day before she removes any. $c_{n+1} = 1.5(c_n - 20)$ or $c_{n+1} = 1.5c_n - 30$

b. Find the number of cells she will find on Friday morning. **303**

Algebra Activity

Materials: compass, penny, nickel, dime, quarter

- Tell students that according to Martin Gardner, in *The Scientific American Book of Mathematical Puzzles & Diversions*, the "Tower of Hanoi was invented by the French mathematician Edouard Lucas and sold as a toy in 1883."
- The toy usually has 3 pegs, with a tower of 8 disks on one peg. The task is to transfer all 8 disks to one of the vacant pegs, using the rules in the activity, in the fewest possible moves.

ITERATION

In-Class Example Power Point®

3 Find the first three iterates x_1, x_2, x_3 of the function $f(x) = 3x - 1$ for an initial value of $x_0 = 5$. **14, 41, 122**

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 11.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises…

Organization by Objective
- **Special Sequences:** 13–30
- **Iteration:** 31–39

Odd/Even Assignments

Exercises 13–24 and 31–38 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 13–19 odd, 23, 25–30, 31–35 odd, 39–55

Average: 13–25 odd, 26–30, 31–37 odd, 39–55

Advanced: 14–24 even, 25–30, 32–40 even, 41–49 (optional: 50–55)

Study Tip

Look Back

To review **composition of functions**, see Lesson 7-7.

ITERATION

Iteration is the process of composing a function with itself repeatedly. For example, if you compose a function with itself once, the result is $f \circ f(x)$ or $f(f(x))$. If you compose a function with itself two times, the result is $f \circ f \circ f(x)$ or $f(f(f(x)))$, and so on.

You can use iteration to recursively generate a sequence. Start with an initial value x_0. Let $x_1 = f(x_0)$, $x_2 = f(x_1)$ or $f(f(x_0))$, $x_3 = f(x_2)$ or $f(f(f(x_0)))$, and so on.

Example 3 Iterate a Function

Find the first three iterates x_1, x_2, x_3 of the function $f(x) = 2x + 3$ for an initial value of $x_0 = 1$.

To find the first iterate x_1, find the value of the function for $x_0 = 1$.

$x_1 = f(x_0)$ Iterate the function.

$= f(1)$ $x_0 = 1$

$= 2(1) + 3$ or 5 Simplify.

To find the second iterate x_2, substitute x_1 for x.

$x_2 = f(x_1)$ Iterate the function.

$= f(5)$ $x_1 = 5$

$= 2(5) + 3$ or 13 Simplify.

Substitute x_2 for x to find the third iterate.

$x_3 = f(x_2)$ Iterate the function.

$= f(13)$ $x_2 = 13$

$= 2(13) + 3$ or 29 Simplify.

Therefore, 1, 5, 13, 29 is an example of a sequence generated using iteration.

Check for Understanding

Concept Check

1. **Write** recursive formulas for the *n*th terms of arithmetic and geometric sequences. **$a_n = a_{n-1} + d$; $a_n = r \cdot a_{n-1}$**
2. **OPEN ENDED** Write a recursive formula for a sequence whose first three terms are 1, 1, and 3. **Sample answer: $a_n = 2a_{n-1} + a_{n-2}$**
3. **State** whether the statement $x_n \neq x_{n-1}$ is *sometimes*, *always*, or *never* true if $x_n = f(x_{n-1})$. Explain. **Sometimes; see margin for explanation.**

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4–7	1, 2
8–10	3
11, 12	2

Find the first five terms of each sequence. **5. −3, −2, 0, 3, 7**

4. $a_1 = 12, a_{n+1} = a_n - 3$ **12, 9, 6, 3, 0**
5. $a_1 = -3, a_{n+1} = a_n + n$
6. $a_1 = 0, a_{n+1} = -2a_n - 4$ **0, − 4, 4, −12, 20**
7. $a_1 = 1, a_2 = 2, a_{n+2} = 4a_{n+1} - 3a_n$ **1, 2, 5, 14, 41**

Find the first three iterates of each function for the given initial value.

8. $f(x) = 3x - 4, x_0 = 3$ **5, 11, 29**
9. $f(x) = -2x + 5, x_0 = 2$ **1, 3, −1**
10. $f(x) = x^2 + 2, x_0 = -1$ **3, 11, 123**

Application

BANKING **For Exercises 11 and 12, use the following information.**
Rita has deposited $1000 in a bank account. At the end of each year, the bank posts interest to her account in the amount of 5% of the balance, but then takes out a $10 annual fee.

11. Let b_0 be the amount Rita deposited. Write a recursive equation for the balance b_n in her account at the end of n years. **$b_n = 1.05b_{n-1} - 10$**
12. Find the balance in the account after four years. **$1172.41**

DAILY INTERVENTION

Differentiated Instruction

Kinesthetic Have students make and play a Tower of Hanoi game as described in the notes for the Algebra Activity. Students can cut 8 cardboard squares of graduated sizes and move them between three circles to represent the pegs. Students can also do additional research about this classic puzzle.

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
13–30	1–2
31–39	3

Extra Practice

See page 853.

13. −6, −3, 0, 3, 6
14. 13, 18, 23, 28, 33
15. 2, 1, −1, −4, −8
16. 6, 10, 15, 21, 28
17. 9, 14, 24, 44, 84
18. 4, 6, 12, 30, 84
19. −1, 5, 4, 9, 13
20. 4, −3, 5, −1, 9
21. $\frac{7}{2}, \frac{7}{4}, \frac{7}{6}, \frac{7}{8}, \frac{7}{10}$
22. $\frac{3}{4}, \frac{3}{2}, \frac{15}{4}, \frac{25}{2}, \frac{425}{8}$

Find the first five terms of each sequence.

13. $a_1 = -6, a_{n+1} = a_n + 3$
14. $a_1 = 13, a_{n+1} = a_n + 5$
15. $a_1 = 2, a_{n+1} = a_n - n$
16. $a_1 = 6, a_{n+1} = a_n + n + 3$
17. $a_1 = 9, a_{n+1} = 2\,a_n - 4$
18. $a_1 = 4, a_{n+1} = 3a_n - 6$
19. $a_1 = -1, a_2 = 5, a_{n+1} = a_n + a_{n-1}$
20. $a_1 = 4, a_2 = -3, a_{n+2} = a_{n+1} + 2a_n$

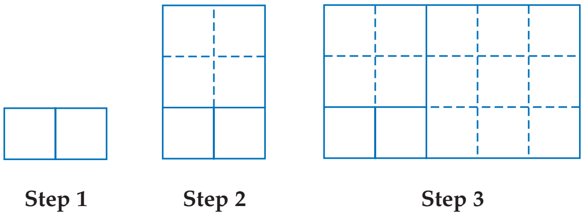
★ 21. $a_1 = \frac{7}{2}, a_{n+1} = \frac{n}{n+1} \cdot a_n$

★ 22. $a_1 = \frac{3}{4}, a_{n+1} = \frac{n^2+1}{n} \cdot a_n$

23. If $a_0 = 7$ and $a_{n+1} = a_n + 12$ for $n \geq 0$, find the value of a_5. **67**

24. If $a_0 = 1$ and $a_{n+1} = -2.1$ for $n \geq 0$, then what is the value of a_4? **−2.1**

GEOMETRY **For Exercises 25 and 26, use the following information.**
Join two 1-unit by 1-unit squares to form a rectangle. Next, draw a larger square along a long side of the rectangle. Continue this process of drawing a square along a long side of the rectangle formed at the previous step.



25. Write the sequence of the lengths of the sides of the squares you added at each step. Begin the sequence with the lengths of the sides of the two original squares. **1, 1, 2, 3, 5, …**

26. Identify the sequence in Exercise 25. **the Fibonacci sequence**

Career Choices

Real Estate Agent

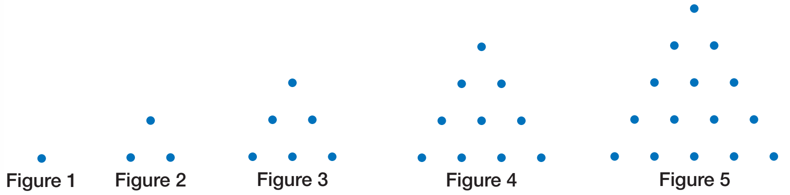
Most real estate agents are independent business-people who earn their income from commission.

Online Research
To learn more about a career in real estate, visit: www.algebra2.com/careers

27. **LOANS** The Cruz family is taking out a mortgage loan for \$100,000 to buy a house. Their monthly payment is \$678.79. The recursive formula $b_n = 1.006\ b_{n-1} - 678.79$ describes the balance left on the loan after n payments. Find the balances of the loan after each of the first eight payments.

27. \$99,921.21, \$99,841.95, \$99,762.21, \$99,681.99, \$99,601.29, \$99,520.11, \$99,438.44, \$99,356.28

GEOMETRY **For Exercises 28–30, study the triangular numbers shown below.**



28. Write a sequence of the first five triangular numbers. **1, 3, 6, 10, 15**

29. Write a recursive formula for the nth triangular number t_n. $t_n = t_{n-1} + n$

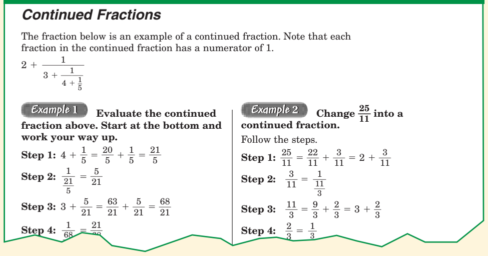
30. What is the 200th triangular number? **20,100**

www.algebra2.com/self_check_quiz

Answers

3. If $f(x) = x^2$ and $x_1 = 2$, then $x_2 = 2^2$ or 4, so $x_2 \neq x_1$. But, if $x_1 = 1$, then $x_2 = 1$, so $x_2 = x_1$.

Enrichment, p. 666



Continued Fractions

The fraction below is an example of a continued fraction. Note that each fraction in the continued fraction has a numerator of 1.

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}$$

Example 1 **Evaluate the continued fraction above. Start at the bottom and work your way up.**

Step 1: $4 + \frac{1}{5} = \frac{20}{5} + \frac{1}{5} = \frac{21}{5}$

Step 2: $\frac{1}{\frac{21}{5}} = \frac{5}{21}$

Step 3: $3 + \frac{5}{21} = \frac{63}{21} + \frac{5}{21} = \frac{68}{21}$

Step 4: $\frac{1}{\frac{68}{21}} = \frac{21}{68}$

Example 2 **Change $\frac{25}{11}$ into a continued fraction.**

Follow the steps.

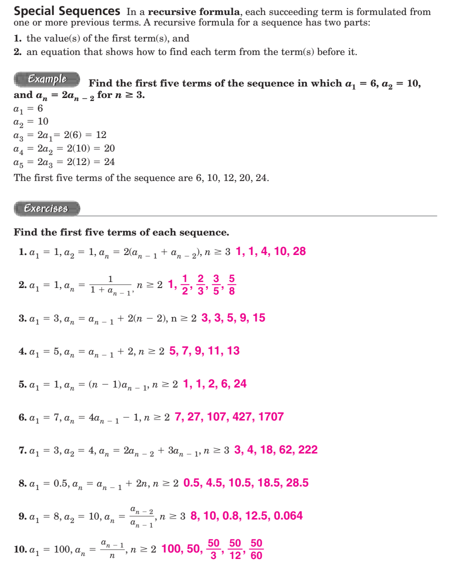
Step 1: $\frac{25}{11} = \frac{22}{11} + \frac{3}{11} = 2 + \frac{3}{11}$

Step 2: $\frac{3}{11} = \frac{1}{\frac{11}{3}}$

Step 3: $\frac{11}{3} = \frac{9}{3} + \frac{2}{3} = 3 + \frac{2}{3}$

Step 4: $\frac{2}{3} = \frac{1}{3}$

Study Guide and Intervention, p. 661 (shown) and p. 662



Special Sequences In a **recursive formula**, each succeeding term is formulated from one or more previous terms. A recursive formula for a sequence has two parts:

1. the value(s) of the first term(s), and
2. an equation that shows how to find each term from the term(s) before it.

Example **Find the first five terms of the sequence in which $a_1 = 6$, $a_2 = 10$, and $a_n = 2a_{n-2}$ for $n \geq 3$.**

$a_1 = 6$
$a_2 = 10$
$a_3 = 2a_1 = 2(6) = 12$
$a_4 = 2a_2 = 2(10) = 20$
$a_5 = 2a_3 = 2(12) = 24$

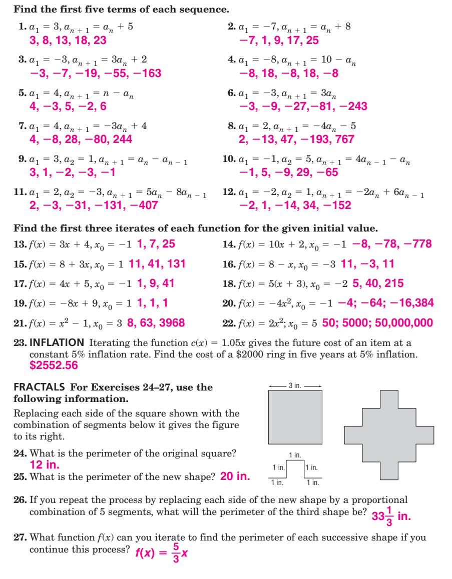
The first five terms of the sequence are 6, 10, 12, 20, 24.

Exercises

Find the first five terms of each sequence.

1. $a_1 = 1, a_2 = 1, a_n = 2(a_{n-1} + a_{n-2}), n \geq 3$ **1, 1, 4, 10, 28**
2. $a_1 = 1, a_n = \frac{1}{1 + a_{n-1}}, n \geq 2$ $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}$
3. $a_1 = 3, a_n = a_{n-1} + 2(n - 2)$, n ≥ 2 **3, 3, 5, 9, 15**
4. $a_1 = 5, a_n = a_{n-1} + 2, n \geq 2$ **5, 7, 9, 11, 13**
5. $a_1 = 1, a_n = (n - 1)a_{n-1}, n \geq 2$ **1, 1, 2, 6, 24**
6. $a_1 = 7, a_n = 4a_{n-1} - 1, n \geq 2$ **7, 27, 107, 427, 1707**
7. $a_1 = 3, a_2 = 4, a_n = 2a_{n-2} + 3a_{n-1}, n \geq 3$ **3, 4, 18, 62, 222**
8. $a_1 = 0.5, a_n = a_{n-1} + 2n, n \geq 2$ **0.5, 4.5, 10.5, 18.5, 28.5**
9. $a_1 = 8, a_2 = 10, a_n = \frac{a_{n-2}}{a_{n-1}}, n \geq 3$ **8, 10, 0.8, 12.5, 0.064**
10. $a_1 = 100, a_n = \frac{a_{n-1}}{n}, n \geq 2$ $100, 50, \frac{50}{3}, \frac{50}{12}, \frac{50}{60}$

Skills Practice, p. 663 and Practice, p. 664 (shown)



Find the first five terms of each sequence.

1. $a_1 = 3, a_{n+1} = a_n + 5$ **3, 8, 13, 18, 23**
2. $a_1 = -7, a_{n+1} = a_n + 8$ **−7, 1, 9, 17, 25**
3. $a_1 = -3, a_{n+1} = 3a_n + 2$ **−3, −7, −19, −55, −163**
4. $a_1 = -8, a_{n+1} = 10 - a_n$ **−8, 18, −8, 18, −8**
5. $a_1 = 4, a_{n+1} = n - a_n$ **4, −3, 5, −2, 6**
6. $a_1 = -3, a_{n+1} = 3a_n$ **−3, −9, −27,−81, −243**
7. $a_1 = 4, a_{n+1} = -3a_n + 4$ **4, −8, 28, −80, 244**
8. $a_1 = 2, a_{n+1} = -4a_n - 5$ **2, −13, 47, −193, 767**
9. $a_1 = 3, a_2 = 1, a_{n+1} = a_n - a_{n-1}$ **3, 1, −2, −3, −1**
10. $a_1 = -1, a_2 = 5, a_{n+1} = 4a_{n-1} - a_n$ **−1, 5, −9, 29, −65**
11. $a_1 = 2, a_2 = -3, a_{n+1} = 5a_n - 8a_{n-1}$ **2, −3, −31, −131, −407**
12. $a_1 = -2, a_2 = 1, a_{n+1} = -2a_n + 6a_{n-1}$ **−2, 1, −14, 34, −152**

Find the first three iterates of each function for the given initial value.

13. $f(x) = 3x + 4, x_0 = -1$ **1, 7, 25**
14. $f(x) = 10x + 2, x_0 = -1$ **−8, −78, −778**
15. $f(x) = 8 + 3x, x_0 = 1$ **11, 41, 131**
16. $f(x) = 8 - x, x_0 = -3$ **11, −3, 11**
17. $f(x) = 4x + 5, x_0 = -1$ **1, 9, 41**
18. $f(x) = 5(x + 3), x_0 = -2$ **5, 40, 215**
19. $f(x) = -8x + 9, x_0 = 1$ **1, 1, 1**
20. $f(x) = -4x^2, x_0 = -1$ **−4; −64; −16,384**
21. $f(x) = x^2 - 1, x_0 = 3$ **8, 63, 3968**
22. $f(x) = 2x^2; x_0 = 5$ **50; 5000; 50,000,000**

23. **INFLATION** Iterating the function $c(x) = 1.05x$ gives the future cost of an item at a constant 5% inflation rate. Find the cost of a \$2000 ring in five years at 5% inflation. **\$2552.56**

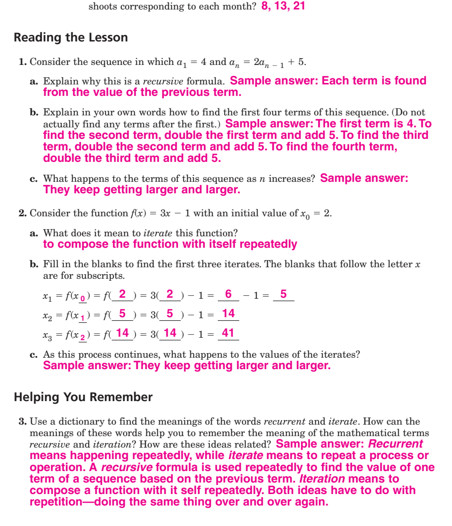
FRACTALS For Exercises 24–27, use the following information.

Replacing each side of the square shown with the combination of segments below it gives the figure to its right.

24. What is the perimeter of the original square? **12 in.**
25. What is the perimeter of the new shape? **20 in.**
26. If you repeat the process by replacing each side of the new shape by a proportional combination of 5 segments, what will the perimeter of the third shape be? $33\frac{1}{3}$ **in.**
27. What function $f(x)$ can you iterate to find the perimeter of each successive shape if you continue this process? $f(x) = \frac{5}{3}x$

Reading to Learn Mathematics, p. 665

ELL



Pre-Activity **How is the Fibonacci sequence illustrated in nature?**

Read the introduction to Lesson 11-6 at the top of page 606 in your textbook.

What are the next three numbers in the sequence that gives the number of shoots corresponding to each month? **8, 13, 21**

Reading the Lesson

1. Consider the sequence in which $a_1 = 4$ and $a_n = 2a_{n-1} + 5$.
   - a. Explain why this is a *recursive* formula. **Sample answer: Each term is found from the value of the previous term.**
   - b. Explain in your own words how to find the first four terms of this sequence. (Do not actually find any terms after the first.) **Sample answer: The first term is 4. To find the second term, double the first term and add 5. To find the third term, double the second term and add 5. To find the fourth term, double the third term and add 5.**
   - c. What happens to the terms of this sequence as n increases? **Sample answer: They keep getting larger and larger.**
2. Consider the function $f(x) = 3x - 1$ with an initial value of $x_0 = 2$.
   - a. What does it mean to *iterate* this function? **to compose the function with itself repeatedly**
   - b. Fill in the blanks to find the first three iterates. The blanks that follow the letter x are for subscripts.

     $x_1 = f(x_{\underline{0}}) = f(\underline{\ 2\ }) = 3(\underline{\ 2\ }) - 1 = \underline{\ 6\ } - 1 = \underline{\ 5\ }$

     $x_2 = f(x_{\underline{1}}) = f(\underline{\ 5\ }) = 3(\underline{\ 5\ }) - 1 = \underline{\ 14\ }$

     $x_3 = f(x_{\underline{2}}) = f(\underline{\ 14\ }) = 3(\underline{\ 14\ }) - 1 = \underline{\ 41\ }$
   - c. As this process continues, what happens to the values of the iterates? **Sample answer: They keep getting larger and larger.**

Helping You Remember

3. Use a dictionary to find the meanings of the words *recurrent* and *iterate*. How can the meanings of these words help you to remember the meaning of the mathematical terms *recursive* and *iteration*? How are these ideas related? **Sample answer: *Recurrent* means happening repeatedly, while *iterate* means to repeat a process or operation. A *recursive* formula is used repeatedly to find the value of one term of a sequence based on the previous term. *Iteration* means to compose a function with it self repeatedly. Both ideas have to do with repetition—doing the same thing over and over again.**

4 Assess

Open-Ended Assessment

Speaking Have students explain, with examples, what it means to say that a formula or a function is recursive.

Tips for New Teachers

Intervention Make sure that students understand the language used in this lesson, particularly *iteration*, *iterative*, and *iterate*.

Getting Ready for Lesson 11-7

BASIC SKILL Students will use the Binomial Theorem in Lesson 11-7. This will involve their simplifying factorial expressions. Use Exercises 50–55 to determine your students' familiarity with evaluating the kinds of expressions they will encounter when simplifying factorials.

Assessment Options

Quiz (Lessons 11-5 and 11-6) is available on p. 694 of the *Chapter 11 Resource Masters.*

Answer

41. Under certain conditions, the Fibonacci sequence can be used to model the number of shoots on a plant. Answers should include the following.
    - The 13th term of the sequence is 233, so there are 233 shoots on the plant during the 13th month.
    - The Fibonacci sequence is not arithmetic because the differences (0, 1, 1, 2, …) of the terms are not constant. The Fibonacci sequence is not geometric because the ratios $\left(1, 2, \frac{3}{2}, \ldots\right)$ of the terms are not constant.

Find the first three iterates of each function for the given initial value.

31. $f(x) = 9x - 2, x_0 = 2$ **16, 142, 1276**

32. $f(x) = 4x - 3, x_0 = 2$ **5, 17, 65**

33. $f(x) = 3x + 5, x_0 = -4$ **−7, −16, −43**

34. $f(x) = 5x + 1, x_0 = -1$ **−4, −19, −94**

35. $f(x) = 2x^2 - 5, x_0 = -1$ **−3, 13, 333**

36. $f(x) = 3x^2 - 4, x_0 = 1$ **−1, −1, −1**

37. $\frac{5}{2}, \frac{37}{2}, \frac{1445}{2}$

38. $\frac{4}{3}, \frac{10}{3}, \frac{76}{3}$

★ **37.** $f(x) = 2x^2 + 2x + 1, x_0 = \frac{1}{2}$

★ **38.** $f(x) = 3x^2 - 3x + 2, x_0 = \frac{1}{3}$

39. ECONOMICS If the rate of inflation is 2%, the cost of an item in future years can be found by iterating the function $c(x) = 1.02x$. Find the cost of a \$70 portable stereo in four years if the rate of inflation remains constant. **\$75.77**

40. No; according to the first two iterates, $f(4) = 4$. According to the second and third iterates, $f(4) = 7$. Since $f(x)$ is a function, it cannot have two values when $x = 4$.

40. CRITICAL THINKING Are there a function $f(x)$ and an initial value x_0 such that the first three iterates, in order, are 4, 4, and 7? If so, state such a function and initial value. If not, explain.

41. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. **See margin.**

How is the Fibonacci sequence illustrated in nature?

Include the following in your answer:

- the 13th term in the Fibonacci sequence, with an explanation of what it tells you about the plant described, and
- an explanation of why the Fibonacci sequence is neither arithmetic nor geometric.

Standardized Test Practice

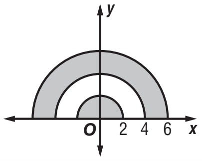
42. If a is positive, what percent of $4a$ is 8? **D**

Ⓐ $\frac{a}{100}\%$ Ⓑ $\frac{a}{2}\%$ Ⓒ $\frac{8}{a}\%$ Ⓓ $\frac{200}{a}\%$

43. The figure at the right is made of three concentric semicircles. What is the total area of the shaded regions? **C**

Ⓐ 4π units^2 Ⓑ 10π units^2

Ⓒ 12π units^2 Ⓓ 20π units^2



Maintain Your Skills

Mixed Review

Find the sum of each infinite geometric series, if it exists. *(Lesson 11-5)*

44. $9 + 6 + 4 + \ldots$ **27**

45. $\frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \ldots$ $\frac{1}{6}$

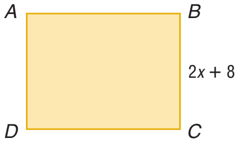
46. $4 - \frac{8}{3} + \frac{16}{9} + \ldots$ $\frac{12}{5}$

Find the sum of each geometric series. *(Lesson 11-4)*

47. $2 - 10 + 50 - \ldots$ to 6 terms **−5208**

48. $3 + 1 + \frac{1}{3} + \ldots$ to 7 terms $\frac{1093}{243}$

49. GEOMETRY The area of rectangle *ABCD* is $6x^2 + 38x + 56$ square units. Its width is $2x + 8$ units. What is the length of the rectangle? *(Lesson 5-3)* **3x + 7 units**



Getting Ready for the Next Lesson

BASIC SKILL **Evaluate each expression.** **51. 5040**

50. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ **120**

51. $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

52. $\frac{4 \cdot 3}{2 \cdot 1}$ **6**

53. $\frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$ **20**

54. $\frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}$ **126**

55. $\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$ **210**

Algebra Activity

A Follow-Up of Lesson 11-6

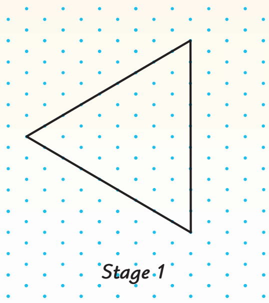
Fractals

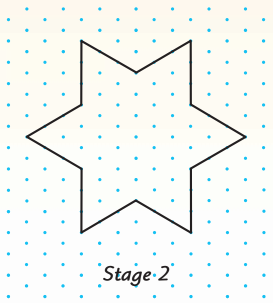
Fractals are sets of points that often involve intricate geometric shapes. Many fractals have the property that when small parts are magnified, the detail of the fractal is not lost. In other words, the magnified part is made up of smaller copies of itself. Such fractals can be constructed recursively.

You can use isometric dot paper to draw stages of the construction of a fractal called the *von Koch snowflake*.

Stage 1 Draw an equilateral triangle with sides of length 9 units on the dot paper.

Stage 2 Now remove the middle third of each side of the triangle from Stage 1 and draw the other two sides of an equilateral triangle pointing outward.


Stage 1


Stage 2

Imagine continuing this process indefinitely. The von Koch snowflake is the shape that these stages approach.

Model and Analyze

3. $s_n = 3 \cdot 4^{n-1}$, $\ell_n = 9\left(\frac{1}{3}\right)^{n-1}$, $P_n = 27\left(\frac{4}{3}\right)^{n-1}$

1. Copy and complete the table. Draw Stage 3, if necessary.

Stage	1	2	3	4
Number of Segments	3	12	48	192
Length of each Segment	9	3	1	$\frac{1}{3}$
Perimeter	27	36	48	64

2. $s_n = 4s_{n-1}$, $\ell_n = \frac{1}{3}\ell_{n-1}$, $P_n = \frac{4}{3}P_{n-1}$

2. Write recursive formulas for the number s_n of segments in Stage n, the length ℓ_n of each segment in Stage n, and the perimeter P_n of Stage n.

3. Write nonrecursive formulas for s_n, ℓ_n, and P_n.

4. What is the perimeter of the von Koch snowflake? Explain. **See pp. 629A–629F.**

5. Explain why the area of the von Koch snowflake can be represented by the infinite series $\frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + 3\sqrt{3} + \frac{4\sqrt{3}}{3} + \ldots$. **See pp. 629A–629F.**

6. Find the sum of the series in Exercise 5. Explain your steps. **See pp. 629A–629F.**

7. Do you think the results of Exercises 4 and 6 are contradictory? Explain. **See pp. 629A–629F.**

Algebra Activity

A Follow-Up of Lesson 11-6

Getting Started

Objective To apply iterations to various aspects of the Koch snowflake fractal.

Materials
isometric dot paper

Teach

- Have students explore the difference between using dot paper and graph paper in terms of counting the units in the perimeter.
- Tell students that one of the characteristics of the Koch snowflake is that the area of the interior is finite but its perimeter is infinite. Invite them to explore fractals using the many related sites on the Internet.

Assess

In **Exercises 1–3**, students should

- be able to see the iterative nature of these data.
- make the generalizations that will form the parts of the formulas.

In **Exercises 4–7**, students should

- apply the formulas of the previous lessons.
- understand that fractals have mathematical characteristics that distinguish them from ordinary polygons.

Resource Manager

Teaching Algebra with Manipulatives
- p. 19 (isometric dot paper)
- p. 284 (student recording sheet)

Glencoe Mathematics Classroom Manipulative Kit
- isometric dot grid stamp

Study Notebook

You may wish to have students summarize this activity and what they learned from it.

11-7 Lesson Notes

1 Focus

5-Minute Check Transparency 11-7 Use as a quiz or review of Lesson 11-6.

Mathematical Background notes are available for this lesson on p. 576D.

How **does a power of a binomial describe the numbers of boys and girls in a family?**

Ask students:

- In this problem, does the order make a difference, or is any family with 2 girls and 2 boys the same as any other? **Order does make a difference.**
- What does b^2g^2 represent in the triangle shown? **Any sequence with 2 boys and 2 girls.**

11-7 The Binomial Theorem

What You'll Learn

- Use Pascal's triangle to expand powers of binomials.
- Use the Binomial Theorem to expand powers of binomials.

Vocabulary

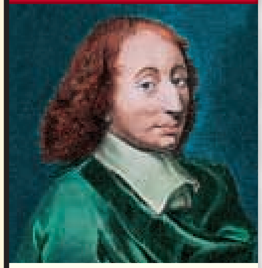
- Pascal's triangle
- Binomial Theorem
- factorial

How does a power of a binomial describe the numbers of boys and girls in a family?

According to the U.S. Census Bureau, ten percent of families have three or more children. If a family has four children, there are six sequences of births of boys and girls that result in two boys and two girls. These sequences are listed below.

BBGG BGBG BGGB GBBG GBGB GGBB

More About. . .



Pascal's Triangle

Although he did not discover it, Pascal's triangle is named for the French mathematician Blaise Pascal (1623–1662).

PASCAL'S TRIANGLE You can use the coefficients in powers of binomials to count the number of possible sequences in situations such as the one above. Remember that a binomial is a polynomial with two terms. Expand a few powers of the binomial $b + g$.

$$\begin{aligned}
(b+g)^0 &= 1b^0g^0\\
(b+g)^1 &= 1b^1g^0 + 1b^0g^1\\
(b+g)^2 &= 1b^2g^0 + 2b^1g^1 + 1b^0g^1\\
(b+g)^3 &= 1b^3g^0 + 3b^2g^1 + 3b^1g^2 + 1b^0g^3\\
(b+g)^4 &= 1b^4g^0 + 4b^3g^1 + 6b^2g^2 + 4b^1g^3 + 1b^0g^4
\end{aligned}$$

The coefficient 6 of the b^2g^2 term in the expansion of $(b + g)^4$ gives the number of sequences of births that result in two boys and two girls. As another example, the coefficient 4 of the b^1g^3 term gives the number of sequences with one boy and 3 girls.

Here are some patterns that can be seen in any binomial expansion of the form $(a + b)^n$.

1. There are $n + 1$ terms.
2. The exponent n of $(a + b)^n$ is the exponent of a in the first term and the exponent of b in the last term.
3. In successive terms, the exponent of a decreases by one, and the exponent of b increases by one.
4. The sum of the exponents in each term is n.
5. The coefficients are symmetric. They increase at the beginning of the expansion and decrease at the end.

The coefficients form a pattern that is often displayed in a triangular formation. This is known as **Pascal's triangle**. Notice that each row begins and ends with 1. Each coefficient is the sum of the two coefficients above it in the previous row.

$(a+b)^0$						1					
$(a+b)^1$					1		1				
$(a+b)^2$				1		2		1			
$(a+b)^3$			1		3		3		1		
$(a+b)^4$		1		4		6		4		1	
$(a+b)^5$	1		5		10		10		5		1

Resource Manager

Workbook and Reproducible Masters

Chapter 11 Resource Masters

- Study Guide and Intervention, pp. 667–668
- Skills Practice, p. 669
- Practice, p. 670
- Reading to Learn Mathematics, p. 671
- Enrichment, p. 672

Transparencies

5-Minute Check Transparency 11-7
Answer Key Transparencies

Technology

Interactive Chalkboard

Example 1 Use Pascal's Triangle

Expand $(x + y)^7$.

Write two more rows of Pascal's triangle.

$$\begin{array}{cccccccc} & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \end{array}$$

Use the patterns of a binomial expansion and the coefficients to write the expansion of $(x + y)^7$.

$$\begin{aligned}(x + y)^7 &= 1x^7y^0 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7x^1y^6 + 1x^0y^7 \\ &= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7\end{aligned}$$

THE BINOMIAL THEOREM Another way to show the coefficients in a binomial expansion is to write them in terms of the previous coefficients.

$$\begin{array}{llllll}(a+b)^0 & 1 & & & & \\ (a+b)^1 & 1 & \frac{1}{1} & & & \\ (a+b)^2 & 1 & \frac{2}{1} & \frac{2 \cdot 1}{1 \cdot 2} & & \\ (a+b)^3 & 1 & \frac{3}{1} & \frac{3 \cdot 2}{1 \cdot 2} & \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} & \\ (a+b)^4 & 1 & \frac{4}{1} & \frac{4 \cdot 3}{1 \cdot 2} & \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} & \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}\end{array}$$

Eliminate common factors that are shown in color.

This pattern provides the coefficients of $(a + b)^n$ for any nonnegative integer n. The pattern is summarized in the **Binomial Theorem**.

Key Concept — Binomial Theorem

If n is a nonnegative integer, then

$$(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \ldots + 1a^0b^n.$$

Example 2 Use the Binomial Theorem

Expand $(a - b)^6$.

The expansion will have seven terms. Use the sequence $1, \frac{6}{1}, \frac{6 \cdot 5}{1 \cdot 2}, \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$ to find the coefficients for the first four terms. Then use symmetry to find the remaining coefficients.

$$\begin{aligned}(a - b)^6 &= 1a^6(-b)^0 + \frac{6}{1}a^5(-b)1 + \frac{6 \cdot 5}{1 \cdot 2}a^4(-b)^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}a^3(-b)^3 + \ldots + 1a^0(-b)^6 \\ &= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6\end{aligned}$$

Notice that in terms having the same coefficients, the exponents are reversed, as in $15a^4b^2$ and $15a^2b^4$.

Study Tip

Graphing Calculators

On a TI-83 Plus, the factorial symbol, !, is located on the MATH PRB menu.

The factors in the coefficients of binomial expansions involve special products called **factorials**. For example, the product $4 \cdot 3 \cdot 2 \cdot 1$ is written $4!$ and is read *4 factorial*. In general, if n is a positive integer, then $n! = n(n - 1)(n - 2)(n - 3) \ldots 2 \cdot 1$. *By definition, $0! = 1$.*

www.algebra2.com/extra_examples

2 Teach

PASCAL'S TRIANGLE

In-Class Example Power Point®

1 Expand $(p + q)^5$. $p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

THE BINOMIAL THEOREM

In-Class Example Power Point®

2 Expand $(t - s)^8$. $t^8 - 8t^7s + 28t^6s^2 - 56t^5s^3 + 70t^4s^4 - 56t^3s^5 + 28t^2s^6 - 8ts^7 + s^8$

Teaching Tip Have students discuss each of the various patterns in these examples to make sure they see what happens with coefficients, exponents, and signs.

In-Class Examples

Power Point®

3 Evaluate $\frac{6!}{2!4!}$. **15**

Teaching Tip Encourage students to write the factors and simplify before they calculate.

4 Expand $(3x - y)^4$. $\mathbf{81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4}$

Teaching Tip Make sure students understand that 0! = 1 by definition, and also that 1! = 1.

Example 3 Factorials

Evaluate $\frac{8!}{3!5!}$.

$$\frac{8!}{3!5!} = \frac{8\cdot7\cdot6\cdot\overset{1}{\cancel{5\cdot4\cdot3\cdot2\cdot1}}}{3\cdot2\cdot1\cdot\underset{1}{\cancel{5\cdot4\cdot3\cdot2\cdot1}}}$$

Note that $8! = 8\cdot7\cdot6\cdot5!$, so $\frac{8!}{3!5!} = \frac{8\cdot7\cdot6\cdot5!}{3!5!}$ or $\frac{8\cdot7\cdot6}{3\cdot2\cdot1}$

$$= \frac{8\cdot7\cdot6}{3\cdot2\cdot1} \text{ or } 56$$

Study Tip

Missing Steps

If you don't understand a step like $\frac{6\cdot5\cdot4}{1\cdot2\cdot3} = \frac{6!}{3!3!}$, work it out on a piece of scrap paper.

$$\frac{6\cdot5\cdot4}{1\cdot2\cdot3} = \frac{6\cdot5\cdot4\cdot3!}{1\cdot2\cdot3\cdot3!}$$
$$= \frac{6!}{3!3!}$$

An expression such as $\frac{6\cdot5\cdot4}{1\cdot2\cdot3}$ in Example 2 can be written as a quotient of factorials. In this case, $\frac{6\cdot5\cdot4}{1\cdot2\cdot3} = \frac{6!}{3!3!}$. Using this idea, you can rewrite the expansion of $(a + b)^6$ using factorials.

$$(a + b)^6 = \frac{6!}{6!0!}a^6b^0 + \frac{6!}{5!1!}a^5b^1 + \frac{6!}{4!2!}a^4b^2 + \frac{6!}{3!3!}a^3b^3 + \frac{6!}{2!4!}a^2b^4 + \frac{6!}{1!5!}a^1b^5 + \frac{6!}{0!6!}a^0b^6$$

You can also write this series using sigma notation.

$$(a + b)^6 = \sum_{k=0}^{6}\frac{6!}{(6 - k)!k!}a^{6-k}b^k$$

In general, the Binomial Theorem can be written both in factorial notation and in sigma notation.

Key Concept — Binomial Theorem, Factorial Form

$$(a + b)^n = \frac{n!}{n!0!}a^nb^0 + \frac{n!}{(n-1)!1!}a^{n-1}b^1 + \frac{n!}{(n-2)!2!}a^{n-2}b^2 + \ldots + \frac{n!}{0!n!}a^0b^n$$

$$= \sum_{k=0}^{n}\frac{n!}{(n-k)!k!}a^{n-k}b^k$$

Example 4 Use a Factorial Form of the Binomial Theorem

Expand $(2x + y)^5$.

$$(2x + y)^5 = \sum_{k=0}^{5}\frac{5!}{(5-k)!k!}(2x)^{5-k}y^k$$ Binomial Theorem, factorial form

$$= \frac{5!}{5!0!}(2x)^5y^0 + \frac{5!}{4!1!}(2x)^4y^1 + \frac{5!}{3!2!}(2x)^3y^2 + \frac{5!}{2!3!}(2x)^2y^3 + \frac{5!}{1!4!}(2x)^1y^4 + \frac{5!}{0!5!}(2x)^0y^5$$ Let $k = 0, 1, 2, 3, 4,$ and 5.

TEACHING TIP

A common mistake students make is forgetting to evaluate $(2x)^5$ as 2^5x^5. Inserting the line $1(32x^5) + 5(16x^4)y + 10(8x^3)y^2 + 10(4x^2)y^3 + 5(2x)y^4 + y^5$ might help.

$$= \frac{5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1\cdot1}(2x)^5 + \frac{5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1\cdot1}(2x)^4y + \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1}(2x)^3y^2 + \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1}(2x)^2y^3 + \frac{5\cdot4\cdot3\cdot2\cdot1}{1\cdot4\cdot3\cdot2\cdot1}(2x)y^4 + \frac{5\cdot4\cdot3\cdot2\cdot1}{1\cdot5\cdot4\cdot3\cdot2\cdot1}y^5$$

$$= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$$ Simplify.

Sometimes you need to know only a particular term of a binomial expansion. Note that when the Binomial Theorem is written in sigma notation, $k = 0$ for the first term, $k = 1$ for the second term, and so on. In general, the value of k is always one less than the number of the term you are finding.

Answers

7. $p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$
8. $t^6 + 12t^5 + 60t^4 + 160t^3 + 240t^2 + 192t + 64$
9. $x^4 - 12x^3y + 54x^2y^2 - 108xy^3 + 81y^4$
19. $a^3 - 3a^2b + 3ab^2 - b^3$
20. $m^4 + 4m^3n + 6m^2n^2 + 4mn^3 + n^4$
21. $r^8 + 8r^7s + 28r^6s^2 + 56r^5s^3 + 70r^4s^4 + 56r^3s^5 + 28r^2s^6 + 8rs^7 + s^8$
22. $m^5 - 5m^4a + 10m^3a^2 - 10m^2a^3 + 5ma^4 - a^5$
23. $x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243$
24. $a^4 - 8a^3 + 24a^2 - 32a + 16$
25. $16b^4 - 32b^3x + 24b^2x^2 - 8bx^3 + x^4$
26. $64a^6 + 192a^5b + 240a^4b^2 + 160a^3b^3 + 60a^2b^4 + 12ab^5 + b^6$
27. $243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5$
28. $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$
29. $\frac{a^5}{32} + \frac{5a^4}{8} + 5a^3 + 20a^2 + 40a + 32$
30. $243 + 135m + 30m^2 + \frac{10m^3}{3} + \frac{5m^4}{27} + \frac{m^5}{243}$

Example 5 Find a Particular Term

Find the fifth term in the expansion of $(p + q)^{10}$.

First, use the Binomial Theorem to write the expansion in sigma notation.

$$(p + q)^{10} = \sum_{k=0}^{10} \frac{10!}{(10 - k)!k!} p^{10 - k} q^k$$

In the fifth term, $k = 4$.

$$\frac{10!}{(10-k)!k!} p^{10 - k} q^k = \frac{10!}{(10-4)!4!} p^{10 - 4} q^4 \qquad k = 4$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} p^6 q^4 \qquad \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!4!} \text{ or } \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 210p^6q^4 \qquad \text{Simplify.}$$

Check for Understanding

Concept Check

1. 1, 8, 28, 56, 70, 56, 28, 8, 1

1. **List** the coefficients in the row of Pascal's triangle corresponding to $n = 8$.
2. **Identify** the coefficient of $a^{n-1}b$ in the expansion of $(a + b)^n$. n
3. **OPEN ENDED** Write a power of a binomial for which the first term of the expansion is $625x^4$. Sample answer: $(5x + y)^4$

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4–6	3
7–9, 12	1, 2, 4
10, 11	5

Evaluate each expression.

4. 8! 40,320
5. $\frac{13!}{9!}$ 17,160
6. $\frac{12!}{2!10!}$ 66

Expand each power. 7–9. See margin.

7. $(p + q)^5$
8. $(t + 2)^6$
9. $(x - 3y)^4$

Find the indicated term of each expansion.

10. fourth term of $(a + b)^8$ $56a^5b^3$
11. fifth term of $(2a + 3b)^{10}$ $1{,}088{,}640a^6b^4$

Application

12. **SCHOOL** Mr. Hopkins is giving a five-question true-false quiz. How many ways could a student answer the questions with three trues and two falses? 10

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
13–18	3
19–33	1, 2, 4
34–41	5

Extra Practice

See page 853.

Evaluate each expression.

13. 9! 362,880
14. 13! 6,227,020,800
15. $\frac{9!}{7!}$ 72
16. $\frac{7!}{4!}$ 210
17. $\frac{12!}{8!4!}$ 495
18. $\frac{14!}{5!9!}$ 2002

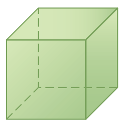
Expand each power. 19-30. See margin.

19. $(a - b)^3$
20. $(m + n)^4$
21. $(r + s)^8$
22. $(m - a)^5$
23. $(x + 3)^5$
24. $(a - 2)^4$
25. $(2b - x)^4$
26. $(2a + b)^6$
27. $(3x - 2y)^5$
28. $(3x + 2y)^4$

★ 29. $\left(\frac{a}{2} + 2\right)^5$

★ 30. $\left(3 + \frac{m}{3}\right)^5$

31. **GEOMETRY** Write an expanded expression for the volume of the cube at the right. $27x^3 + 54x^2 + 36x + 8 \text{ cm}^3$



www.algebra2.com/self_check_quiz

DAILY INTERVENTION

Differentiated Instruction

ELL

Verbal/Linguistic Have pairs of students work together to make up a jingle or a poem that describes the patterns in the Binomial Theorem. The poem should describe at least three of the five patterns in the Binomial Theorem described on p. 612 of the Student Edition.

In-Class Example

Power Point®

5 Find the fourth term in the expansion of $(a + 3b)^4$. $108ab^3$

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 11.
- add the Study Tip on p. 613 to their list of tips about the graphing calculator.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises…

Organization by Objective

- **Pascal's Triangle:** 19–22, 34–41
- **The Binomial Theorem:** 13–18, 23–33

Odd/Even Assignments

Exercises 13–30 and 34–41 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

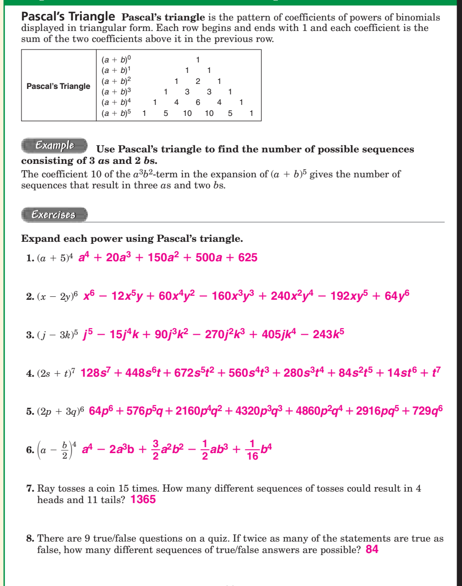
Basic: 13–27 odd, 31–39 odd, 42–62

Average: 13–39 odd, 41–62

Advanced: 14–40 even, 42–58 (optional: 59–62)

All: Practice Quiz 2 (1–10)

Study Guide and Intervention, p. 667 (shown) and p. 668



Pascal's Triangle **Pascal's triangle** is the pattern of coefficients of powers of binomials displayed in triangular form. Each row begins and ends with 1 and each coefficient is the sum of the two coefficients above it in the previous row.

Pascal's Triangle	$(a + b)^0$	1
	$(a + b)^1$	1 1
	$(a + b)^2$	1 2 1
	$(a + b)^3$	1 3 3 1
	$(a + b)^4$	1 4 6 4 1
	$(a + b)^5$	1 5 10 10 5 1

Example **Use Pascal's triangle to find the number of possible sequences consisting of 3 *a*s and 2 *b*s.**

The coefficient 10 of the a^3b^2-term in the expansion of $(a + b)^5$ gives the number of sequences that result in three *a*s and two *b*s.

Exercises

Expand each power using Pascal's triangle.

1. $(a + 5)^4$ $a^4 + 20a^3 + 150a^2 + 500a + 625$

2. $(x - 2y)^6$ $x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6$

3. $(j - 3k)^5$ $j^5 - 15j^4k + 90j^3k^2 - 270j^2k^3 + 405jk^4 - 243k^5$

4. $(2s + t)^7$ $128s^7 + 448s^6t + 672s^5t^2 + 560s^4t^3 + 280s^3t^4 + 84s^2t^5 + 14st^6 + t^7$

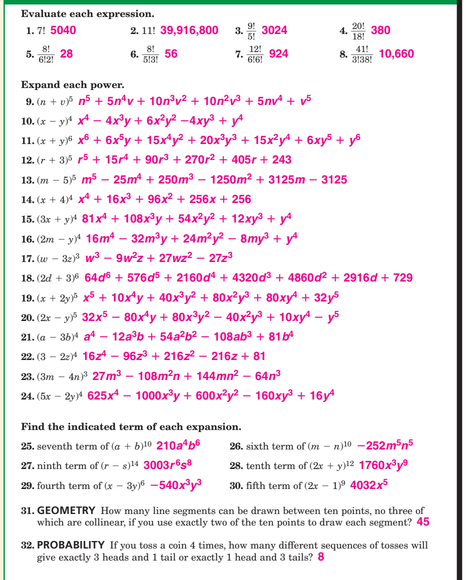
5. $(2p + 3q)^6$ $64p^6 + 576p^5q + 2160p^4q^2 + 4320p^3q^3 + 4860p^2q^4 + 2916pq^5 + 729q^6$

6. $\left(a - \frac{b}{2}\right)^4$ $a^4 - 2a^3b + \frac{3}{2}a^2b^2 - \frac{1}{2}ab^3 + \frac{1}{16}b^4$

7. Ray tosses a coin 15 times. How many different sequences of tosses could result in 4 heads and 11 tails? 1365

8. There are 9 true/false questions on a quiz. If twice as many of the statements are true as false, how many different sequences of true/false answers are possible? 84

Skills Practice, p. 669 and Practice, p. 670 (shown)



Evaluate each expression.

1. 7! 5040

2. 11! 39,916,800

3. $\frac{9!}{5!}$ 3024

4. $\frac{20!}{18!}$ 380

5. $\frac{8!}{6!2!}$ 28

6. $\frac{8!}{5!3!}$ 56

7. $\frac{12!}{6!6!}$ 924

8. $\frac{41!}{3!38!}$ 10,660

Expand each power.

9. $(n + v)^5$ $n^5 + 5n^4v + 10n^3v^2 + 10n^2v^3 + 5nv^4 + v^5$

10. $(x - y)^4$ $x^4 - 4x^3y + 6x^2y^2 -4xy^3 + y^4$

11. $(x + y)^6$ $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

12. $(r + 3)^5$ $r^5 + 15r^4 + 90r^3 + 270r^2 + 405r + 243$

13. $(m - 5)^5$ $m^5 - 25m^4 + 250m^3 - 1250m^2 + 3125m - 3125$

14. $(x + 4)^4$ $x^4 + 16x^3 + 96x^2 + 256x + 256$

15. $(3x + y)^4$ $81x^4 + 108x^3y + 54x^2y^2 + 12xy^3 + y^4$

16. $(2m - y)^4$ $16m^4 - 32m^3y + 24m^2y^2 - 8my^3 + y^4$

17. $(w - 3z)^3$ $w^3 - 9w^2z + 27wz^2 - 27z^3$

18. $(2d + 3)^6$ $64d^6 + 576d^5 + 2160d^4 + 4320d^3 + 4860d^2 + 2916d + 729$

19. $(x + 2y)^5$ $x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

20. $(2x - y)^5$ $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$

21. $(a - 3b)^4$ $a^4 - 12a^3b + 54a^2b^2 - 108ab^3 + 81b^4$

22. $(3 - 2z)^4$ $16z^4 - 96z^3 + 216z^2 - 216z + 81$

23. $(3m - 4n)^3$ $27m^3 - 108m^2n + 144mn^2 - 64n^3$

24. $(5x - 2y)^4$ $625x^4 - 1000x^3y + 600x^2y^2 - 160xy^3 + 16y^4$

Find the indicated term of each expansion.

25. seventh term of $(a + b)^{10}$ $210a^4b^6$

26. sixth term of $(m - n)^{10}$ $-252m^5n^5$

27. ninth term of $(r - s)^{14}$ $3003r^6s^8$

28. tenth term of $(2x + y)^{12}$ $1760x^3y^9$

29. fourth term of $(x - 3y)^6$ $-540x^3y^3$

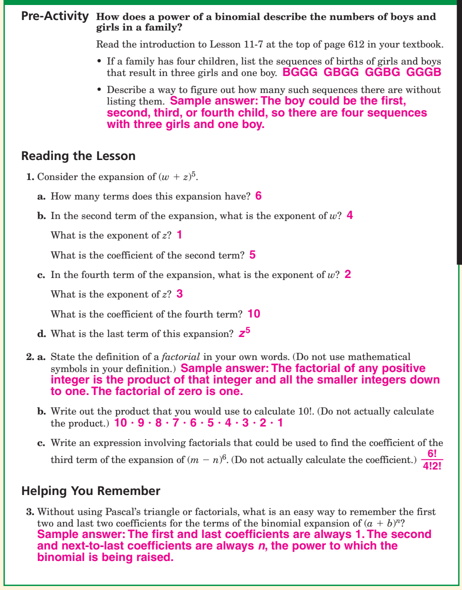
30. fifth term of $(2x - 1)^9$ $4032x^5$

31. GEOMETRY How many line segments can be drawn between ten points, no three of which are collinear, if you use exactly two of the ten points to draw each segment? 45

32. PROBABILITY If you toss a coin 4 times, how many different sequences of tosses will give exactly 3 heads and 1 tail or exactly 1 head and 3 tails? 8

Reading to Learn Mathematics, p. 671

ELL



Pre-Activity **How does a power of a binomial describe the numbers of boys and girls in a family?**

Read the introduction to Lesson 11-7 at the top of page 612 in your textbook.

- If a family has four children, list the sequences of births of girls and boys that result in three girls and one boy. BGGG GBGG GGBG GGGB
- Describe a way to figure out how many such sequences there are without listing them. Sample answer: The boy could be the first, second, third, or fourth child, so there are four sequences with three girls and one boy.

Reading the Lesson

1. Consider the expansion of $(w + z)^5$.

a. How many terms does this expansion have? 6

b. In the second term of the expansion, what is the exponent of *w*? 4

What is the exponent of *z*? 1

What is the coefficient of the second term? 5

c. In the fourth term of the expansion, what is the exponent of *w*? 2

What is the exponent of *z*? 3

What is the coefficient of the fourth term? 10

d. What is the last term of this expansion? z^5

2. a. State the definition of a *factorial* in your own words. (Do not use mathematical symbols in your definition.) Sample answer: The factorial of any positive integer is the product of that integer and all the smaller integers down to one. The factorial of zero is one.

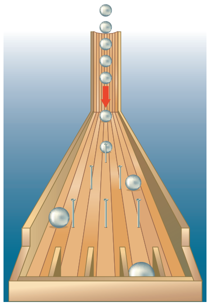
b. Write out the product that you would use to calculate 10!. (Do not actually calculate the product.) $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

c. Write an expression involving factorials that could be used to find the coefficient of the third term of the expansion of $(m - n)^6$. (Do not actually calculate the coefficient.) $\frac{6!}{4!2!}$

Helping You Remember

3. Without using Pascal's triangle or factorials, what is an easy way to remember the first two and last two coefficients for the terms of the binomial expansion of $(a + b)^n$? Sample answer: The first and last coefficients are always 1. The second and next-to-last coefficients are always *n*, the power to which the binomial is being raised.

32. GAMES The diagram shows the board for a game in which ball bearings are dropped down a chute. A pattern of nails and dividers causes the bearings to take various paths to the sections at the bottom. For each section, how many paths through the board lead to that section? 1, 4, 6, 4, 1



WebQuest

Pascal's triangle displays many patterns. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

33. INTRAMURALS Ofelia is taking ten shots in the intramural free-throw shooting competition. How many sequences of makes and misses are there that result in her making eight shots and missing two? 45

Find the indicated term of each expansion.

34. sixth term of $(x - y)^9$ $-126x^4y^5$

35. seventh term of $(x + y)^{12}$ $924x^6y^6$

36. fourth term of $(x + 2)^7$ $280x^4$

37. fifth term of $(a - 3)^8$ $5670a^4$

38. fifth term of $(2a + 3b)^{10}$ $1{,}088{,}640a^6b^4$

39. fourth term of $(2x + 3y)^9$ $145{,}152x^6y^3$

★ **40.** fourth term of $\left(x + \frac{1}{3}\right)^7$ $\frac{35}{27}x^4$

★ **41.** sixth term of $\left(x - \frac{1}{2}\right)^{10}$ $-\frac{63}{8}x^5$

42. CRITICAL THINKING Explain why $\frac{12!}{7!5!} + \frac{12!}{6!6!} = \frac{13!}{7!6!}$ without finding the value of any of the expressions. See pp. 629A–629F.

43. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson. See pp. 629A–629F.

How does a power of a binomial describe the numbers of boys and girls in a family?

Include the following in your answer:

- the expansion of $(b + g)^5$ and what it tells you about sequences of births of boys and girls in families with five children, and
- an explanation of how to find a formula for the number of sequences of births that have exactly *k* girls in a family of *n* children.

Standardized Test Practice

Ⓐ Ⓑ Ⓒ Ⓓ

44. Which of the following represents the values of *x* that are solutions of the inequality $x^2 < x + 20$? D

Ⓐ $x > -4$

Ⓑ $x < 5$

Ⓒ $-5 < x < 4$

Ⓓ $-4 < x < 5$

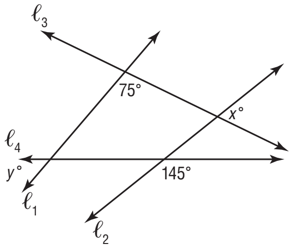
45. If four lines intersect as shown in the figure at the right, $x + y =$ C

Ⓐ 70.

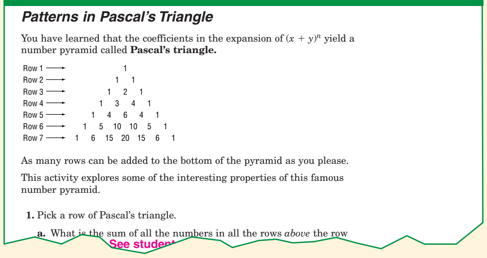
Ⓑ 115.

Ⓒ 140.

Ⓓ It cannot be determined from the information given.



Enrichment, p. 672



Patterns in Pascal's Triangle

You have learned that the coefficients in the expansion of $(x + y)^n$ yield a number pyramid called **Pascal's triangle.**

Row 1 ⟶ 1
Row 2 ⟶ 1 1
Row 3 ⟶ 1 2 1
Row 4 ⟶ 1 3 4 1
Row 5 ⟶ 1 4 6 4 1
Row 6 ⟶ 1 5 10 10 5 1
Row 7 ⟶ 1 6 15 20 15 6 1

As many rows can be added to the bottom of the pyramid as you please.

This activity explores some of the interesting properties of this famous number pyramid.

1. Pick a row of Pascal's triangle.

a. What is the sum of all the numbers in all the rows *above* the row

See student

Maintain Your Skills

Mixed Review

Find the first five terms of each sequence. *(Lesson 11-6)*

46. $a_1 = 7, a_{n+1} = a_n - 2$ **7, 5, 3, 1, −1** **47.** $a_1 = 3, a_{n+1} = 2a_n - 1$ **3, 5, 9, 17, 33**

48. **CLOCKS** The spring in Juanita's old grandfather clock is broken. When you try to set the pendulum in motion by holding it against the wall of the clock and letting go, it follows a swing pattern of 25 centimeters, 20 centimeters, 16 centimeters, and so on until it comes to rest. What is the total distance the pendulum swings before coming to rest? *(Lesson 11-5)* **125 cm**

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. *(Lesson 10-4)*

49. $\log_2 5$ $\frac{\log 5}{\log 2}$; **2.3219** **50.** $\log_3 10$ $\frac{1}{\log 3}$; **2.0959** **51.** $\log_5 8$ $\frac{\log 8}{\log 5}$; **1.2920**

Determine any vertical asymptotes and holes in the graph of each rational function. *(Lesson 9-3)* **54. hole: $x = -3$**

52. $f(x) = \frac{1}{x^2 + 5x + 6}$ **53.** $f(x) = \frac{x + 2}{x^2 + 3x - 4}$ **54.** $f(x) = \frac{x^2 + 4x + 3}{x + 3}$

52. asymptotes: $x = -2$, $x = -3$

53. asymptotes: $x = -4$, $x = 1$

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. *(Lesson 8-6)*

55. $x^2 - 6x - y^2 - 3 = 0$ **hyperbola** **56.** $4y - x + y^2 = 1$ **parabola**

Determine whether each pair of functions are inverse functions. *(Lesson 7-8)*

57. $f(x) = x + 3$ **yes**
$g(x) = x - 3$

58. $f(x) = 2x + 1$ **no**
$g(x) = \frac{x + 1}{2}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **State whether each statement is *true* or *false* when $n = 1$. Explain.** *(To review **evaluating expressions**, see Lesson 1-1.)*

59–62. See margin for explanations.

59. $1 = \frac{n(n + 1)}{2}$ **true** **60.** $1 = \frac{(n + 1)(2n + 1)}{2}$ **false**

61. $1 = \frac{n^2(n + 1)^2}{4}$ **true** **62.** $3^n - 1$ is even. **true**

Practice Quiz 2

Lessons 11-4 through 11-7

Find the sum of each geometric series. *(Lessons 11-4 and 11-5)*

1. $a_1 = 5, r = 3, n = 12$ **1,328,600** **2.** $\sum_{n-1}^{6} 2(-3)^{n-1}$ **−364**

3. $\sum_{n=1}^{\infty} 8\left(\frac{2}{3}\right)^{n-1}$ **24** **4.** $5 + 1 + \frac{1}{5} + \ldots$ $\frac{25}{4}$

Find the first five terms of each sequence. *(Lesson 11-6)*

5. $a_1 = 1, a_{n+1} = 2a_n + 3$ **1, 5, 13, 29, 61** **6.** $a_1 = 2, a_{n+1} = a_n + 2n$ **2, 4, 8, 14, 22**

7. Find the first three iterates of the function $f(x) = -3x + 2$ for an initial value of $x_0 = -1$. *(Lesson 11-6)* **5, −13, 41**

Expand each power. *(Lesson 11-7)* **8.** $243x^5 + 405x^4y + 270x^3y^2 + 90x^2y^3 + 15xy^4 + y^5$

8. $(3x + y)^5$ **9.** $(a + 2)^6$ $a^6 + 12a^5 + 60a^4 + 160a^3 + 240a^2 + 192a + 64$

10. Find the fifth term of the expansion of $(2a + b)^9$. *(Lesson 11-7)* $4032a^5b^4$

4 Assess

Open-Ended Assessment

Writing Have students write their own example of a binomial expansion, using colored pens or pencils to emphasize the patterns.

Tips for New Teachers

Intervention Invite students to share their confusions about this material in small groups or with a partner in order to clear up any problems.

Getting Ready for Lesson 11-8

PREREQUISITE SKILL Students will prove statements using mathematical induction in Lesson 11-8. This will include their showing that a statement is true for $n = 1$ by evaluating an equation for that value. Use Exercises 59–62 to determine your students' familiarity with evaluating equations for a given value.

Assessment Options

Practice Quiz 2 The quiz provides students with a brief review of the concepts and skills in Lessons 11-4 through 11-7. Lesson numbers are given to the right of the exercises or instruction lines so students can review concepts not yet mastered.

Answers

59. $\frac{1(1 + 1)}{2} = \frac{1(2)}{2}$ or 1

60. $\frac{(1 + 1)(2 \cdot 1 + 1)}{2} = \frac{2(3)}{2}$ or 3

61. $\frac{1^2(1 + 1)^2}{4} = \frac{1(4)}{4}$ or 1

62. $3^1 - 1 = 2$, which is even

11-8 Lesson Notes

1 Focus

5-Minute Check Transparency 11-8 Use as a quiz or review of Lesson 11-7.

Mathematical Background notes are available for this lesson on p. 576D.

How does the concept of a ladder help you prove statements about numbers?

Ask students:

Why is it not enough to prove only Step 2 and Step 3? **Steps 2 and 3 prove the statement for the next integer, given that it is true for some integer. You must prove the statement for some specific value of *n* in order to prove that the statement is true for any value of *k* that is greater than or equal to *n*.**

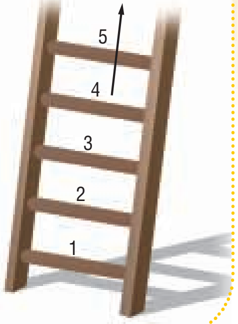
11-8 Proof and Mathematical Induction

What You'll Learn

- Prove statements by using mathematical induction.
- Disprove statements by finding a counterexample.

How does the concept of a ladder help you prove statements about numbers?

Imagine the positive integers as a ladder that goes upward forever. You know that you cannot leap to the top of the ladder, but you can stand on the first step, and no matter which step you are on, you can always climb one step higher. Is there any step you cannot reach?



Vocabulary

- mathematical induction
- inductive hypothesis

MATHEMATICAL INDUCTION **Mathematical induction** is used to prove statements about positive integers. An induction proof consists of three steps.

Study Tip

Step 1
In many cases, it will be helpful to let $n = 1$.

Key Concept — Mathematical Induction

Step 1 Show that the statement is true for some integer n.

Step 2 Assume that the statement is true for some positive integer k, where $k \geq n$. This assumption is called the **inductive hypothesis**.

Step 3 Show that the statement is true for the next integer $k + 1$.

Example 1 Summation Formula

Prove that the sum of the squares of the first n positive integers is $\frac{n(n+1)(2n+1)}{6}$. That is, prove that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Step 1 When $n = 1$, the left side of the given equation is 1^2 or 1. The right side is $\frac{1(1+1)[2(1)+1]}{6}$ or 1. Thus, the equation is true for $n = 1$.

Step 2 Assume $1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$ for a positive integer k.

Step 3 Show that the given equation is true for $n = k + 1$.

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{Add } (k+1)^2 \text{ to each side.}\\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{Add.}\\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} && \text{Factor.}\\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} && \text{Simplify.}\\
&= \frac{(k+1)(k+2)(2k+3)}{6} && \text{Factor.}\\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}
\end{aligned}$$

Resource Manager

Workbook and Reproducible Masters

Chapter 11 Resource Masters

- Study Guide and Intervention, pp. 673–674
- Skills Practice, p. 675
- Practice, p. 676
- Reading to Learn Mathematics, p. 677
- Enrichment, p. 678
- Assessment, p. 694

Transparencies

5-Minute Check Transparency 11-8
Answer Key Transparencies

Technology

Interactive Chalkboard

The last expression on page 618 is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus, the equation is true for $n = k + 1$.

This proves that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$ for all positive integers n.

Example 2 Divisibility

Prove that $7^n - 1$ is divisible by 6 for all positive integers n.

Step 1 When $n = 1$, $7^n - 1 = 7^1 - 1$ or 6. Since 6 is divisible by 6, the statement is true for $n = 1$.

Step 2 Assume that $7^k - 1$ is divisible by 6 for some positive integer k. This means that there is a whole number r such that $7^k - 1 = 6r$.

Step 3 Show that the statement is true for $n = k + 1$.

$$
\begin{aligned}
7^k - 1 &= 6r && \text{Inductive hypothesis}\\
7^k &= 6r + 1 && \text{Add 1 to each side.}\\
7(7^k) &= 7(6r + 1) && \text{Multiply each side by 7.}\\
7^{k+1} &= 42r + 7 && \text{Simplify.}\\
7^{k+1} - 1 &= 42r + 6 && \text{Subtract 1 from each side.}\\
7^{k+1} - 1 &= 6(7r + 1) && \text{Factor.}
\end{aligned}
$$

Since r is a whole number, $7r + 1$ is a whole number. Therefore, $7^{k+1} - 1$ is divisible by 6. Thus, the statement is true for $n = k + 1$.

This proves that $7^n - 1$ is divisible by 6 for all positive integers n.

TEACHING TIP

Point out that, since k is a positive integer, $7^k - 1$ is a positive integer. Therefore, r must also be a positive integer. This type of reasoning will help students analyze the factorizations that they obtain in Exercises 7, 19, and 20.

COUNTEREXAMPLES

Of course, not every formula that you can write is true. A formula that works for a few positive integers may not work for *every* positive integer. You can show that a formula is not true by finding a *counterexample*. This often involves trial and error.

Study Tip

Reading Math

One of the meanings of *counter* is *to oppose*, so a counterexample is an example that opposes a hypothesis.

Example 3 Counterexample

Find a counterexample for the formula $1^4 + 2^4 + 3^4 + \ldots + n^4 = 1 + (4n - 4)^2$.

Check the first few positive integers.

n	Left Side of Formula	Right Side of Formula	
1	1^4 or 1	$1 + [4(1) - 4]^2 = 1 + 0^2$ or 1	true
2	$1^4 + 2^4 = 1 + 16$ or 17	$1 + [4(2) - 4]^2 = 1 + 4^2$ or 17	true
3	$1^4 + 2^4 + 3^4 = 1 + 16 + 81$ or 98	$1 + [4(3) - 4]^2 = 1 + 64$ or 65	false

The value $n = 3$ is a counterexample for the formula.

Check for Understanding

Concept Check

1–2. See pp. 629A–629F.

1. **Describe** some of the types of statements that can be proved by using mathematical induction.
2. **Explain** the difference between mathematical induction and a counterexample.
3. **OPEN ENDED** Write an expression of the form $b^n - 1$ that is divisible by 2 for all positive integers n. Sample answer: $3^n - 1$

www.algebra2.com/extra_examples

2 Teach

MATHEMATICAL INDUCTION

In-Class Examples

Power Point®

1 Prove that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$. Step 1: When $n = 1$, the left side of the given equation is 1. The right side is 1^2. Since $1 = 1^2$, the equation is true for $n = 1$. Step 2: Assume $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ for a positive integer k. Step 3: Does $1 + 3 + 5 + \ldots + [2(k + 1) - 1] = (k + 1)^2$? Yes, the left side simplifies to $k^2 + 2k + 1$ or $(k + 1)^2$ which is equal to the right side.

2 Prove that $6^n - 1$ is divisible by 5 for all positive integers n. Proof uses steps similar to those in Example 2 in the Student Edition.

COUNTEREXAMPLES

In-Class Example

Power Point®

3 Find a counterexample for the formula that $n^2 + n + 5$ is always a prime number for any positive integer n. $n = 4$

3 Practice/Apply

Study Notebook

Have students—

- complete the definitions/examples for the remaining terms on their Vocabulary Builder worksheets for Chapter 11.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

DAILY INTERVENTION

Differentiated Instruction

Visual/Spatial Have students demonstrate proof by induction by laying out a “train” of dominoes. Have them relate the steps in an inductive proof to the requirements that (1) the first domino must fall and (2) if any one domino falls, the next one must fall.

About the Exercises...

Organization by Objective

- **Mathematical Induction:** 11–24
- **Counterexamples:** 25–30

Odd/Even Assignments

Exercises 11–20 and 25–30 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 11–27 odd, 31–42

Average: 11–29 odd, 31–42

Advanced: 12–30 even, 31–42

4 Assess

Open-Ended Assessment

Speaking Have students explain how you can prove or disprove statements by using induction and counterexamples.

Tips for New Teachers

Intervention Use simple examples to help students understand the principles of an inductive proof before they get involved in elaborate calculations.

Assessment Options

Quiz (Lessons 11-7 and 11-8) is available on p. 694 of the *Chapter 11 Resource Masters*.

Guided Practice

GUIDED PRACTICE KEY

Exercises	Examples
4, 5, 10	1
6, 7	2
8, 9	3

Prove that each statement is true for all positive integers. 4–7. See pp. 629A–629F.

4. $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

5. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

6. $4^n - 1$ is divisible by 3.

7. $5^n + 3$ is divisible by 4.

Find a counterexample for each statement.

8. $1 + 2 + 3 + \ldots + n = n^2$ Sample answer: $n = 2$

9. $2^n + 2n$ is divisible by 4. Sample answer: $n = 3$

Application

10. PARTIES Suppose that each time a new guest arrives at a party, he or she shakes hands with each person already at the party. Prove that after n guests have arrived, a total of $\frac{n(n-1)}{2}$ handshakes have taken place. See pp. 629A–629F.

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
11–23, 31	1
24	1, 2
25–30	3

Extra Practice

See page 853.

Prove that each statement is true for all positive integers.

11. $1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$ 11–20. See pp. 629A–629F.

12. $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$

13. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

14. $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \underline{n(2n-1)(2n+1)}$

15. $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$

16. $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^n} = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$

17. $8^n - 1$ is divisible by 7.

18. $9^n - 1$ is divisible by 8.

19. $12^n + 10$ is divisible by 11.

20. $13^n + 11$ is divisible by 12.

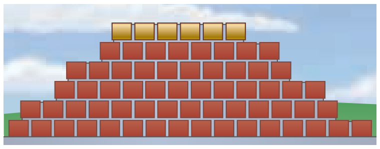
More About. . .

Architecture

The Vietnam Veterans Memorial lists the names of 58,220 deceased or missing soldiers.

Source: National Parks Service

21. ARCHITECTURE A memorial being constructed in a city park will be a brick wall, with a top row of six gold-plated bricks engraved with the names of six local war veterans. Each row has two more bricks than the row above it. Prove that the number of bricks in the top n rows is $n^2 + 5n$. See pp. 629A–629F.

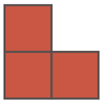


22. GEOMETRIC SERIES Use mathematical induction to prove the formula $a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1} = \frac{a_1(1 - r^n)}{1 - r}$ for the sum of a finite geometric series.

23. ARITHMETIC SERIES Use mathematical induction to prove the formula $a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + [a_1 + (n - 1)d] = \frac{n}{2}[2\,a_1 + (n - 1)d]$ for the sum of an arithmetic series.

22–23. See pp. 629A–629F.

24. PUZZLES Show that a 2^n by 2^n checkerboard with the top right square missing can always be covered by nonoverlapping L-shaped tiles like the one at the right. See pp. 629A–629F.



Answer

31. Write 7^n as $(6 + 1)^n$. Then use the Binomial Theorem.

$$\begin{aligned} 7^n - 1 &= (6 + 1)^n - 1 \\ &= 6^n + n \cdot 6^{n-1} + \frac{n(n-1)}{2}6^{n-2} + \ldots + n \cdot 6 + 1 - 1 \\ &= 6^n + n \cdot 6^{n-1} + \frac{n(n-1)}{2}6^{n-2} + \ldots + n \cdot 6 \end{aligned}$$

Since each term in the last expression is divisible by 6, the whole expression is divisible by 6. Thus, $7^n - 1$ is divisible by 6.

Find a counterexample for each statement.

25. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(3n-1)}{2}$ **Sample answer: $n = 3$**

26. $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = 12n^3 - 23n^2 + 12n$ **Sample answer: $n = 4$**

27. $3^n + 1$ is divisible by 4. **Sample answer: $n = 2$**

28. $2^n + 2n^2$ is divisible by 4. **Sample answer: $n = 3$**

★ 29. $n^2 - n + 11$ is prime. **Sample answer: $n = 11$**

★ 30. $n^2 + n + 41$ is prime. **Sample answer: $n = 41$**

31. See margin.

31. **CRITICAL THINKING** Refer to Example 2. Explain how to use the Binomial Theorem to show that $7^n - 1$ is divisible by 6 for all positive integers n.

32. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See pp. 629A–629F.**

How does the concept of a ladder help you prove statements about numbers?

Include the following in your answer:

- an explanation of which part of an inductive proof corresponds to stepping onto the bottom step of the ladder, and
- an explanation of which part of an inductive proof corresponds to climbing from one step on the ladder to the next.

Standardized Test Practice

Ⓐ Ⓑ Ⓒ Ⓓ

33. $\dfrac{x - \frac{4}{x}}{1 - \frac{4}{x} + \frac{4}{x^2}} =$ **C**

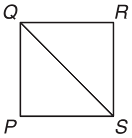
Ⓐ $\dfrac{x}{x-2}$ Ⓑ $\dfrac{x^2+2}{x-2}$ Ⓒ $\dfrac{x^2+2x}{x-2}$ Ⓓ $\dfrac{x^2+2x}{(x-2)^2}$

34. **Quantitative Comparison**

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

Ⓐ **the quantity in Column A is greater,**
Ⓑ **the quantity in Column B is greater,**
Ⓒ **the two quantities are equal, or**
Ⓓ **the relationship cannot be determined from the information given.**

PQRS is a square. **A**



Column A	Column B
2	$\dfrac{\text{length of } \overline{QS}}{\text{length of } \overline{RS}}$

Maintain Your Skills

Mixed Review **Expand each power.** *(Lesson 11-7)* **35–37. See margin.**

35. $(x + y)^6$ 36. $(a - b)^7$ 37. $(2x + y)^8$

Find the first three iterates of each function for the given initial value. *(Lesson 11-6)*

38. $f(x) = 3x - 2$, $x_0 = 2$ **4, 10, 28**

39. $f(x) = 4x^2 - 2$, $x_0 = 1$ **2, 14, 782**

40. **BIOLOGY** Suppose an amoeba divides into two amoebas once every hour. How long would it take for a single amoeba to become a colony of 4096 amoebas? *(Lesson 10-2)* **12 h**

Solve each equation. Check your solutions. *(Lesson 9-6)*

41. $\frac{1}{y+1} - \frac{3}{y-3} = 2$ **0, 1**

42. $\frac{6}{a-7} = \frac{a-49}{a^2-7a} + \frac{1}{a}$ **−14**

www.algebra2.com/self_check_quiz

Answers

35. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

36. $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$

37. $256x^8 + 1024x^7y + 1792x^6y^2 + 1792x^5y^3 + 1120x^4y^4 + 448x^3y^5 + 112x^2y^6 + 16xy^7 + y^8$

Enrichment, p. 678

Proof by Induction

Mathematical induction is a useful tool when you want to prove that a statement is true for all natural numbers.

The three steps in using induction are:

1. Prove that the statement is true for $n = 1$.
2. Prove that if the statement is true for the natural number n, it must also be true for $n + 1$.
3. Conclude that the statement is true for all natural numbers.

Follow the steps to complete each proof.

Theorem A: The sum of the first n odd natural numbers is equal to n^2.

1. Show that the theorem is true for $n = 1$.
$1 = (1)^2$

2. Suppose $1 + 3 + 5 + \ldots + (2n-1) = n^2$. Show that

Study Guide and Intervention, p. 673 (shown) and p. 674

Mathematical Induction Mathematical induction is a method of proof used to prove statements about positive integers.

Mathematical Induction Proof	**Step 1** Show that the statement is true for some integer n. **Step 2** Assume that the statement is true for some positive integer k where $k \geq n$. This assumption is called the **inductive hypothesis**. **Step 3** Show that the statement is true for the next integer $k + 1$.

Example **Prove that $5 + 11 + 17 + \ldots + (6n - 1) = 3n^2 + 2n$.**

Step 1 When $n = 1$, the left side of the given equation is $6(1) - 1 = 5$. The right side is $3(1)^2 + 2(1) = 5$. Thus the equation is true for $n = 1$.

Step 2 Assume that $5 + 11 + 17 + \ldots + (6k - 1) = 3k^2 + 2k$ for some positive integer k.

Step 3 Show that the equation is true for $n = k + 1$. First, add $[6(k + 1) - 1]$ to each side.

$$\begin{aligned} 5 + 11 + 17 + \ldots + (6k-1) + [6(k+1)-1] &= 3k^2 + 2k + [6(k+1)-1] \\ &= 3k^2 + 2k + 6k + 5 && \text{Add.} \\ &= 3k^2 + 6k + 3 + 2k + 2 && \text{Rewrite.} \\ &= 3(k^2 + 2k + 1) + 2(k + 1) && \text{Factor.} \\ &= 3(k+1)^2 + 2(k+1) && \text{Factor.} \end{aligned}$$

The last expression above is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus the equation is true for $n = k + 1$.

This proves that $5 + 11 + 17 + \ldots + (6n - 1) = 3n^2 + 2n$ for all positive integers n.

Exercises

Prove that each statement is true for all positive integers.

1. $3 + 7 + 11 + \ldots + (4n - 1) = 2n^2 + n$.

Step 1 The statement is true for $n = 1$ since $4(1) - 1 = 3$ and $2(1)^2 + 1 = 3$.
Step 2 Assume that $3 + 7 + 11 + \ldots + (4k - 1) = 2k^2 + k$ for some positive integer k.
Step 3 Adding the $(k + 1)$st term to each side from step 2, we get $3 + 7 + 11 + \ldots + (4k - 1) + [4(k + 1) - 1] = 2k^2 + k + [4(k + 1) - 1]$.
Simplifying the right side of the equation gives $2(k + 1)^2 + (k + 1)$, which is the statement to be proved.

2. $500 + 100 + 20 + \ldots + 4 \cdot 5^{4-n} = 625\left(1 - \frac{1}{5^n}\right)$.

Step 1 The statement is true for $n = 1$, since $4 \cdot 5^{4-1} = 4 \cdot 5^3 = 500$ and $625\left(1 - \frac{1}{5^1}\right) = \frac{4}{5}(625) = 500$.
Step 2 Assume that $500 + 100 + 20 + \ldots + 4 \cdot 5^{4-k} = 625\left(1 - \frac{1}{5^k}\right)$ for some positive integer k.
Step 3 Adding the $(k + 1)$st term to each side from step 2 and simplifying gives $500 + 100 + 20 + \ldots + 4 \cdot 5^{4-k} + 4 \cdot 5^{3-k} = 625\left(1 - \frac{1}{5^k}\right) + 4 \cdot 5^{3-k} = 625\left(1 - \frac{1}{5^{k+1}}\right)$, which is the statement to be proved.

Skills Practice, p. 675 and Practice, p. 676 (shown)

Prove that each statement is true for all positive integers.

1. $1 + 2 + 4 + 8 + \ldots + 2^{n-1} = 2^n - 1$

Step 1: When $n = 1$, then $2^{n-1} = 2^{1-1} = 2^0 = 1 = 2^1 - 1$. So, the equation is true for $n = 1$.
Step 2: Assume that $1 + 2 + 4 + 8 + \ldots + 2^{k-1} = 2^k - 1$ for some positive integer k.
Step 3: Show that the given equation is true for $n = k + 1$.
$1 + 2 + 4 + 8 + \ldots + 2^{k-1} + 2^{(k+1)-1} = (2^k - 1) + 2^{(k+1)-1}$
$= 2^k - 1 + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1$
So, $1 + 2 + 4 + 8 + \ldots + 2^{n-1} = 2^n - 1$ for all positive integers n.

2. $1 + 4 + 9 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1: When $n = 1$, $n^2 = 1^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$; true for $n = 1$.
Step 2: Assume that $1 + 4 + 9 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$ for some positive integer k.
Step 3: Show that the given equation is true for $n = k + 1$.

$$\begin{aligned} 1 + 4 + 9 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)[(k+2)(2k+3)]}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} \end{aligned}$$

So, $1 + 4 + 9 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n.

3. $18^n - 1$ is a multiple of 17.

Step 1: When $n = 1$, $18^n - 1 = 18 - 1$ or 17; true for $n = 1$.
Step 2: Assume that $18^k - 1$ is divisible by 17 for some positive integer k. This means that there is a whole number r such that $18^k - 1 = 17r$.
Step 3: Show that the statement is true for $n = k + 1$.
$18^k - 1 = 17r$, so $18^k = 17r + 1$, and $18(18^k) = 18(17r + 1)$. This is equivalent to $18^{k+1} = 306r + 18$, so $18^{k+1} - 1 = 306r + 17$, and $18^{k+1} - 1 = 17(18r + 1)$.
Since r is a whole number, $18r + 1$ is a whole number, and $18^{k+1} - 1$ is divisible by 17. The statement is true for $n = k + 1$. So, $18^n - 1$ is divisible by 17 for all positive integers n.

Find a counterexample for each statement.

4. $1 + 4 + 7 + \ldots + (3n - 2) = n^3 - n^2 + 1$
Sample answer: $n = 3$

5. $5^n - 2n - 3$ is divisible by 3.
Sample answer: $n = 3$

6. $1 + 3 + 5 + \ldots + (2n - 1) = \frac{n^2 + 3n - 2}{2}$
Sample answer: $n = 3$

7. $1^3 + 2^3 + 3^3 + \ldots + n^3 = n^4 - n^3 + 1$
Sample answer: $n = 3$

Reading to Learn Mathematics, p. 677

ELL

Pre-Activity **How does the concept of a ladder help you prove statements about numbers?**

Read the introduction to Lesson 11-8 at the top of page 618 in your textbook.

What are two ways in which a ladder could be constructed so that you could not reach every step of the ladder?

Sample answer: 1. The first step could be too far off the ground for you to climb on it. 2. The steps could be too far apart for you to go up from one step to the next.

Reading the Lesson

1. Fill in the blanks to describe the three steps in a proof by mathematical induction.

Step 1 Show that the statement is **true** for the number **1**.

Step 2 Assume that the statement is **true** for some positive **integer** k.

This assumption is called the **inductive hypothesis**.

Step 3 Show that the statement is **true** for the next integer **$k + 1$**.

2. Suppose that you wanted to prove that the following statement is true for all positive integers.

$$3 + 6 + 9 + \ldots + 3n = \frac{3n(n+1)}{2}$$

a. Which of the following statements shows that the statement is true for $n = 1$? **ii**

i. $3 = \frac{3 \cdot 2 + 1}{2}$ **ii.** $3 = \frac{3 \cdot 1 \cdot 2}{2}$ **iii.** $3 = \frac{3 + 1 + 2}{2}$

b. Which of the following is the statement for $n = k + 1$? **iv**

i. $3 + 6 + 9 + \ldots + 3^k = \frac{3k(k+1)}{2}$

ii. $3 + 6 + 9 + \ldots + 3^{k+1} = \frac{3k(k+1)}{2}$

iii. $3 + 6 + 9 + \ldots + 3^{k+1} = 3(k+1)(k+2)$

iv. $3 + 6 + 9 + \ldots + 3(k+1) = \frac{3(k+1)(k+2)}{2}$

Helping You Remember

3. Many students confuse the roles of n and k in a proof by mathematical induction. What is a good way to remember the difference in the ways these variables are used in such a proof? **Sample answer: The letter *n* stands for "number" and is used as a variable to represent *any* natural number. The letter *k* is used to represent a particular value of *n*.**

Chapter 11 Study Guide and Review

Vocabulary and Concept Check

- This alphabetical list of vocabulary terms in Chapter 11 includes a page reference where each term was introduced.
- **Assessment** A vocabulary test/review for Chapter 11 is available on p. 692 of the *Chapter 11 Resource Masters*.

Lesson-by-Lesson Review

For each lesson,

- the main ideas are summarized,
- additional examples review concepts, and
- practice exercises are provided.

Vocabulary PuzzleMaker

ELL The Vocabulary PuzzleMaker software improves students' mathematics vocabulary using four puzzle formats—crossword, scramble, word search using a word list, and word search using clues. Students can work on a computer screen or from a printed handout.

MindJogger Videoquizzes

ELL MindJogger Videoquizzes provide an alternative review of concepts presented in this chapter. Students work in teams in a game show format to gain points for correct answers. The questions are presented in three rounds.

Round 1 Concepts (5 questions)
Round 2 Skills (4 questions)
Round 3 Problem Solving (4 questions)

Chapter 11 Study Guide and Review

Vocabulary and Concept Check

arithmetic means (p. 580)
arithmetic sequence (p. 578)
arithmetic series (p. 583)
Binomial Theorem (p. 613)
common difference (p. 578)
common ratio (p. 588)
factorial (p. 613)
Fibonacci sequence (p. 606)
geometric means (p. 590)
geometric sequence (p. 588)
geometric series (p. 594)
index of summation (p. 585)
inductive hypothesis (p. 618)
infinite geometric series (p. 599)
iteration (p. 608)
mathematical induction (p. 618)
partial sum (p. 599)
Pascal's triangle (p. 612)
recursive formula (p. 606)
sequence (p. 578)
series (p. 583)
sigma notation (p. 585)
term (p. 578)

Choose the term from the list above that best completes each statement.

1. A(n) ____________ of an infinite series is the sum of a certain number of terms. **partial sum**
2. If a sequence has a common ratio, then it is a(n) ______________. **geometric sequence**
3. Using _________, the series 2 + 5 + 8 + 11 + 14 can be written as $\sum_{n=1}^{5}(3n - 1)$. **sigma notation**
4. Eleven and 17 are the two ____ between 5 and 23 in the sequence 5, 11, 17, 23. **arithmetic means**
5. Using the _________, $(a - 2)^4$ can be expanded to $a^4 - 8a^3 + 24a^2 - 32a + 16$. **Binomial Theorem**
6. The _________ of the sequence 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, $\frac{16}{27}$ is $\frac{2}{3}$. **common ratio**
7. The ________ 11 + 16.5 + 22 + 27.5 + 33 has a sum of 110. **arithmetic series**
8. A(n) ____ is expressed as $n! = n(n - 1)(n - 2) \ldots 2 \cdot 1$. **factorial**

Lesson-by-Lesson Review

11-1 Arithmetic Sequences

See pages 578–582.

Concept Summary

- An arithmetic sequence is formed by adding a constant to each term to get the next term.
- The *n*th term a_n of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$, where n is any positive integer.

Examples

1 **Find the 12th term of an arithmetic sequence if $a_1 = -17$ and $d = 4$.**

$a_n = a_1 + (n - 1)d$ Formula for the *n*th term

$a_{12} = -17 + (12 - 1)4$ $n = 12, a_1 = -17, d = 4$

$a_{12} = 27$ Simplify.

2 **Find the two arithmetic means between 4 and 25.**

$a_n = a_1 + (n - 1)d$ Formula for the *n*th term

$a_4 = 4 + (4 - 1)d$ $n = 4, a_1 = 4$

$25 = 4 + 3d$ $a_4 = 25$

$7 = d$ The arithmetic means are 4 + 7 or 11 and 11 + 7 or 18.

www.algebra2.com/vocabulary_review

FOLDABLES™ Study Organizer

For more information about Foldables, see *Teaching Mathematics with Foldables.*

Students are usually interested in doing well on various kinds of tests. One way to achieve this goal is by writing their own questions about the material. Have student volunteers read some of their questions. Have other student volunteers answer, and have the writer of the question comment on the answer. Ask students to use what they have learned in this discussion to revise their own Foldables.

Encourage students to refer to their Foldables while completing the Study Guide and Review and to use them in preparing for the Chapter Test.

Exercises **Find the indicated term of each arithmetic sequence.** *See Example 2 on p. 579.*

9. $a_1 = 6, d = 8, n = 5$ **38**

10. $a_1 = -5, d = 7, n = 22$ **142**

11. $a_1 = 5, d = -2, n = 9$ **−11**

12. $a_1 = -2, d = -3, n = 15$ **−44**

Find the arithmetic means in each sequence. *See Example 4 on page 580.* **15. 6, 3, 0, −3**

13. $-7, \underline{\ ?\ }, \underline{\ ?\ }, \underline{\ ?\ }, 9$ **−3, 1, 5**

14. $12, \underline{\ ?\ }, \underline{\ ?\ }, 4$ $\mathbf{\frac{28}{3}, \frac{20}{3}}$

15. $9, \underline{\ ?\ }, \underline{\ ?\ }, \underline{\ ?\ }, \underline{\ ?\ }, -6$

16. $56, \underline{\ ?\ }, \underline{\ ?\ }, \underline{\ ?\ }, 28$ **49, 42, 35**

11-2 Arithmetic Series

See pages 583–587.

Concept Summary

- The sum S_n of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$.

Example **Find S_n for the arithmetic series with $a_1 = 34$, $a_n = 2$, and $n = 9$.**

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_9 = \frac{9}{2}(34 + 2)$ $n = 9, a_1 = 34, a_n = 2$

$S_9 = 162$ Simplify.

Exercises **Find S_n for each arithmetic series.** *See Examples on pages 584 and 585.*

17. $a_1 = 12, a_n = 117, n = 36$ **2322**

18. $4 + 10 + 16 + \ldots + 106$ **990**

19. $10 + 4 + (-2) + \ldots + (-50)$ **−220**

20. $\sum_{n=2}^{13}(3n + 1)$ **282**

11-3 Geometric Sequences

See pages 588–592.

Concept Summary

- A geometric sequence is one in which each term after the first is found by multiplying the previous term by a common ratio.
- The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1 \cdot r^{n-1}$, where n is any positive integer.

Examples **1** **Find the fifth term of a geometric sequence for which $a_1 = 7$ and $r = 3$.**

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_5 = 7 \cdot 3^{5-1}$ $n = 5, a_1 = 7, r = 3$

$a_5 = 567$ The fifth term is 567.

2 **Find two geometric means between 1 and 8.**

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_4 = 1 \cdot r^{4-1}$ $n = 4$ and $a_1 = 1$

$8 = r^3$ $a_4 = 8$

$2 = r$ The geometric means are 1(2) or 2 and 2(2) or 4.

Exercises **Find the indicated term of each geometric sequence.**
See Example 2 on page 589.

21. $a_1 = 2, r = 2, n = 5$ **32**

22. $a_1 = 7, r = 2, n = 4$ **56**

23. $a_1 = 243, r = -\frac{1}{3}, n = 5$ **3**

24. a_6 for $\frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$ $\frac{64}{3}$

Find the geometric means in each sequence. *See Example 5 on page 590.*

25. 3, __?__, __?__, 24 **6, 12**

26. 7.5, __?__, __?__, __?__, 120 **±15, 30, ±60**

27. 8, __?__, __?__, __?__, __?__, $\frac{1}{4}$ **4, 2, 1,** $\frac{1}{2}$

28. 5, __?__, __?__, __?__, 80 **±10, 20, ±40**

11-4 Geometric Series

See pages 594–598.

Concept Summary

- The sum S_n of the first n terms of a geometric series is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \text{ or } S_n = \frac{a_1 - a_1 r^n}{1 - r}, \text{ where } r \neq 1.$$

Example **Find the sum of a geometric series for which $a_1 = 7$, $r = 3$, and $n = 14$.**

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ Sum formula

$S_{14} = \frac{7 - 7 \cdot 3^{14}}{1 - 3}$ $n = 14, a_1 = 7, r = 3$

$S_{14} = 16{,}740{,}388$ Use a calculator.

Exercises **Find S_n for each geometric series.** *See Examples 1 and 3 on pages 595 and 596.*

29. $a_1 = 12, r = 3, n = 5$ **1452**

30. $4 - 2 + 1 - \ldots$ to 6 terms $\frac{21}{8}$

31. $256 + 192 + 144 + \ldots$ to 7 terms $\frac{14{,}197}{16}$

32. $\sum_{n=1}^{5}\left(-\frac{1}{2}\right)^{n-1}$ $\frac{11}{16}$

11-5 Infinite Geometric Series

See pages 599–604.

Concept Summary

- The sum S of an infinite geometric series with $-1 < r < 1$ is given by $S = \frac{a_1}{1 - r}$.

Example **Find the sum of the infinite geometric series for which $a_1 = 18$ and $r = -\frac{2}{7}$.**

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{18}{1 - \left(-\frac{2}{7}\right)}$ $a_1 = 18, r = -\frac{2}{7}$

$= \frac{18}{\frac{9}{7}}$ or 14 Simplify.

Exercises **Find the sum of each infinite geometric series, if it exists.**
See Example 1 on page 600. **34. does not exist**

33. $a_1 = 6, r = \frac{11}{12}$ **72** **34.** $\frac{1}{8} - \frac{3}{16} + \frac{9}{32} - \frac{27}{64} + \ldots$ **35.** $\sum_{n=1}^{\infty} -2\left(-\frac{5}{8}\right)^{n-1}$ $-\frac{16}{13}$

11-6 Recursion and Special Sequences

See pages 606–610.

Concept Summary

- In a recursive formula, each term is formulated from one or more previous terms.
- Iteration is the process of composing a function with itself repeatedly.

Examples

1 **Find the first five terms of the sequence in which $a_1 = 2$ and $a_{n+1} = 2a_n - 1$.**

$a_{n+1} = 2a_n - 1$ Recursive formula

$a_{1+1} = 2a_1 - 1$ $n = 1$

$a_2 = 2(2) - 1$ or 3 $a_1 = 2$

$a_{2+1} = 2a_2 - 1$ $n = 2$

$a_3 = 2(3) - 1$ or 5 $a_2 = 3$

$a_{3+1} = 2a_3 - 1$ $n = 3$

$a_4 = 2(5) - 1$ or 9 $a_3 = 5$

$a_{4+1} = 2a_4 - 1$ $n = 4$

$a_5 = 2(9) - 1$ or 17 $a_4 = 9$

The first five terms of the sequence are 2, 3, 5, 9, and 17.

2 **Find the first three iterates of $f(x) = -5x - 1$ for an initial value of $x_0 = -1$.**

$x_1 = f(x_0)$
$= f(-1)$
$= -5(-1) - 1$ or 4

$x_2 = f(x_1)$
$= f(4)$
$= -5(4) - 1$ or -21

$x_3 = f(x_2)$
$= f(-21)$
$= -5(-21) - 1$ or 104

The first three iterates are 4, −21, and 104.

Exercises **Find the first five terms of each sequence.** *See Example 1 on page 606.*

36. $a_1 = -2, a_{n+1} = a_n + 5$ **37.** $a_1 = 3, a_{n+1} = 4a_n - 10$

38. $a_1 = 2, a_{n+1} = a_n + 3n$ **2, 5, 11, 20, 32** **39.** $a_1 = 1, a_2 = 3, a_{n+2} = a_{n+1} + a_n$

36. −2, 3, 8, 13, 18 37. 3, 2, −2, −18, −82 39. 1, 3, 4, 7, 11

Find the first three iterates of each function for the given initial value.
See Example 3 on page 608. **43. −1, 4, −31**

40. $f(x) = -2x + 3, x_0 = 1$ **1, 1, 1** **41.** $f(x) = 7x - 4, x_0 = 2$ **10, 66, 458**

42. $f(x) = x^2 - 6, x_0 = -1$ **−5, 19, 355** **43.** $f(x) = -2x^2 - x + 5, x_0 = -2$

11-7 The Binomial Theorem

See pages 612–617.

Concept Summary

- Pascal's triangle can be used to find the coefficients in a binomial expansion.
- The Binomial Theorem: $(a + b)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^{n-k} b^k$

Chapter 11 **For More ...**

- Extra Practice, see pages 851–855.
- Mixed Problem Solving, see page 872.

Example **Expand $(a - 2b)^4$.**

$$(a-2b)^4 = \sum_{k=0}^{4} \frac{4!}{(4-k)!k!} a^{4-k}(-2b)^k \quad \text{Binomial Theorem}$$

$$= \frac{4!}{4!0!}a^4(-2b)^0 + \frac{4!}{3!1!}a^3(-2b)^1 + \frac{4!}{2!2!}a^2(-2b)^2 + \frac{4!}{1!3!}a^1(-2b)^3 + \frac{4!}{0!4!}a^0(-2b)^4$$

$$= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4 \quad \text{Simplify.}$$

Exercises **Expand each power.** *See Examples 1, 2, and 4 on pages 613 and 614.*

44. $(x + y)^3$ **45.** $(x - 2)^4$ **46.** $(3r + s)^5$

44–46. See margin. **48.** $-13{,}107{,}200x^9$

Find the indicated term of each expansion. *See Example 5 on page 615.*

47. fourth term of $(x + 2y)^6$ $160x^3y^3$ **48.** second term of $(4x - 5)^{10}$

11-8 Proof and Mathematical Induction

See pages 618–621.

Concept Summary

- Mathematical induction is a method of proof used to prove statements about the positive integers.

Example **Prove $1 + 5 + 25 + \ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)$ for all positive integers n.**

Step 1 When $n = 1$, the left side of the given equation is 1. The right side is $\frac{1}{4}(5^1 - 1)$ or 1. Thus, the equation is true for $n = 1$.

Step 2 Assume that $1 + 5 + 25 + \ldots + 5^{k-1} = \frac{1}{4}(5^k - 1)$ for some positive integer k.

Step 3 Show that the given equation is true for $n = k + 1$.

$$1 + 5 + 25 + \ldots + 5^{k-1} + 5^{(k+1)-1} = \frac{1}{4}(5^k - 1) + 5^{(k+1)-1} \quad \text{Add } 5^{(k+1)-1} \text{ to each side.}$$

$$= \frac{1}{4}(5^k - 1) + 5^k \quad \text{Simplify the exponent.}$$

$$= \frac{5^k - 1 + 4 \cdot 5^k}{4} \quad \text{Common denominator}$$

$$= \frac{5 \cdot 5^k - 1}{4} \quad \text{Distributive Property}$$

$$= \frac{1}{4}(5^{k+1} - 1) \quad 5 \cdot 5^k = 5^{k+1}$$

The last expression above is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus, the equation is true for $n = k + 1$.

This proves that $1 + 5 + 25 + \ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)$ for all positive integers n.

Exercises **Prove that each statement is true for all positive integers.** *See Examples 1 and 2 on pages 618 and 619.* **49–50. See margin.**

49. $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$ **50.** $6^n - 1$ is divisible by 5.

Answers

44. $x^3 + 3x^2y + 3xy^2 + y^3$

45. $x^4 - 8x^3 + 24x^2 - 32x + 16$

46. $243r^5 + 405r^4s + 270r^3s^2 + 90r^2s^3 + 15rs^4 + s^5$

49. Step 1: When $n = 1$, the left side of the given equation is 1. The right side is $2^1 - 1$ or 1, so the equation is true for $n = 1$.

Step 2: Assume $1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1$ for some positive integer k.

Step 3: $1 + 2 + 4 + \ldots + 2^{k-1} + 2^{(k+1)-1}$

$= 2^k - 1 + 2^{(k+1)-1}$

$= 2^k - 1 + 2^k$

$= 2 \cdot 2^k - 1$

$= 2^{k+1} - 1$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

Therefore, $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$ for all positive integers n.

50. Step 1: $6^1 - 1 = 5$, which is divisible by 5. The statement is true for $n = 1$.

Step 2: Assume that $6^k - 1$ is divisible by 5 for some positive integer k. This means that $6^k - 1 = 5r$ for some whole number r.

Step 3:

$$\begin{aligned} 6^k - 1 &= 5r \\ 6^k &= 5r + 1 \\ 6(6^k) &= 6(5r + 1) \\ 6^{k+1} &= 30r + 6 \\ 6^{k+1} - 1 &= 30r + 5 \\ 6^{k+1} - 1 &= 5(6r + 1) \end{aligned}$$

Since r is a whole number, $6r + 1$ is a whole number. Thus, $6^{k+1} - 1$ is divisible by 5, so the statement is true for $n = k + 1$.

Therefore, $6^n - 1$ is divisible by 5 for all positive integers n.

Chapter 11 Practice Test

Vocabulary and Concepts

Choose the correct term to complete each sentence.

1. A sequence in which each term after the first is found by adding a constant to the previous term is called a(n) (*arithmetic*, *geometric*) sequence.
2. A (*Fibonacci sequence*, *series*) is a sum of terms of a sequence.
3. (*Pascal's triangle*, *Recursive formulas*) and the Binomial Theorem can be used to expand powers of binomials.

Skills and Applications

4. Find the next four terms of the arithmetic sequence 42, 37, 32, … . **27, 22, 17, 12**
5. Find the 27th term of an arithmetic sequence for which $a_1 = 2$ and $d = 6$. **158**
6. Find the three arithmetic means between −4 and 16. **1, 6, 11**
7. Find the sum of the arithmetic series for which $a_1 = 7$, $n = 31$, and $a_n = 127$. **2077**
8. Find the next two terms of the geometric sequence $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots$. **$\frac{1}{3}$, 1**
9. Find the sixth term of the geometric sequence for which $a_1 = 5$ and $r = -2$. **−160**
10. Find the two geometric means between 7 and 189. **21, 63**
11. Find the sum of the geometric series for which $a_1 = 125$, $r = \frac{2}{5}$, and $n = 4$. **203**

Find the sum of each series, if it exists. **13. does not exist**

12. $\sum_{k=3}^{15}(14 - 2k)$ **−52** 13. $\sum_{n=1}^{\infty}\frac{1}{3}(-2)^{n-1}$ 14. $91 + 85 + 79 + \ldots + (-29)$ **651** 15. $12 + (-6) + 3 - \frac{3}{2} + \ldots$ **8**

Find the first five terms of each sequence.

16. $a_1 = 1, a_{n+1} = a_n + 3$ **1, 4, 7, 10, 13**
17. $a_1 = -3, a_{n+1} = a_n + n^2$ **−3, −2, 2, 11, 27**

18. Find the first three iterates of $f(x) = x^2 - 3x$ for an initial value of $x_0 = 1$. **−2, 10, 70**
19. Expand $(2s - 3t)^5$. **$32s^5 - 240s^4t + 720s^3t^2 - 1080s^2t^3 + 810st^4 - 243t^5$**
20. Find the third term of the expansion of $(x + y)^{10}$. **$45x^8y^2$**

Prove that each statement is true for all positive integers. **21–22. See pp. 629A–629F.**

21. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
22. $14^n - 1$ is divisible by 13.

23. **DESIGN** A landscaper is designing a wall of white brick and red brick. The pattern starts with 20 red bricks on the bottom row. Each row above it contains 3 fewer red bricks than the preceding row. If the top row contains no red bricks, how many rows are there and how many red bricks were used? **8 rows, 77 bricks**
24. **RECREATION** One minute after it is released, a gas-filled balloon has risen 100 feet. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will the balloon rise in 5 minutes? **193.75 ft**
25. **STANDARDIZED TEST PRACTICE** Find the next term in the geometric sequence $8, 6, \frac{9}{2}, \frac{27}{8}, \ldots$. **D**

Ⓐ $\frac{11}{8}$ Ⓑ $\frac{27}{16}$ Ⓒ $\frac{9}{4}$ Ⓓ $\frac{81}{32}$

www.algebra2.com/chapter_test

Chapter 11 Practice Test

Assessment Options

Vocabulary Test A vocabulary test/review for Chapter 11 can be found on p. 692 of the *Chapter 11 Resource Masters*.

Chapter Tests There are six Chapter 11 Tests and an Open-Ended Assessment task available in the *Chapter 11 Resource Masters*.

Chapter 11 Tests			
Form	**Type**	**Level**	**Pages**
1	MC	basic	679–680
2A	MC	average	681–682
2B	MC	average	683–684
2C	FR	average	685–686
2D	FR	average	687–688
3	FR	advanced	689–690

MC = multiple-choice questions
FR = free-response questions

Open-Ended Assessment Performance tasks for Chapter 11 can be found on p. 691 of the *Chapter 11 Resource Masters*. A sample scoring rubric for these tasks appears on p. A31.

TestCheck and Worksheet Builder

This **networkable software** has three modules for assessment.

- **Worksheet Builder** to make worksheets and tests.
- **Student Module** to take tests on-screen.
- **Management System** to keep student records.

Portfolio Suggestion

Introduction Throughout this course, you have been working in groups to solve problems.

Ask Students What roles do you play in the group?

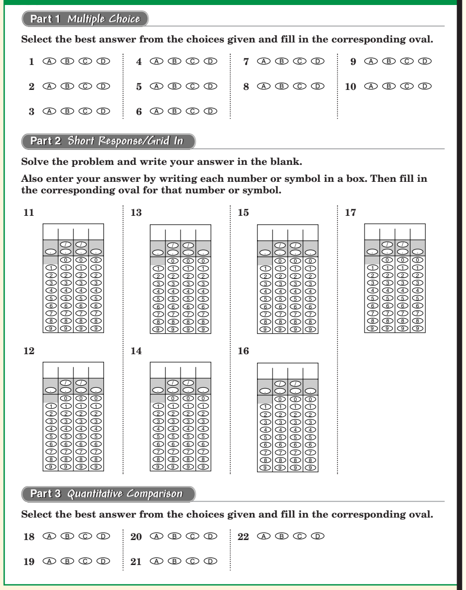
- Do you help to keep your group on task? ask questions? just listen and copy down answers?
- List some ways you are a good group member and some ways you could do better.

Place your responses in your portfolio.

These two pages contain practice questions in the various formats that can be found on the most frequently given standardized tests.

A practice answer sheet for these two pages can be found on p. A1 of the *Chapter 11 Resource Masters*.

Standardized Test Practice
Student Recording Sheet, p. A1



Part 1 *Multiple Choice*

Select the best answer from the choices given and fill in the corresponding oval.

1 Ⓐ Ⓑ Ⓒ Ⓓ
2 Ⓐ Ⓑ Ⓒ Ⓓ
3 Ⓐ Ⓑ Ⓒ Ⓓ
4 Ⓐ Ⓑ Ⓒ Ⓓ
5 Ⓐ Ⓑ Ⓒ Ⓓ
6 Ⓐ Ⓑ Ⓒ Ⓓ
7 Ⓐ Ⓑ Ⓒ Ⓓ
8 Ⓐ Ⓑ Ⓒ Ⓓ
9 Ⓐ Ⓑ Ⓒ Ⓓ
10 Ⓐ Ⓑ Ⓒ Ⓓ

Part 2 *Short Response/Grid In*

Solve the problem and write your answer in the blank.

Also enter your answer by writing each number or symbol in a box. Then fill in the corresponding oval for that number or symbol.

11 13 15 17 12 14 16

Part 3 *Quantitative Comparison*

Select the best answer from the choices given and fill in the corresponding oval.

18 Ⓐ Ⓑ Ⓒ Ⓓ
19 Ⓐ Ⓑ Ⓒ Ⓓ
20 Ⓐ Ⓑ Ⓒ Ⓓ
21 Ⓐ Ⓑ Ⓒ Ⓓ
22 Ⓐ Ⓑ Ⓒ Ⓓ

Additional Practice

See pp. 697–698 in the *Chapter 11 Resource Masters* for additional standardized test practice.

Chapter 11 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. For all positive integers, let $\boxed{n} = n + g$, where g is the greatest factor of n, and $g < n$. If $\boxed{18} = x$, then $\boxed{x} =$ **C**

Ⓐ 9. Ⓑ 8.
Ⓒ 27. Ⓓ 36.

2. If p is positive, what percent of $6p$ is 12? **D**

Ⓐ $\frac{p}{100}\%$ Ⓑ $\frac{p}{2}\%$
Ⓒ $\frac{12}{p}\%$ Ⓓ $\frac{200}{p}\%$

3. A box is 12 units tall, 6 units long, and 8 units wide. A designer is creating a new box that must have the same volume as the first box. If the length and width of the new box are each 50% greater than the length and width of the first box, about how many units tall will the new box be? **A**

Ⓐ 5.3 Ⓑ 6.8
Ⓒ 7.1 Ⓓ 8.5

4. Which of the following statements must be true when $0 < m < 1$? **A**

I $\frac{\sqrt{m}}{m} > 1$ **II** $4m < 1$ **III** $m^2 - m^3 < 0$

Ⓐ I only
Ⓑ III only
Ⓒ I and II only
Ⓓ I, II, and III

5. If $3kx - \frac{4s}{t} = 3ky$, then $x - y =$? **D**

Ⓐ $-\frac{4s}{3kt}$ Ⓑ $\frac{-4s}{t} + \frac{1}{3k}$
Ⓒ $\frac{4s}{3t} - k$ Ⓓ $\frac{4s}{3kt}$

6. For all $n \neq 0$, what is the slope of the line passing through $(3n, -k)$ and $(-n, -k)$? **A**

Ⓐ 0 Ⓑ $\frac{k}{2n}$
Ⓒ $\frac{2n}{k}$ Ⓓ undefined

7. Which is the graph of the equation $x^2 + (y - 4)^2 = 20$? **C**

Ⓐ line Ⓑ parabola
Ⓒ circle Ⓓ ellipse

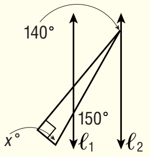
8. $\dfrac{x - \frac{9}{x}}{1 - \frac{6}{x} + \frac{9}{x^2}} =$ **C**

Ⓐ $\frac{x}{x - 3}$ Ⓑ $\frac{x^2 + 3}{x - 3}$
Ⓒ $\frac{x^2 + 3x}{x - 3}$ Ⓓ $\frac{x^2 + 3x}{(x - 3)^2}$

9. What is the sum of the positive even factors of 30? **C**

Ⓐ 18 Ⓑ 30
Ⓒ 48 Ⓓ 72

10. If ℓ_1 is parallel to ℓ_2 in the figure, what is the value of x? **D**



Ⓐ 30
Ⓑ 40
Ⓒ 70
Ⓓ 80

The Princeton Review

Test-Taking Tip

Question 5 Some questions ask you to find the value of an expression. It is often not necessary to find the value of each variable in the expression. For example, to answer Question 5, it is not necessary to find the values of x and y. Isolate the expression $x - y$ on one side of the equation.

The Princeton Review

Log On **for Test Practice**

The Princeton Review offers additional test-taking tips and practice problems at their web site. Visit www.princetonreview.com or www.review.com

TestCheck and Worksheet Builder

Special banks of standardized test questions similar to those on the SAT, ACT, TIMSS 8, NAEP 8, and Algebra 1 End-of-Course tests can be found on this CD-ROM.

Aligned and verified by The Princeton Review

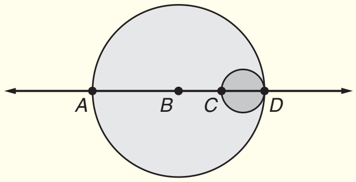
Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. $$\begin{array}{r} AA \\ +\ BB \\ \hline CC \end{array}$$

If A, B, and C are each digits and $A = 3B$, then what is one possible value of C? **4 or 8**

12. In the figure, each arc is a semicircle. If B is the midpoint of $\overline{AD}$ and C is the midpoint of $\overline{BD}$, what is the ratio of the area of the semicircle $\overset{\frown}{CD}$ to the area of the semicircle $\overset{\frown}{AD}$? **1/16**



13. Two people are 17.5 miles apart. They begin to walk toward each other along a straight line at the same time. One walks at the rate of 4 miles per hour, and the other walks at the rate of 3 miles per hour. In how many hours will they meet? **2.5 or 5/2**

14. If $\frac{x+y}{x} = \frac{5}{4}$, then $\frac{y}{x} =$ **1/4 or .25**

15. A car's gasoline tank is $\frac{1}{2}$ full. After adding 7 gallons of gas, the gauge shows that the tank is $\frac{3}{4}$ full. How many gallons does the tank hold? **28**

16. If $a = 15 - b$, what is the value of $3a + 3b$? **45**

17. If $x^9 = \frac{45}{y}$ and $x^7 = \frac{1}{5y}$, and $x > 0$, what is the value of x? **15**

www.algebra2.com/standardized_test

Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- Ⓐ **the quantity in Column A is greater,**
- Ⓑ **the quantity in Column B is greater,**
- Ⓒ **the two quantities are equal, or**
- Ⓓ **the relationship cannot be determined from the information given.**

	Column A	Column B
18.	the arithmetic mean of three consecutive integers where x is the median	the arithmetic mean of five consecutive integers where x is the median

C

19. The area of Square B is equal to nine times the area of Square A. **C**

three times the perimeter of Square A	the perimeter of Square B

20. $\boxed{n} = n(n + 1)$ if n is even

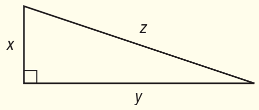
$\boxed{n} = n(n - 1)$ if n is odd **C**

$\boxed{8}$	$\boxed{9}$

21.	$(1 - \sqrt{3})(1 - \sqrt{3})$	$(1 - \sqrt{3})(1 + \sqrt{3})$

A

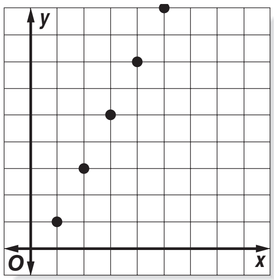
22. **B**



$\frac{x+y}{2}$	$\frac{x+z}{2}$

Page 577, Chapter 11 Getting Started

7.



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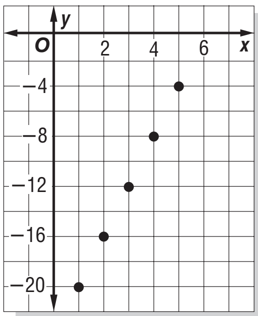

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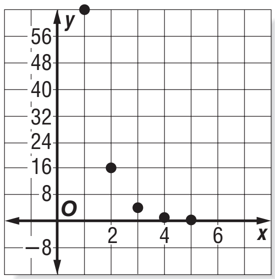

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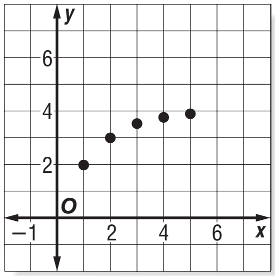
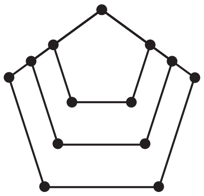
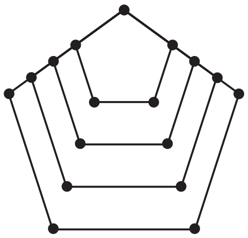
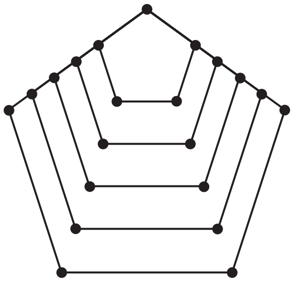


Page 582, Lesson 11-1

49.







Page 587, Lesson 11-2

48. Arithmetic series can be used to find the seating capacity of an amphitheater. Answers should include the following.

- The sequence represents the numbers of seats in the rows. The sum of the first n terms of the series is the seating capacity of the first n rows.
- One method is to write out the terms and add them: $18 + 22 + 26 + 30 + 34 + 38 + 42 + 46 + 50 + 54 = 360$. Another method is to use the formula $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$:

  $S_{10} = \frac{10}{2}[2(18) + (10 - 1)4]$ or 360.

Page 596, Lesson 11-4

3. Sample answer: The first term is $a_1 = 2$. Divide the second term by the first to find that the common ratio is $r = 6$. Therefore, the nth term of the series is given by $2 \cdot 6^{n-1}$. There are five terms, so the series can be written as $\sum_{n=1}^{5} 2 \cdot 6^{n-1}$.

Page 603, Lesson 11-5

48.

$$\begin{array}{rrl} & S &= a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots \\ (-) & rS &= \phantom{a_1 + {}} a_1r + a_1r^2 + a_1r^3 + a_1r^4 + \ldots \\ \hline & S - rS &= a_1 + \ 0 \ + \ 0 \ + \ 0 \ + \ 0 \ + \ldots \\ & S(1 - r) &= a_1 \\ & S &= \frac{a_1}{1 - r} \end{array}$$

49. The total distance that a ball bounces, both up and down, can be found by adding the sums of two infinite geometric series. Answers should include the following.

- $a_n = a_1 \cdot r^{n-1}$, $S_n = \frac{a_1(1 - r^n)}{1 - r}$, or $S = \frac{a_1}{1 - r}$
- The total distance the ball falls is given by the infinite geometric series $3 + 3(0.6) + 3(0.6)^2 + \ldots$. The sum of this series is $\frac{3}{1 - 0.6}$ or 7.5. The total distance the ball bounces up is given by the infinite geometric series $1.8(0.6) + 1.8(0.6)^2 + 1.8(0.6)^3 + \ldots$. The sum of this series is $\frac{1.8(0.6)}{1 - 0.6}$ or 2.7. Thus, the total distance the ball travels is 7.5 + 2.7 or 10.2 feet.

Page 611, Follow-Up of Lesson 11-6
Algebra Activity

4. The von Koch snowflake has infinite perimeter. As n increases, the perimeter P_n of Stage n increases without bound. That is, the limit of $27\left(\frac{4}{3}\right)^{n-1}$ is ∞.

5. Stage 1 is an equilateral triangle with sides of length 9 units, so its area is $\frac{81\sqrt{3}}{4}$ units². Each subsequent stage encloses $3 \cdot 4^{n-2}$ additional equilateral triangular regions of area $\frac{81\sqrt{3}}{4 \cdot 3^{2n-2}}$ units². Thus, the additional area at each stage is $3 \cdot 4^{n-2} \cdot \frac{81\sqrt{3}}{4 \cdot 3^{2n-2}}$ or $\frac{4^{n-3}\sqrt{3}}{3^{2n-7}}$ units². This is the general term of the series for $n \geq 2$.

6. Beginning with the second term, the terms of the series in Exercise 5 form an infinite geometric series with common ratio $\frac{4}{9}$. Therefore, the sum of the whole series in Exercise 5 is $\frac{81\sqrt{3}}{4} + \frac{\frac{27\sqrt{3}}{4}}{1 - \frac{4}{9}}$ or $\frac{162\sqrt{3}}{5}$. The area of the von Koch snowflake is $\frac{162\sqrt{3}}{5}$ units².

7. Sample answer: No, they show that it is possible for a figure with infinite perimeter to enclose only a finite amount of area.

Page 616, Lesson 11-7

42. $\frac{12!}{7!5!}$ and $\frac{12!}{6!6!}$ represent the sixth and seventh entries in the row for $n = 12$ in Pascal's triangle. $\frac{13!}{7!6!}$ represents the seventh entry in the row for $n = 13$.

Since $\frac{13!}{7!6!}$ is below $\frac{12!}{7!5!}$ and $\frac{12!}{6!6!}$ in Pascal's triangle, $\frac{12!}{7!5!} + \frac{12!}{6!6!} = \frac{13!}{7!6!}$.

43. The coefficients in a binomial expansion give the numbers of sequences of births resulting in given numbers of boys and girls. Answers should include the following.

- $(b + g)^5 = b^5 + 5b^4g + 10b^3g^2 + 10b^2g^3 + 5bg^4 + g^5$; There is one sequence of births with all five boys, five sequences with four boys and one girl, ten sequences with three boys and two girls, ten sequences with two boys and three girls, five sequences with one boy and four girls, and one sequence with all five girls.
- The number of sequences of births that have exactly k girls in a family of n children is the coefficient of $b^{n-k}g^k$ in the expansion of $(b + g)^n$. According to the Binomial Theorem, this coefficient is $\frac{n!}{(n-k)!k!}$.

Pages 619–621, Lesson 11-8

1. Sample answers: formulas for the sums of powers of the first n positive integers and statements that expressions involving exponents of n are divisible by certain numbers

2. Mathematical induction is used to show that a statement is true. A counterexample is used to show that a statement is false.

4. **Step 1:** When $n = 1$, the left side of the given equation is 1. The right side is $\frac{1(1+1)}{2}$ or 1, so the equation is true for $n = 1$.

Step 2: Assume $1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$ for some positive integer k.

Step 3: $1 + 2 + 3 + \dots + k + (k + 1)$

$$\begin{aligned} &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

Therefore, $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ for all positive integers n.

5. **Step 1:** When $n = 1$, the left side of the given equation is $\frac{1}{2}$. The right side is $1 - \frac{1}{2}$ or $\frac{1}{2}$, so the equation is true for $n = 1$.

Step 2: Assume $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$ for some positive integer k.

Step 3:

$$\begin{aligned} \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^{k+1}} \end{aligned}$$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

Therefore, $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ for all positive integers n.

6. **Step 1:** $4^1 - 1 = 3$, which is divisible by 3. The statement is true for $n = 1$.

Step 2: Assume that $4^k - 1$ is divisible by 3 for some positive integer k. This means that $4^k - 1 = 3r$ for some whole number r.

Step 3:

$$\begin{aligned} 4^k - 1 &= 3r \\ 4^k &= 3r + 1 \\ 4^{k+1} &= 12r + 4 \\ 4^{k+1} - 1 &= 12r + 3 \\ 4^{k+1} - 1 &= 3(4r + 1) \end{aligned}$$

Since r is a whole number, $4r + 1$ is a whole number. Thus, $4^{k+1} - 1$ is divisible by 3, so the statement is true for $n = k + 1$. Therefore, $4^n - 1$ is divisible by 3 for all positive integers n.

7. **Step 1:** $5^1 + 3 = 8$, which is divisible by 4. The statement is true for $n = 1$.

Step 2: Assume that $5^k + 3$ is divisible by 4 for some positive integer k. This means that $5k + 3 = 4r$ for some positive integer r.

Step 3:

$$\begin{aligned} 5^k + 3 &= 4r \\ 5^k &= 4r - 3 \\ 5^{k+1} &= 20r - 15 \\ 5^{k+1} + 3 &= 20r - 12 \\ 5^{k+1} + 3 &= 4(5r - 3) \end{aligned}$$

Since r is a positive integer, $5r - 3$ is a positive integer. Thus, $5^{k+1} + 3$ is divisible by 4, so the statement is true for $n = k + 1$.

Therefore, $5^n + 3$ is divisible by 4 for all positive integers n.

10. **Step 1:** After the first guest has arrived, no handshakes have taken place. $\frac{1(1-1)}{2} = 0$, so the formula is correct for $n = 1$.

Step 2: Assume that after k guests have arrived, a total of $\frac{k(k-1)}{2}$ handshakes have take place, for some positive integer k.

Step 3: When the $(k + 1)$st guest arrives, he or she shakes hands with the k guests already there, so the total number of handshakes that have then taken place is $\frac{k(k-1)}{2} + k$.

$$\begin{aligned} \frac{k(k-1)}{2} + k &= \frac{k(k-1) + 2k}{2} \\ &= \frac{k[(k-1) + 2]}{2} \\ &= \frac{k(k+1)}{2} \text{ or } \frac{(k+1)k}{2} \end{aligned}$$

The last expression is the formula to be proved, where $n = k + 1$. Thus, the formula is true for $n = k + 1$.

Therefore, the total number of handshakes is $\frac{n(n-1)}{2}$ for all positive integers n.

11. **Step 1:** When $n = 1$, the left side of the given equation is 1. The right side is $1[2(1) - 1]$ or 1, so the equation is true for $n = 1$.

    Step 2: Assume $1 + 5 + 9 + \ldots + (4k - 3) = k(2k - 1)$ for some positive integer k.

    Step 3: $1 + 5 + 9 + \ldots + (4k - 3) + [4(k + 1) - 3]$

    $$\begin{aligned}&= k(2k - 1) + [4(k + 1) - 3]\\&= 2k^2 - k + 4k + 4 - 3\\&= 2k^2 + 3k + 1\\&= (k + 1)(2k + 1)\\&= (k + 1)[2(k + 1) - 1]\end{aligned}$$

    The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

    Therefore, $1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$ for all positive integers n.

12. **Step 1:** When $n = 1$, the left side of the given equation is 2. The right side is $\frac{1[3(1) + 1]}{2}$ or 2, so the equation is true for $n = 1$.

    Step 2: Assume $2 + 5 + 8 + \ldots + (3k - 1) = \frac{k(3k + 1)}{2}$ for some positive integer k.

    Step 3: $2 + 5 + 8 + \ldots + (3k - 1) + [3(k + 1) - 1]$

    $$\begin{aligned}&= \frac{k(3k + 1)}{2} + [3(k + 1) - 1]\\&= \frac{k(3k + 1) + 2[3(k + 1) - 1]}{2}\\&= \frac{3k^2 + k + 6k + 6 - 2}{2}\\&= \frac{3k^2 + 7k + 4}{2}\\&= \frac{(k + 1)(3k + 4)}{2}\\&= \frac{(k + 1)[(3(k + 1) + 1]}{2}\end{aligned}$$

    The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

    Therefore, $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n + 1)}{2}$ for all positive integers n.

13. **Step 1:** When $n = 1$, the left side of the given equation is 1^3 or 1. The right side is $\frac{1^2(1 + 1)^2}{4}$ or 1, so the equation is true for $n = 1$.

    Step 2: Assume $1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k + 1)^2}{4}$ for some positive integer k.

    Step 3: $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3$

    $$\begin{aligned}&= \frac{k^2(k + 1)^2}{4} + (k + 1)^3\\&= \frac{k^2(k + 1)^2 + 4(k + 1)^3}{4}\\&= \frac{(k + 1)^2[k^2 + 4(k + 1)]}{4}\\&= \frac{(k + 1)^2(k^2 + 4k + 4)}{4}\\&= \frac{(k + 1)^2(k + 2)^2}{4}\\&= \frac{(k + 1)^2[(k + 1) + 1]^2}{4}\end{aligned}$$

    The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

    Therefore, $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}$ for all positive integers n.

14. **Step 1:** When $n = 1$, the left side of the given equation is 1^2 or 1. The right side is $\frac{1[2(1) - 1][2(1) + 1]}{3}$ or 1, so the equation is true for $n = 1$.

    Step 2: Assume $1^2 + 3^2 + 5^2 + \ldots + (2k - 1)^2 = \frac{k(2k - 1)(2k + 1)}{3}$ for some positive integer k.

    Step 3: $1^2 + 3^2 + 5^2 + \ldots + (2k - 1)^2 + [2(k + 1) - 1]^2$

    $$\begin{aligned}&= \frac{k(2k - 1)(2k + 1)}{3} + [2(k + 1) - 1]^2\\&= \frac{k(2k - 1)(2k + 1) + 3(2k + 1)^2}{3}\\&= \frac{(2k + 1)[k(2k - 1) + 3(2k + 1)]}{3}\\&= \frac{(2k + 1)(2k^2 - k + 6k + 3)}{3}\\&= \frac{(2k + 1)(2k^2 + 5k + 3)}{3}\\&= \frac{(2k + 1)(k + 1)(2k + 3)}{3}\\&= \frac{(k + 1)[2(k + 1) - 1][2(k + 1) + 1]}{3}\end{aligned}$$

    The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

    Therefore, $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}$ for all positive integers n.

15. **Step 1:** When $n = 1$, the left side of the given equation is $\frac{1}{3}$. The right side is $\frac{1}{2}\left(1 - \frac{1}{3}\right)$ or $\frac{1}{3}$, so the equation is true for $n = 1$.

    Step 2: Assume $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^k} = \frac{1}{2}\left(1 - \frac{1}{3^k}\right)$ for some positive integer k.

Step 3: $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^k} + \frac{1}{3^{k+1}}$

$$\begin{aligned} &= \frac{1}{2}\left(1 - \frac{1}{3^k}\right) + \frac{1}{3^{k+1}} \\ &= \frac{1}{2} - \frac{1}{2 \cdot 3^k} + \frac{1}{3^{k+1}} \\ &= \frac{3^{k+1} - 3 + 2}{2 \cdot 3^{k+1}} \\ &= \frac{3^{k+1} - 1}{2 \cdot 3^{k+1}} \\ &= \frac{1}{2}\left(\frac{3^{k+1} - 1}{3^{k+1}}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{3^{k+1}}\right) \end{aligned}$$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

Therefore, $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$ for all positive integers n.

16. **Step 1:** When $n = 1$, the left side of the given equation is $\frac{1}{4}$. The right side is $\frac{1}{3}\left(1 - \frac{1}{4}\right)$ or $\frac{1}{4}$, so the equation is true for $n = 1$.

Step 2: Assume $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^k} = \frac{1}{3}\left(1 - \frac{1}{4^k}\right)$ for some positive integer k.

Step 3: $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^k} + \frac{1}{4^{k+1}}$

$$\begin{aligned} &= \frac{1}{3}\left(1 - \frac{1}{4^k}\right) + \frac{1}{4^{k+1}} \\ &= \frac{1}{3} - \frac{1}{3 \cdot 4^k} + \frac{1}{4^{k+1}} \\ &= \frac{4^{k+1} - 4 + 3}{3 \cdot 4^{k+1}} \\ &= \frac{4^{k+1} - 1}{3 \cdot 4^{k+1}} \\ &= \frac{1}{3}\left(\frac{4^{k+1} - 1}{4^{k+1}}\right) \\ &= \frac{1}{3}\left(1 - \frac{1}{4^{k+1}}\right) \end{aligned}$$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

Therefore, $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^n} = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$ for all positive integers n.

17. **Step 1:** $8^1 - 1 = 7$, which is divisible by 7. The statement is true for $n = 1$.

Step 2: Assume that $8^k - 1$ is divisible by 7 for some positive integer k. This means that $8^k - 1 = 7r$ for some whole number r.

Step 3: $8^k - 1 = 7r$

$$\begin{aligned} 8^k &= 7r + 1 \\ 8^{k+1} &= 56r + 8 \\ 8^{k+1} - 1 &= 56r + 7 \\ 8^{k+1} - 1 &= 7(8r + 1) \end{aligned}$$

Since r is a whole number, $8r + 1$ is a whole number. Thus, $8^{k+1} - 1$ is divisible by 7, so the statement is true for $n = k + 1$.

Therefore, $8^n - 1$ is divisible by 7 for all positive integers n.

18. **Step 1:** $9^1 - 1 = 8$, which is divisible by 8. The statement is true for $n = 1$.

Step 2: Assume that $9^k - 1$ is divisible by 8 for some positive integer k. This means that $9^k - 1 = 8r$ for some whole number r.

Step 3: $9^k - 1 = 8r$

$$\begin{aligned} 9^k &= 8r + 1 \\ 9^{k+1} &= 72r + 9 \\ 9^{k+1} - 1 &= 72r + 8 \\ 9^{k+1} - 1 &= 8(9r + 1) \end{aligned}$$

Since r is a whole number, $9r + 1$ is a whole number. Thus, $9^{k+1} - 1$ is divisible by 8, so the statement is true for $n = k + 1$.

Therefore, $9^n - 1$ is divisible by 8 for all positive integers n.

19. **Step 1:** $12^1 + 10 = 22$, which is divisible by 11. The statement is true for $n = 1$.

Step 2: Assume that $12^k + 10$ is divisible by 11 for some positive integer k. This means that $12^k + 10 = 11r$ for some positive integer r.

Step 3: $12^k + 10 = 11r$

$$\begin{aligned} 12^k &= 11r - 10 \\ 12^{k+1} &= 132r - 120 \\ 12^{k+1} + 10 &= 132r - 110 \\ 12^{k+1} + 10 &= 11(12r - 10) \end{aligned}$$

Since r is a positive integer, $12r - 10$ is a positive integer. Thus, $12^{k+1} + 10$ is divisible by 11, so the statement is true for $n = k + 1$.

Therefore, $12^n + 10$ is divisible by 11 for all positive integers n.

20. **Step 1:** $13^1 + 11 = 24$, which is divisible by 12. The statement is true for $n = 1$.

Step 2: Assume that $13^k + 11$ is divisible by 12 for some positive integer k. This means that $13^k + 11 = 12r$ for some positive integer r.

Step 3: $13^k + 11 = 12r$

$$\begin{aligned} 13^k &= 12r - 11 \\ 13^{k+1} &= 156r - 143 \\ 13^{k+1} + 11 &= 156r - 132 \\ 13^{k+1} + 11 &= 12(13r - 11) \end{aligned}$$

Since r is a positive integer, $13r - 11$ is a positive integer. Thus, $13^{k+1} + 11$ is divisible by 12, so the statement is true for $n = k + 1$.

Therefore, $13^n + 11$ is divisible by 12 for all positive integers n.

21. **Step 1:** There are 6 bricks in the top row, and $1^2 + 5(1) = 6$, so the formula is true for $n = 1$.

Step 2: Assume that there are $k^2 + 5k$ bricks in the top k rows for some positive integer k.

Step 3: Since each row has 2 more bricks than the one above, the numbers of bricks in the rows form an arithmetic sequence. The number of bricks in the $(k + 1)$st row is $6 + [(k + 1) - 1](2)$ or $2k + 6$. Then the number of bricks in the top $k + 1$ rows is $k^2 + 5k + (2k + 6)$ or $k^2 + 7k + 6$.

$k^2 + 7k + 6 = (k + 1)^2 + 5(k + 1)$, which is the formula to be proved, where $n = k + 1$. Thus, the formula is true for $n = k + 1$.

Therefore, the number of bricks in the top n rows is $n^2 + 5n$ for all positive integers n.

22. **Step 1:** When $n = 1$, the left side of the given equation is a_1. The right side is $\frac{a_1(1 - r^1)}{1 - r}$ or a_1, so the equation is true for $n = 1$.

Step 2: Assume $a_1 + a_1r + a_1r^2 + \ldots + a_1r^{k-1} = \frac{a_1(1 - r^k)}{1 - r}$ for some positive integer k.

Step 3: $a_1 + a_1r + a_1r^2 + \ldots + a_1r^{k-1} + a_1r^k$

$$= \frac{a_1(1 - r^k)}{1 - r} + a_1r^k$$
$$= \frac{a_1(1 - r^k) + (1 - r)a_1r^k}{1 - r}$$
$$= \frac{a_1 - a_1r^k + a_1r^k - a_1r^{k+1}}{1 - r}$$
$$= \frac{a_1(1 - r^{k+1})}{1 - r}$$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

Therefore, $a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1} = \frac{a_1(1 - r^n)}{1 - r}$ for all positive integers n.

23. **Step 1:** When $n = 1$, the left side of the given equation is a_1. The right side is $\frac{1}{2}[2a_1 + (1 - 1)d]$ or a_1, so the equation is true for $n = 1$.

Step 2: Assume $a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + [a_1 + (k - 1)d] = \frac{k}{2}[2a_1 + (k - 1)d]$ for some positive integer k.

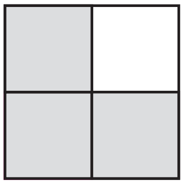
Step 3: $a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + [a_1 + (k - 1)d] + [a_1 + (k + 1 - 1)d]$

$$= \frac{k}{2}[2a_1 + (k - 1)d] + [a_1 + (k + 1 - 1)d]$$
$$= \frac{k}{2}[2a_1 + (k - 1)d] + a_1 + kd$$
$$= \frac{k[2a_1 + (k - 1)d] + 2(a_1 + kd)}{2}$$
$$= \frac{k \cdot 2a_1 + (k^2 - k)d + 2a_1 + 2kd}{2}$$
$$= \frac{(k + 1)2a_1 + (k^2 - k + 2k)d}{2}$$
$$= \frac{(k + 1)2a_1 + k(k + 1)d}{2}$$
$$= \frac{k + 1}{2}(2a_1 + kd)$$
$$= \frac{k + 1}{2}[2a_1 + (k + 1 - 1)d]$$

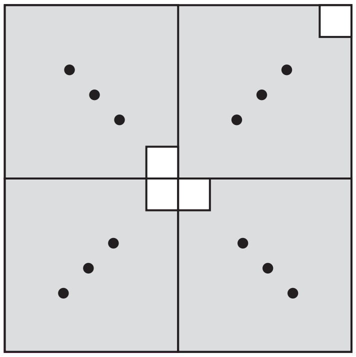
The last expression is the right side of the formula to be proved, where $n = k + 1$. Thus, the formula is true for $n = k + 1$.

Therefore, $a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + [a_1 + (n - 1)d] = \frac{n}{2}[2a_1 + (n - 1)d]$ for all positive integers n.

24. **Step 1:** The figure below shows how to cover a 2^1 by 2^1 board, so the statement is true for $n = 1$.



Step 2: Assume that a 2^k by 2^k board can be covered for some positive integer k.



Step 3: Divide a 2^{k+1} by 2^{k+1} board into four quadrants. By the inductive hypothesis, the first quadrant can be covered. Rotate the design that covers Quadrant I 90° clockwise and use it to cover Quadrant II. Use the design that covers Quadrant I to cover Quadrant III. Rotate the design that covers Quadrant I 90° counterclockwise and use it to cover Quadrant IV. This leaves three empty squares near the center of the board, as shown. Use one more L-shaped tile to cover these 3 squares. Thus, a 2^{k+1} by 2^{k+1} board can be covered. The statement is true for $n = k + 1$.

Therefore, a 2^n by 2^n checkerboard with the top right square missing can be covered for all positive integers n.

32. An analogy can be made between mathematical induction and a ladder with the positive integers on the steps. Answers should include the following.

- Showing that the statement is true for $n = 1$ (Step 1).
- Assuming that the statement is true for some positive integer k and showing that it is true for $k + 1$ (Steps 2 and 3).

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21. **Step 1:** When $n = 1$, the left side of the given equation is 1. The right side is 1^2 or 1, so the equation is true for $n = 1$.

Step 2: Assume $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ for some positive integer k.

Step 3: $1 + 3 + 5 + \ldots + (2k - 1) + [2(k + 1) - 1]$

$$= k^2 + [2(k + 1) - 1]$$
$$= k^2 + 2k + 2 - 1$$
$$= k^2 + 2k + 1$$
$$= (k + 1)^2$$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$.

Therefore, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for all positive integers n.

22. **Step 1:** $14^1 - 1 = 13$, which is divisible by 13. The statement is true for $n = 1$.

Step 2: Assume that $14^k - 1$ is divisible by 13 for some positive integer k. This means that $14^k - 1 = 13r$ for some whole number r.

Step 3:
$$\begin{aligned} 14^k - 1 &= 13r \\ 14^k &= 13r + 1 \\ 14^{k+1} &= 182r + 14 \\ 14^{k+1} - 1 &= 182r + 13 \\ 14^{k+1} - 1 &= 13(14r + 1) \end{aligned}$$

Since r is a whole number, $14r + 1$ is a whole number. Thus, $14^{k+1} - 1$ is divisible by 13, so the statement is true for $n = k + 1$.

Therefore, $14^n - 1$ is divisible by 13 for all positive integers n.