UNIT 5 Notes

Introduction

In this unit, students investigate the six trigonometric functions, both as ratios in right triangles and as circular functions, which they graph. The Law of Sines and the Law of Cosines are used to solve problems, as are the inverse trigonometric functions.

The unit concludes with lessons in which students verify and use trigonometric identities, and solve trigonometric equations.

Assessment Options

Unit 5 Test Pages 899–900 of the *Chapter 14 Resource Masters* may be used as a test or review for Unit 5. This assessment contains both multiple-choice and short answer items.

TestCheck and Worksheet Builder

This CD-ROM can be used to create additional unit tests and review worksheets.

UNIT 5 Trigonometry

Trigonometry is used in navigation, physics, and construction, among other fields. In this unit, you will learn about trigonometric functions, graphs, and identities.

Chapter 13
Trigonometric Functions

Chapter 14
Trigonometric Graphs and Identities

WebQuest Internet Project

Trig Class Angles for Lessons in Lit

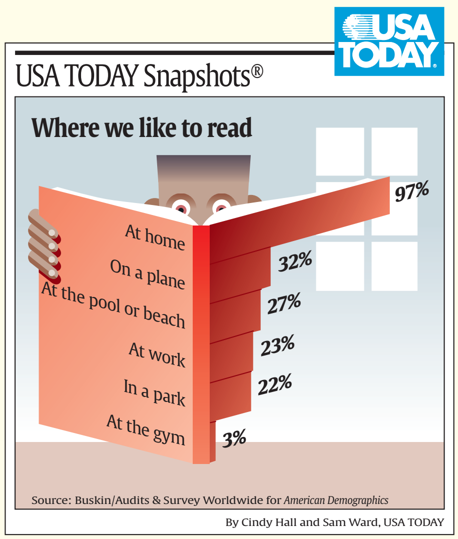
Source: *USA TODAY*, November 21, 2000

"The groans from the trigonometry students immediately told teacher Michael Buchanan what the class thought of his idea to read Homer Hickam's *October Sky*. In the story, in order to accomplish what they would like, the kids had to teach themselves trig, calculus, and physics." In this project, you will research applications of trigonometry as it applies to a possible career for you.

Log on to www.algebra2.com/webquest. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 5.

Lesson	13-1	14-2
Page	708	775



Teaching Suggestions

Have students study the USA TODAY Snapshot®.

- Ask students to discuss their own reading habits, and whether for recreational reading they prefer fictional or non-fictional material.
- Ask students to share any experience they have with teaching themselves a subject or skill.
- Point out to students that good reading skills have a high positive correlation with academic and vocational success.

Additional USA TODAY Snapshots® appearing in Unit 5:

Chapter 13 Sports radio extends coverage area (p. 715)

Chapter 14 Strongest quakes in 20th century (p. 797)

WebQuest Internet Project

A WebQuest is an online project in which students do research on the Internet, gather data, and make presentations using word processing, graphing, page-making, or presentation software. In each chapter, students advance to the next step in their WebQuest. At the end of Chapter 14, the project culminates with a presentation of their findings.

Teaching suggestions and sample answers are available in the *WebQuest and Project Resources.*

Chapter 13

Trigonometric Functions
Chapter Overview and Pacing

LESSON OBJECTIVES	PACING (days) Regular: Basic/Average	Regular: Advanced	Block: Basic/Average	Block: Advanced
13-1 Right Triangle Trigonometry *(pp. 700–708)* ***Preview:*** Special Right Triangles • Find values of trigonometric functions for acute angles. • Solve problems involving right triangles.	2 (with 13-1 Preview)	2 (with 13-1 Preview)	1	1
13-2 Angles and Angle Measure *(pp. 709–716)* • Change radian measure to degree measure and vice versa. • Identify coterminal angles. ***Follow-Up:*** Investigating Regular Polygons Using Trigonometry	1	1	0.5	0.5
13-3 Trigonometric Functions of General Angles *(pp. 717–724)* • Find values of trigonometric functions for general angles. • Use reference angles to find values of trigonometric functions.	2 (with 13-2 Follow-Up)	2 (with 13-2 Follow-Up)	1	1
13-4 Law of Sines *(pp. 725–732)* • Solve problems by using the Law of Sines. • Determine whether a triangle has one, two, or no solutions.	2	2	1	1
13-5 Law of Cosines *(pp. 733–738)* • Solve problems by using the Law of Cosines. • Determine whether a triangle can be solved by first using the Law of Sines or the Law of Cosines.	2	2	1	1
13-6 Circular Functions *(pp. 739–745)* • Define and use the trigonometric functions based on the unit circle. • Find the exact values of trigonometric functions of angles.	2	2	1	1
13-7 Inverse Trigonometric Functions *(pp. 746–751)* • Solve equations by using inverse trigonometric functions. • Find values of expressions involving trigonometric functions.	1	1	0.5	0.5
Study Guide and **Practice Test** *(pp. 752–757)* **Standardized Test Practice** *(pp. 758–759)*	1	1	0.5	0.5
Chapter Assessment	1	1	1	0.5
TOTAL	**14**	**14**	**7.5**	**7**

Pacing suggestions for the entire year can be found on pages T20–T21.

Timesaving Tools
TeacherWorks™
All-In-One Planner and Resource Center
See pages T12–T13.

Chapter Resource Manager

Chapter 13 Resource Masters: Study Guide and Intervention	Chapter 13 Resource Masters: Practice (Skills and Average)	Chapter 13 Resource Masters: Reading to Learn Mathematics	Chapter 13 Resource Masters: Enrichment	Chapter 13 Resource Masters: Assessment	Applications*	5-Minute Check Transparencies	Interactive Chalkboard	Alge2PASS: Tutorial Plus (lessons)	Materials
775–776	777–778	779	780		GCS 52, SM 37–40	13-1	13-1		(*Preview:* spreadsheet program)
781–782	783–784	785	786	831		13-2	13-2		yardsticks (*Follow-Up:* compass, ruler, protractor, calculator)
787–788	789–790	791	792			13-3	13-3	24, 25	
793–794	795–796	797	798	831, 833	GCS 51, SC 25	13-4	13-4		coffee stirrer sticks
799–800	801–802	803	804		SC 26	13-5	13-5	26	
805–806	807–808	809	810	832		13-6	13-6		graphing calculator, geoboard and rubber bands
811–812	813–814	815	816	832		13-7	13-7		
				817–830, 834–836					

**Key to Abbreviations:* GCS = Graphing Calculator and Speadsheet Masters,
SC = School-to-Career Masters,
SM = Science and Mathematics Lab Manual

Chapter 13

Mathematical Connections and Background

Continuity of Instruction

Prior Knowledge

Students have used the Pythagorean Theorem. They are familiar with applying formulas to solve problems. Also, they introduced new notation for inverse functions when they explored logarithmic functions, and they restricted domains when they found inverses for functions such as $y = x^2$.

This Chapter

Students explore trigonometric functions, first for acute angles in right triangles, then for angles in standard form, and also for points on the unit circle. They derive and use the Law of Sines and the Law of Cosines as applications of trigonometric functions, and they develop inverses for the sine, cosine, and tangent functions.

Future Connections

Students' exploration of trigonometric functions and periodic functions continues in the following chapter. There they will explore amplitude and frequency for periodic functions and will look at translations of their graphs. They will develop and use trigonometric functions for sums or differences of angles and they will solve equations involving trigonometric terms.

13-1 Right Triangle Trigonometry

To solve many of the problems in this lesson, students will need to remember the Pythagorean Theorem, which states that the sum of the squares of the legs of a right triangle equals the square of the hypotenuse. The *hypotenuse* is the side directly across from the right angle. It is also the longest side. The *legs* of a right triangle are the two shorter sides. If the legs of a right triangle have measures a and b and the hypotenuse has a measure of c, then $a^2 + b^2 = c^2$.

When solving real-world problems involving right triangles and trigonometric ratios, first determine what information is given about the triangle's sides and angles. Then determine how the given side or sides of the triangle relate to the given angle or the angle to be considered. In other words, determine if a given side is the hypotenuse, side opposite, or side adjacent to the angle. For example, if the problem gives an angle measure and the measure of the side adjacent to this angle and asks you to find the side opposite this angle, use the tangent ratio.

13-2 Angles and Angle Measure

This lesson will introduce students to concept of negative angle measure. It is important to note that an angle measuring $-210°$ is not less than an angle measuring 210°. The negative angle measure indicates that the direction of the rotation is clockwise instead of counterclockwise.

Remember that in trigonometry an angle is a measure of rotation. One complete rotation, or turn, measures 360°, the number of degrees in a circle. In the real world, objects can rotate about a fixed point many times. Angles measuring more than 360° are used to describe these rotations. To draw such an angle, first subtract 360° from the angle measure and continue doing so until you arrive at a measure that is less than or equal to 360°. Draw this angle and then use an arrow spiraling from the angle's initial side, through as many 360°-increments as you subtracted, until finally reaching the angle's terminal side.

13-3 Trigonometric Functions of General Angles

In Lesson 13-1, students find the exact values of the six trigonometric functions for angles measuring less than 90°, since the angles other than the right angle in a right triangle are both acute angles. Right triangle trigonometry is also used to define the values of the trigonometric functions for angles other than acute angles. From a point $P(x, y)$ on the terminal side of any angle, draw a segment perpendicular, meeting at a right angle, to the x-axis. A right triangle is formed with one side measuring x units, another side measuring y units, and the hypotenuse measuring r units. The value of r can be found using the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$. To determine the values of the six trigonometric functions for the original angle θ, find the value of each function for the angle θ' that is formed by the terminal side and the x-axis. It is important to note that the trigonometric values of angles other than acute angles can be negative.

13-4 Law of Sines

Examine the chart on page 727. In each drawing where A is acute, the positions of the horizontal segment and segment b are fixed, thus allowing the measure of A to remain constant. The measure of segment b in each drawing is also constant. The position of segment a, however, can change like a door on a hinge. Notice also that the length of a in each diagram is different and often compared to the value $b \sin A$.

In the first diagram in the first row, the length of a is less than the value $b \sin A$. In other words, side a is too short to form a right triangle. When a is too short to form a right triangle, no triangle can be formed. When a equals $b \sin A$, a right triangle is formed, as in the second diagram. In the first diagram in the second row, a is greater than $b \sin A$ but still less than b. In other words, side a is too long to form a right triangle, but can still rotate on its hinge and meet the opposite side to form either an obtuse or an acute triangle. If a is greater than or equal to b, side a is again too long to form a right triangle, but is now also too long to rotate back towards $\angle A$. Instead, it can meet the opposite side in only one place, as shown in the second diagram in the second row.

13-5 Law of Cosines

The Law of Cosines involves three sides and one angle of a triangle. The key step in deriving the Law of Cosines is to consider an altitude of length h intersecting a side and dividing it into two parts. Using x to represent one of the two parts, the Pythagorean Theorem gives an equation in terms of x and the three side lengths. Then x can be replaced with an expression involving one side and the cosine of one angle. The result is an expression of the Law of Cosines as three equations:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

The Law of Cosines lets you calculate measures for all the angles and all the sides of a triangle if you are given (1) the length of two sides and the measure of the included angle, or (2) the lengths of three sides. The Law of Sines can be used if the measures of two angles and one side are known (ASA, AAS) or if the measures of two sides and the non-included angle are known (SSA, but there may be 0, 1, or 2 triangles).

13-6 Circular Functions

To memorize the unit circle presented at the bottom of page 740, use the following technique. First memorize the coordinates for the angles in the first quadrant. Notice that all of these coordinates contain values having a denominator of 2, that the x- and y-coordinates for 45° are identical, $\frac{\sqrt{2}}{2}$, and that the x- and y-coordinates for 30° and 60° are reversed. With this quadrant memorized, the coordinates for the other quadrants can be obtained using the signs of x and y in each quadrant and the symmetry of a circle.

The coordinates of the quadrantal angles, 0°, 90°, 180°, and 270°, are easily remembered by recalling that a unit circle has a radius of 1 unit. For example, since 0° is located on the x-axis, its coordinates are (1, 0).

Once the unit circle is completed, it can be used as a reference to obtain the sine or cosine of any of the angles listed. One need only remember that the x-coordinate given for an angle measure is the cosine of the angle and the y-coordinate is the sine of the angle. In other words $(x, y) = (\cos \theta, \sin \theta)$.

13-7 Inverse Trigonometric Functions

In this lesson, students limit the domain of the sine, cosine, and tangent functions, and define an inverse for each function. The graphs of those functions do not pass the horizontal line test, so they would not have inverses. However, the section of the sine and tangent graphs between −180° and 180° and the section of the cosine graph between 0° and 180° do pass the horizontal line test, so inverse functions can be identified for those domains. The function $y = \text{Sin } x$ is the restricted-domain sine function. Its inverse is $x = \text{Sin}^{-1}\, y$ or $x = \text{Arcsin } y$. The function $y = \text{Cos } x$ is the restricted-domain cosine function. Its inverse is $x = \text{Cos}^{-1}\, y$ or $x = \text{Arccos } y$. The function $y = \text{Tan } x$ represents the restricted-domain tangent function. Its inverse function is $x = \text{Tan}^{-1}\, y$ or $x = \text{Arctan } y$.

Chapter 13

DAILY INTERVENTION and Assessment

	Type	Student Edition	Teacher Resources	Technology/Internet
INTERVENTION	Ongoing	Prerequisite Skills, pp. 699, 708, 715, 724, 732, 738, 745 Practice Quiz 1, p. 715 Practice Quiz 2, p. 738	5-Minute Check Transparencies Quizzes, *CRM* pp. 831–832 Mid-Chapter Test, *CRM* p. 833 Study Guide and Intervention, *CRM* pp. 775–776, 781–782, 787–788, 793–794, 799–800, 805–806, 811–812	Alge2PASS: Tutorial Plus www.algebra2.com/self_check_quiz www.algebra2.com/extra_examples
	Mixed Review	pp. 708, 714, 724, 732, 738, 745, 751	Cumulative Review, *CRM* p. 834	
	Error Analysis	Find the Error, pp. 730, 735 Common Misconceptions, p. 703	Find the Error, *TWE* pp. 730, 735 Unlocking Misconceptions, *TWE* pp. 718, 726 Tips for New Teachers, *TWE* pp. 703, 711	
ASSESSMENT	Standardized Test Practice	pp. 702, 706, 708, 714, 724, 732, 737, 738, 745, 751, 757, 758–759	*TWE* p. 702 Standardized Test Practice, *CRM* pp. 835–836	Standardized Test Practice CD-ROM www.algebra2.com/standardized_test
	Open-Ended Assessment	Writing in Math, pp. 708, 714, 724, 732, 737, 744, 751 Open Ended, pp. 706, 712, 722, 729, 736, 742, 749	Modeling: *TWE* pp. 708, 732, 745 Speaking: *TWE* pp. 715, 751 Writing: *TWE* pp. 724, 738 Open-Ended Assessment, *CRM* p. 829	
	Chapter Assessment	Study Guide, pp. 752–756 Practice Test, p. 757	Multiple-Choice Tests (Forms 1, 2A, 2B), *CRM* pp. 817–822 Free-Response Tests (Forms 2C, 2D, 3), *CRM* pp. 823–828 Vocabulary Test/Review, *CRM* p. 830	TestCheck and Worksheet Builder (see below) MindJogger Videoquizzes www.algebra2.com/vocabulary_review www.algebra2.com/chapter_test

Key to Abbreviations: TWE = Teacher Wraparound Edition; CRM = Chapter Resource Masters

Additional Intervention Resources

The Princeton Review's *Cracking the SAT & PSAT*
The Princeton Review's *Cracking the ACT*
ALEKS

TestCheck and Worksheet Builder

This **networkable** software has three modules for intervention and assessment flexibility:

- **Worksheet Builder** to make worksheet and tests
- **Student Module** to take tests on screen (optional)
- **Management System** to keep student records (optional)

Special banks are included for SAT, ACT, TIMSS, NAEP, and End-of-Course tests.

Intervention Technology

Alge2PASS: Tutorial Plus CD-ROM offers a complete, self-paced algebra curriculum.

Algebra 2 Lesson	Alge2PASS Lesson
13-3	**24** *Trigonometric Functions of Coterminal Angles*
13-3	**25** *Trigonometric Functions of Acute and Quadrantal Angles*
13-5	**26** *Trigonometry II*

ALEKS is an online mathematics learning system that adapts assessment and tutoring to the student's needs. Subscribe at www.k12aleks.com.

Intervention at Home

Log on for student study help.

- For each lesson in the Student Edition, there are Extra Examples and Self-Check Quizzes.
  www.algebra2.com/extra_examples
  www.algebra2.com/self_check_quiz
- For chapter review, there is vocabulary review, test practice, and standardized test practice.
  www.algebra2.com/vocabulary_review
  www.algebra2.com/chapter_test
  www.algebra2.com/standardized_test

For more information on Intervention and Assessment, see pp. T8–T11.

Reading and Writing in Mathematics

Glencoe Algebra 2 provides numerous opportunities to incorporate reading and writing into the mathematics classroom.

Student Edition

- Foldables Study Organizer, p. 699
- Concept Check questions require students to verbalize and write about what they have learned in the lesson. (pp. 706, 712, 722, 729, 735, 742, 749, 752)
- Writing in Math questions in every lesson, pp. 708, 714, 724, 732, 737, 744, 751
- Reading Study Tip, pp. 701, 709, 711, 718, 740
- WebQuest, p. 708

Teacher Wraparound Edition

- Foldables Study Organizer, pp. 699, 752
- Study Notebook suggestions, pp. 706, 713, 716, 722, 730, 735, 743, 749
- Modeling activities, pp. 708, 732, 745
- Speaking activities, pp. 715, 751
- Writing activities, pp. 724, 738
- Differentiated Instruction, (Verbal/Linguistic), p. 735
- ELL Resources, pp. 698, 707, 714, 723, 731, 735, 737, 744, 750, 752

Additional Resources

- Vocabulary Builder worksheets require students to define and give examples for key vocabulary terms as they progress through the chapter. (*Chapter 13 Resource Masters*, pp. vii-viii)
- Reading to Learn Mathematics master for each lesson (*Chapter 13 Resource Masters*, pp. 779, 785, 791, 797, 803, 809, 815)
- *Vocabulary PuzzleMaker* software creates crossword, jumble, and word search puzzles using vocabulary lists that you can customize.
- *Teaching Mathematics with Foldables* provides suggestions for promoting cognition and language.
- *Reading and Writing in the Mathematics Classroom*
- *WebQuest and Project Resources*

For more information on Reading and Writing in Mathematics, see pp. T6–T7.

Chapter 13 Notes

What You'll Learn

Have students read over the list of objectives and make a list of any words with which they are not familiar.

Why It's Important

Point out to students that this is only one of many reasons why each objective is important. Others are provided in the introduction to each lesson.

Lesson	NCTM Standards	Local Objectives
13-1 Preview	2, 3, 6	
13-1	2, 3, 6, 8, 9, 10	
13-2	1, 3, 4, 6, 8, 9, 10	
13-2 Follow-Up	1, 3, 4, 6, 8	
13-3	1, 3, 4, 6, 8, 9, 10	
13-4	1, 3, 4, 6, 8, 9, 10	
13-5	1, 3, 4, 6, 8, 9, 10	
13-6	1, 3, 4, 6, 8, 9, 10	
13-7	1, 3, 4, 6, 8, 9, 10	

Key to NCTM Standards:

1=Number & Operations, 2=Algebra, 3=Geometry, 4=Measurement, 5=Data Analysis & Probability, 6=Problem Solving, 7=Reasoning & Proof, 8=Communication, 9=Connections, 10=Representation

Chapter 13 Trigonometric Functions

What You'll Learn

- **Lessons 13-1, 13-2, 13-3, 13-6, and 13-7** Find values of trigonometric functions.
- **Lessons 13-1, 13-4, and 13-5** Solve problems by using right triangle trigonometry.
- **Lessons 13-4 and 13-5** Solve triangles by using the Law of Sines and Law of Cosines.

Key Vocabulary

- solve a right triangle (p. 704)
- radian (p. 710)
- Law of Sines (p. 726)
- Law of Cosines (p. 733)
- circular function (p. 740)

Why It's Important

Trigonometry is the study of the relationships among the angles and sides of right triangles. One of the many real-world applications of trigonometric functions involves solving problems using indirect measurement. For example, surveyors use a trigonometric function to find the heights of buildings. *You will learn how architects who design fountains use a trigonometric function to aim the water jets in Lesson 13-7.*

Vocabulary Builder

ELL

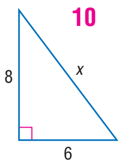
The Key Vocabulary list introduces students to some of the main vocabulary terms included in this chapter. For a more thorough vocabulary list with pronunciations of new words, give students the Vocabulary Builder worksheets found on pages vii and viii of the *Chapter 13 Resource Masters*. Encourage them to complete the definition of each term as they progress through the chapter. You may suggest that they add these sheets to their study notebooks for future reference when studying for the Chapter 13 test.

Getting Started

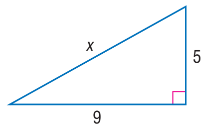
Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 13.

For Lessons 13-1 and 13-3 — Pythagorean Theorem

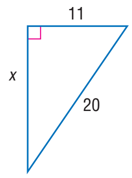
Find the value of x to the nearest tenth. *(For review, see pages 820 and 821.)*

1. 

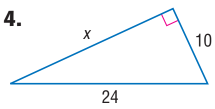
10

2. 

10.3

3. 

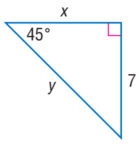
16.7

4. 

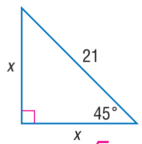
21.8

For Lesson 13-1 — 45°-45°-90° and 30°-60°-90° Triangles

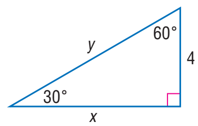
Find each missing measure. Write all radicals in simplest form.

5. 

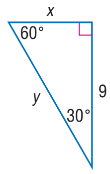
$x = 7,\ y = 7\sqrt{2}$

6. 

$x = \frac{21\sqrt{2}}{2}$

7. 

$x = 4\sqrt{3},\ y = 8$

8. 

$x = 3\sqrt{3},\ y = 6\sqrt{3}$

For Lesson 13-7 — Inverse Functions

Find the inverse of each function. Then graph the function and its inverse.
(For review, see Lesson 7-8.) **9–12. See pp. 759A–759D for graphs.**

9. $f(x) = x + 3$ $\quad f^{-1}(x) = x - 3$

10. $f(x) = \frac{x - 2}{5}$ $\quad f^{-1}(x) = 5x + 2$

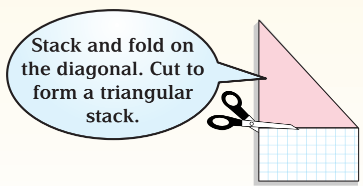
11. $f(x) = x^2 - 4$ $\quad f^{-1}(x) = \pm\sqrt{x + 4}$

12. $f(x) = -7x - 9$ $\quad f^{-1}(x) = \frac{-x - 9}{7}$

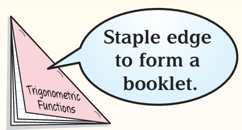
FOLDABLES™ Study Organizer

Make this Foldable to help you organize information about trigonometric functions. Begin with one sheet of construction paper and two pieces of grid paper.

Step 1 Fold and Cut



Step 2 Staple and Label



Reading and Writing As you read and study the chapter, you can write notes, draw diagrams, and record formulas on the grid paper.

Getting Started

This section provides a review of the basic concepts needed before beginning Chapter 13. Page references are included for additional student help.

Prerequisite Skills in the Getting Ready for the Next Lesson section at the end of each exercise set review a skill needed in the next lesson.

For Lesson	Prerequisite Skill
13-2	Dimensional Analysis (p. 708)
13-3	Rationalizing Denominators (p. 715)
13-4	Solving Equations with Trigonometric Functions (p. 724)
13-5	Solving Equations with Trigonometric Functions (p. 732)
13-6	Coterminal Angles (p. 738)
13-7	Finding Angle Measures (p. 745)

FOLDABLES™ Study Organizer

For more information about Foldables, see *Teaching Mathematics with Foldables.*

Organization of Data: Vocabulary and Visuals Have students use their right triangle journals to practice writing concise definitions in their own words and to design visuals that present the information introduced in the lesson in a concrete, easy-to-study format. Encourage students to clearly label their visuals and write captions when needed. Students can use this study guide to review what they know and apply it to what they are currently learning.

Spreadsheet Investigation

A Preview of Lesson 13-1

Getting Started

Cell Format Students can format the cells before they start to enter the data by using the Format Cells command. Format columns C, E, and F for numbers with 8 decimal places. Format columns A, B, and D for integers.

Teach

- Make sure students understand how to enter the formulas shown for columns C through F.
- Have students practice their skills in using a spreadsheet by entering the data in the example.
- Have students complete Exercises 1–3.
- To extend this investigation, ask students to explore the effect on the ratios if $a = b$ when a and b are rational numbers (such as 3.6) instead of integers.

Assess

Ask students:

- What formula could you have used in column B for the 45°-45°-90° triangle instead of entering the data? in column C for the 30°-60°-90° triangle? **$b = a$; $c = 2a$**
- Compare the 30°-60°-90° triangles. What is the same? What is different? **The angle measures are all the same, but the side measures are different. The triangles are similar but not congruent.**

Answer

3. **All of the ratios of side b to side a are approximately 1.73. All of the ratios of side b to side c are approximately 0.87. All of the ratios of side a to side c are 0.5.**

Spreadsheet Investigation

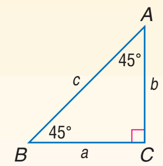
A Preview of Lesson 13-1

Special Right Triangles

You can use a computer spreadsheet program to investigate the relationships among the ratios of the side measures of special right triangles.

Example

The legs of a 45°-45°-90° triangle, a and b, are equal in measure. Use a spreadsheet to investigate the dimensions of 45°-45°-90° triangles. What patterns do you observe in the ratios of the side measures of these triangles?



=SQRT(A2^2 + B2^2) =B2/A2 =B2/C2 =A2/C2

45-45-90 Triangles

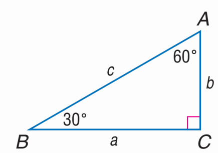
	A	B	C	D	E	F
1	a	b	c	b/a	b/c	a/c
2	1	1	1.41421356	1	0.70710678	0.70710678
3	2	2	2.82842712	1	0.70710678	0.70710678
4	3	3	4.24264069	1	0.70710678	0.70710678
5	4	4	5.65685425	1	0.70710678	0.70710678
6	5	5	7.07106781	1	0.70710678	0.70710678

The spreadsheet shows the formula that will calculate the length of side c. The formula uses the Pythagorean Theorem in the form $c = \sqrt{a^2 + b^2}$. Since 45°-45°-90° triangles share the same angle measures, the triangles listed in the spreadsheet are all similar triangles. Notice that all of the ratios of side b to side a are 1. All of the ratios of side b to side c and of side a to side c are approximately 0.71.

Exercises

For Exercises 1 and 2, use the spreadsheet below for 30°-60°-90° triangles.

If the measure of one leg of a right triangle and the measure of the hypotenuse are in ratio of 1 to 2, then the acute angles of the triangle measure 30° and 60°.



30-60-90 Triangles

	A	B	C	D	E	F
1	a	b	c	b/a	b/c	a/c
2	1	=SQRT(C2^2-A2^2)	2	=B2/A2	=B2/C2	=A2/C2
3	2	=SQRT(C3^2-A3^2)	4	=B3/A3	=B3/C3	=A3/C3
4	3	=SQRT(C4^2-A4^2)	6	=B4/A4	=B4/C4	=A4/C4
5	4	=SQRT(C5^2-A5^2)	8	=B5/A5	=B5/C5	=A5/C5
6	5	=SQRT(C6^2-A6^2)	10	=B6/A6	=B6/C6	=A6/C6

2. The triangles are all similar.

1. Copy and complete the spreadsheet above.
2. Describe the relationship among the 30°-60°-90° triangles whose dimensions are given.
3. What patterns do you observe in the ratios of the side measures of these triangles? **See margin.**

700 Chapter 13 Trigonometric Functions

13-1 Right Triangle Trigonometry

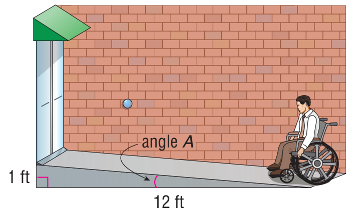
What You'll Learn

- Find values of trigonometric functions for acute angles.
- Solve problems involving right triangles.

How is trigonometry used in building construction?

The Americans with Disabilities Act (ADA) provides regulations designed to make public buildings accessible to all. Under this act, the slope of an entrance ramp designed for those with mobility disabilities must not exceed a ratio of 1 to 12. This means that for every 12 units of horizontal run, the ramp can rise or fall no more than 1 unit.

When viewed from the side, a ramp forms a right triangle. The slope of the ramp can be described by the *tangent* of the angle the ramp makes with the ground. In this example, the tangent of angle A is $\frac{1}{12}$.



Vocabulary

- trigonometry
- trigonometric functions
- sine
- cosine
- tangent
- cosecant
- secant
- cotangent
- solve a right triangle
- angle of elevation
- angle of depression

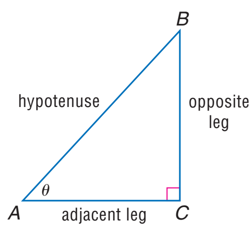
Study Tip

Reading Math

The word *trigonometry* is derived from two Greek words—*trigon* meaning triangle and *metra* meaning measurement.

TRIGONOMETRIC VALUES The tangent of an angle is one of the ratios used in trigonometry. **Trigonometry** is the study of the relationships among the angles and sides of a right triangle.

Consider right triangle ABC in which the measure of acute angle A is identified by the Greek letter *theta*, θ. The sides of the triangle are the *hypotenuse*, the *leg opposite* θ, and the *leg adjacent to* θ.



Using these sides, you can define six **trigonometric functions**: **sine**, **cosine**, **tangent**, **cosecant**, **secant**, and **cotangent**. These functions are abbreviated sin, cos, tan, sec, csc, and cot, respectively.

Key Concept — Trigonometric Functions

If θ is the measure of an acute angle of a right triangle, *opp* is the measure of the leg opposite θ, *adj* is the measure of the leg adjacent to θ, and *hyp* is the measure of the hypotenuse, then the following are true.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Notice that the sine, cosine, and tangent functions are reciprocals of the cosecant, secant, and cotangent functions, respectively. Thus, the following are also true.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

13-1 Lesson Notes

1 Focus

5-Minute Check Transparency 13-1 Use as a quiz or review of Chapter 12.

Mathematical Background notes are available for this lesson on p. 698C.

Building on Prior Knowledge

In previous courses, students learned about the Pythagorean Theorem. In this lesson, students will apply their knowledge to solve triangles.

How is trigonometry used in building construction?

Ask students:

- To meet the required ratio, how long would a ramp have to be to rise from the ground to a door that is 2 feet above ground level? **24 ft**
- Does a ramp with a ratio of 0.08 meet the requirement? **yes**

Resource Manager

Workbook and Reproducible Masters

Chapter 13 Resource Masters
- Study Guide and Intervention, pp. 775–776
- Skills Practice, p. 777
- Practice, p. 778
- Reading to Learn Mathematics, p. 779
- Enrichment, p. 780

Graphing Calculator and Spreadsheet Masters, p. 52

Science and Mathematics Lab Manual, pp. 37–40

Teaching Algebra With Manipulatives Masters, pp. 299–300

Transparencies

5-Minute Check Transparency 13-1
Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

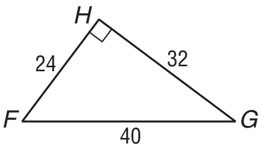
TRIGONOMETRIC VALUES

In-Class Examples

Power Point®

Teaching Tip Discuss with students the relationship between sin A and cos C in the figure. Remind students that A and C are complementary angles.

1 Find the values of the six trigonometric functions for angle G.



$\sin G = \frac{3}{5}$, $\cos G = \frac{4}{5}$, $\tan G = \frac{3}{4}$, $\cot G = \frac{4}{3}$, $\sec G = \frac{5}{4}$, $\csc G = \frac{5}{3}$

2 If $\tan A = \frac{5}{3}$, find the value of $\csc A$. **D**

A $\frac{3}{5}$ **B** $\frac{4}{3}$

C $\sqrt{34}$ **D** $\frac{\sqrt{34}}{5}$

Interactive Chalkboard

PowerPoint® Presentations

This CD-ROM is a customizable Microsoft® PowerPoint® presentation that includes:

- Step-by-step, dynamic solutions of each In-Class Example from the Teacher Wraparound Edition
- Additional, Your Turn exercises for each example
- The 5-Minute Check Transparencies
- Hot links to Glencoe Online Study Tools

Study Tip

Memorize Trigonometric Ratios

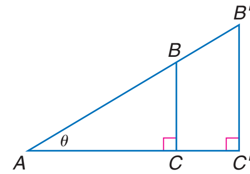
SOH-CAH-TOA is a mnemonic device for remembering the first letter of each word in the ratios for sine, cosine, and tangent.

$\sin \theta = \frac{\text{opp}}{\text{hyp}}$

$\cos \theta = \frac{\text{adj}}{\text{hyp}}$

$\tan \theta = \frac{\text{opp}}{\text{adj}}$

The domain of each of these trigonometric functions is the set of all acute angles θ of a right triangle. The values of the functions depend only on the measure of θ and not on the size of the right triangle. For example, consider $\sin \theta$ in the figure at the right.



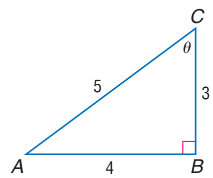
Using $\triangle ABC$: $\sin \theta = \frac{BC}{AB}$

Using $\triangle AB'C'$: $\sin \theta = \frac{B'C'}{AB'}$

The right triangles are similar because they share angle θ. Since they are similar, the ratios of corresponding sides are equal. That is, $\frac{BC}{AB} = \frac{B'C'}{AB'}$. Therefore, you will find the same value for $\sin \theta$ regardless of which triangle you use.

Example 1 Find Trigonometric Values

Find the values of the six trigonometric functions for angle θ.



For this triangle, the leg opposite θ is $\overline{AB}$, and the leg adjacent to θ is $\overline{CB}$. Recall that the hypotenuse is always the longest side of a right triangle, in this case $\overline{AC}$.

Use opp = 4, adj = 3, and hyp = 5 to write each trigonometric ratio.

$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$ $\quad \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$ $\quad \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$

$\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$ $\quad \sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$ $\quad \cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$

Throughout Unit 5, a capital letter will be used to denote both a vertex of a triangle and the measure of the angle at that vertex. The same letter in lowercase will be used to denote the side opposite that angle and its measure.

Standardized Test Practice

A B C D

Example 2 Use One Trigonometric Ratio to Find Another

Multiple-Choice Test Item

If $\cos A = \frac{2}{5}$, find the value of $\tan A$.

Ⓐ $\frac{5}{2}$ Ⓑ $\frac{2\sqrt{21}}{21}$ Ⓒ $\frac{\sqrt{21}}{2}$ Ⓓ $\sqrt{21}$

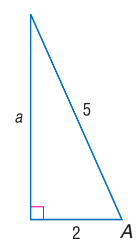
The Princeton Review

Test-Taking Tip

Whenever necessary or helpful, draw a diagram of the situation.

Read the Test Item

Begin by drawing a right triangle and labeling one acute angle A. Since $\cos \theta = \frac{\text{adj}}{\text{hyp}}$ and $\cos A = \frac{2}{5}$ in this case, label the adjacent leg 2 and the hypotenuse 5.



Solve the Test Item

Use the Pythagorean Theorem to find a.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + 2^2 = 5^2$	Replace b with 2 and c with 5.
$a^2 + 4 = 25$	Simplify.
$a^2 = 21$	Subtract 4 from each side.
$a = \sqrt{21}$	Take the square root of each side.

Standardized Test Practice

A B C D

Example 2 Another approach would be to write $\cos A = \frac{2}{5} = \frac{\text{adj}}{\text{hyp}}$. Then write $\tan A = \frac{\text{opp}}{\text{adj}}$. This suggests looking for an answer choice that has a denominator of 2.

Now find tan A.

$\tan A = \frac{opp}{adj}$ Tangent ratio

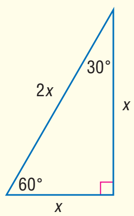
$= \frac{\sqrt{21}}{2}$ Replace *opp* with $\sqrt{21}$ and *adj* with 2.

The answer is C.

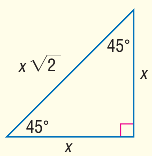
Angles that measure 30°, 45°, and 60° occur frequently in trigonometry. The table below gives the values of the six trigonometric functions for these angles. To remember these values, use the properties of 30°-60°-90° and 45°-45°-90° triangles.

Key Concept *Trigonometric Values for Special Angles*

30°-60°-90° Triangle



45°-45°-90° Triangle



θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

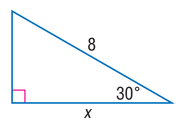
You will verify some of these values in Exercises 27 and 28.

RIGHT TRIANGLE PROBLEMS

You can use trigonometric functions to solve problems involving right triangles.

Example 3 Find a Missing Side Length of a Right Triangle

Write an equation involving sin, cos, or tan that can be used to find the value of x. Then solve the equation. Round to the nearest tenth.



The measure of the hypotenuse is 8. The side with the missing length is *adjacent* to the angle measuring 30°. The trigonometric function relating the adjacent side of a right triangle and the hypotenuse is the cosine function.

$\cos\theta = \frac{adj}{hyp}$ cosine ratio

$\cos 30° = \frac{x}{8}$ Replace θ with 30°, *adj* with x, and *hyp* with 8.

$\frac{\sqrt{3}}{2} = \frac{x}{8}$ $\cos 30° = \frac{\sqrt{3}}{2}$.

$4\sqrt{3} = x$ Multiply each side by 8.

The value of x is $4\sqrt{3}$ or about 6.9.

A calculator can be used to find the value of trigonometric functions for *any* angle, not just the special angles mentioned. Use SIN, COS, and TAN for sine, cosine, and tangent. Use these keys and the reciprocal key, x^{-1}, for cosecant, secant, and cotangent. Be sure your calculator is in degree mode.

Study Tip

Common Misconception

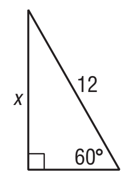
The $\cos^{-1} x$ on a graphing calculator does *not* find $\frac{1}{\cos x}$. To find sec x or $\frac{1}{\cos x}$, find cos x and then use the x^{-1} key.

www.algebra2.com/extra_examples

RIGHT TRIANGLE PROBLEMS

In-Class Example Power Point®

3 Write an equation involving sin, cos, or tan that can be used to find the value of x. Then solve the equation. Round to the nearest tenth.



$\sin 60° = \frac{x}{12}$; $6\sqrt{3}$ or about 10.4

Teaching Tip On a scientific calculator (in contrast to a graphing calculator), the sequence may be to enter the angle measure, such as 20, first, and then press the TAN key.

Tips for New Teachers

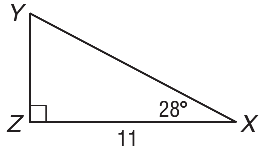
Intervention The lengths of the two special triangles in the Key Concept box on this page are critical. Have students learn how to recreate these triangles themselves from memory. From the triangles, they can generate the entire table shown in the Key Concept box.

Lesson 13-1 Right Triangle Trigonometry 703

In-Class Examples

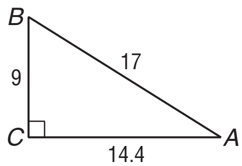
Power Point®

4 Solve $\triangle XYZ$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



$Y = 62°$, $x \approx 5.8$, $z \approx 12.5$

5 Solve $\triangle ABC$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



$A \approx 32°$, $B \approx 58°$

Concept Check

Solving Right Triangles What is the minimum information you have to have about a right triangle to solve it? **two of the following: measure of an acute angle, length of a leg, length of the hypotenuse**

Here are some calculator examples.

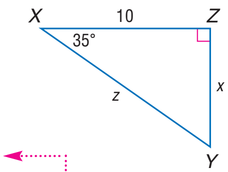
cos 46° **KEYSTROKES:** [COS] 46 [ENTER] .6946583705

cot 20° **KEYSTROKES:** [TAN] 20 [ENTER] [x^{-1}] [ENTER] 2.747477419

If you know the measures of any two sides of a right triangle or the measures of one side and one acute angle, you can determine the measures of all the sides and angles of the triangle. This process of finding the missing measures is known as **solving a right triangle**.

Example 4 Solve a Right Triangle

Solve $\triangle XYZ$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



You know the measures of one side, one acute angle, and the right angle. You need to find x, z, and Y.

Find x and z.

$$\tan 35° = \frac{x}{10} \qquad \sec 35° = \frac{z}{10}$$

$$10 \tan 35° = x \qquad \frac{1}{\cos 35°} = \frac{z}{10}$$

$$7.0 \approx x \qquad \frac{10}{\cos 35°} = z$$

$$12.2 \approx z$$

Find Y.

$35° + Y = 90°$ Angles X and Y are complementary.

$Y = 55°$ Solve for Y.

Therefore, $Y = 55°$, $x \approx 7.0$, and $z \approx 12.2$.

Study Tip

Error in Measurement

The value of z in Example 4 is found using the secant instead of using the Pythagorean Theorem. This is because the secant uses values given in the problem rather than calculated values.

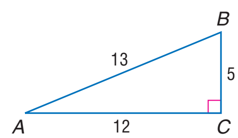
TEACHING TIP

An alternative method that is equally accurate is to solve the equation $\cos 35° = \frac{10}{z}$.

Use the inverse capabilities of your calculator to find the measure of an angle when one of its trigonometric ratios is known. For example, use the SIN^{-1} function to find the measure of an angle when the sine of the angle is known. *You will learn more about inverses of trigonometric functions in Lesson 13-7.*

Example 5 Find Missing Angle Measures of Right Triangles

Solve $\triangle ABC$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



You know the measures of the sides. You need to find A and B.

Find A. $\sin A = \frac{5}{13}$ $\quad \sin A = \frac{\text{opp}}{\text{hyp}}$

Use a calculator and the SIN^{-1} function to find the angle whose sine is $\frac{5}{13}$.

KEYSTROKES: [2nd] [SIN^{-1}] 5 [÷] 13 [)] [ENTER] 22.61986495

To the nearest degree, $A \approx 23°$.

Find B. $23° + B \approx 90°$ Angles A and B are complementary.

$B \approx 67°$ Solve for B.

Therefore, $A \approx 23°$ and $B \approx 67°$.

DAILY INTERVENTION

Differentiated Instruction

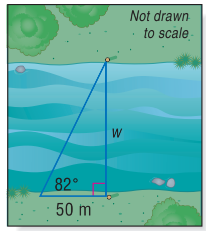
Visual/Spatial Have students use a stack of books and a notebook to model a ramp and investigate how steep the ramp needs to be for a toy car to roll down it without being pushed. Have them report their results in terms of the trigonometric functions of a right triangle.

704 Chapter 13 Trigonometric Functions

Trigonometry has many practical applications. Among the most important is the ability to find distances or lengths that either cannot be measured directly or are not easily measured directly.

Example 6 Indirect Measurement

BRIDGE CONSTRUCTION **In order to construct a bridge across a river, the width of the river at that location must be determined. Suppose a stake is planted on one side of the river directly across from a second stake on the opposite side. At a distance 50 meters to the left of the stake, an angle of 82° is measured between the two stakes. Find the width of the river.**



Let w represent the width of the river at that location. Write an equation using a trigonometric function that involves the ratio of the distance w and 50.

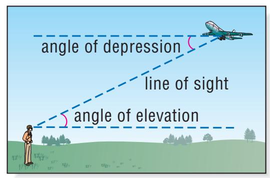
$\tan 82° = \frac{w}{50}$ $\quad$ $\tan \theta = \frac{\text{opp}}{\text{adj}}$

$50 \tan 82° = w$ $\quad$ Multiply each side by 50.

$355.8 \approx w$ $\quad$ Use a calculator.

The width of the river is about 355.8 meters.

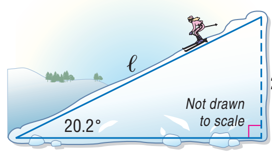
Some applications of trigonometry use an angle of elevation or depression. In the figure at the right, the angle formed by the line of sight from the observer and a line parallel to the ground is called the **angle of elevation**. The angle formed by the line of sight from the plane and a line parallel to the ground is called the **angle of depression**.



The angle of elevation and the angle of depression are congruent since they are alternate interior angles of parallel lines.

Example 7 Use an Angle of Elevation

SKIING **The Aerial run in Snowbird, Utah, has an angle of elevation of 20.2°. Its vertical drop is 2900 feet. Estimate the length of this run.**



Let ℓ represent the length of the run. Write an equation using a trigonometric function that involves the ratio of ℓ and 2900.

$\sin 20.2° = \frac{2900}{\ell}$ $\quad$ $\sin \theta = \frac{\text{opp}}{\text{hyp}}$

$\ell = \frac{2900}{\sin 20.2°}$ $\quad$ Solve for ℓ.

$\ell \approx 8398.5$ $\quad$ Use a calculator.

The length of the run is about 8399 feet.

More About . . .

Skiing

The average annual snowfall in Snowbird, Utah, is 500 inches. The longest designated run there is Chip's Run, at 2.5 miles.

Source: www.utahskiing.com

In-Class Examples

Power Point®

6 **BRIDGE CONSTRUCTION** Suppose, in a situation similar to that of Example 6 in the Student Edition, the angle was measured at a distance 30 meters away from the stake, and found to be 55°. Find the width of the river. **about 42.8 m**

7 **SKIING** A run has an angle of elevation of 15.7° and a vertical drop of 1800 feet. Estimate the length of this run. **about 6652 ft**

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 13.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective
- **Trigonometric Values:** 15–20, 27, 28
- **Right Triangle Problems:** 21–26, 29–46

Odd/Even Assignments
Exercises 15–42 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 15, 17, 21–37 odd, 41, 47–62

Average: 15–41 odd, 43–45, 47–62

Advanced: 16–42 even, 43–58 (optional: 59–62)

Check for Understanding

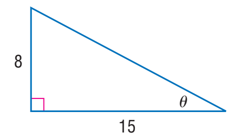
Concept Check

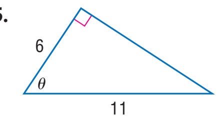
1. **Define** the word *trigonometry*. **1–3. See margin.**
2. **OPEN ENDED** Draw a right triangle. Label one of its acute angles θ. Then, label the hypotenuse, the leg adjacent to θ, and the leg opposite θ.
3. **Find a counterexample** to the following statement.
It is always possible to solve a right triangle.

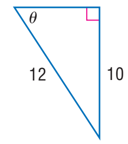
Guided Practice

GUIDED PRACTICE KEY	
Exercises	**Examples**
4–6	1
7	3
8, 11, 12	5
9, 10	4
13	6, 7
14	2

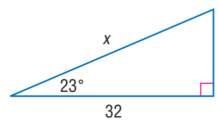
Find the values of the six trigonometric functions for angle θ. **4–6. See margin.**

4. 

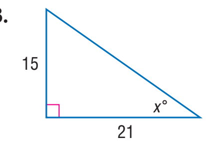
5. 

6. 

Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and angles to the nearest degree.

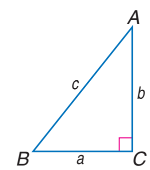
7. 

$\cos 23° = \frac{32}{x}$; $x \approx 34.8$

8. 

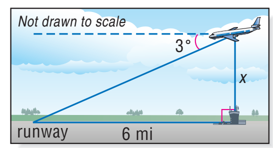
$\tan x° = \frac{15}{21}$; $x \approx 36$

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



9. $A = 45°$, $b = 6$ **9. $B = 45°$, $a = 6$, $c = 8.5$**
10. $B = 56°$, $c = 16$ **10. $A = 34°$, $a \approx 8.9$, $b \approx 13.3$**
11. $b = 7$, $c = 18$ **$a \approx 16.6$, $A \approx 67°$, $B \approx 23°$**
12. $a = 14$, $b = 13$ **$c \approx 19.1$, $A \approx 47°$, $B \approx 43°$**

13. **AVIATION** When landing, a jet will average a 3° angle of descent. What is the altitude x, to the nearest foot, of a jet on final descent as it passes over an airport beacon 6 miles from the start of the runway? **1660 ft**



Standardized Test Practice

14. If $\tan \theta = 3$, find the value of $\sin \theta$. **B**

(A) $\frac{3}{10}$ (B) $\frac{3\sqrt{10}}{10}$ (C) $\frac{10}{3}$ (D) $\frac{1}{3}$

★ indicates increased difficulty

Practice and Apply

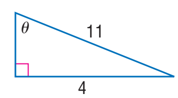
Homework Help

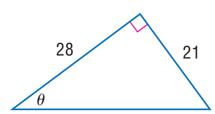
For Exercises	See Examples
15–20	1
21–24, 27, 28	3
25, 26, 37–40	5
29–36	4
41–46	6, 7
49	2

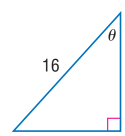
Extra Practice

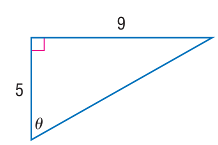
See page 857.

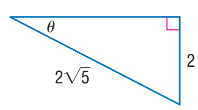
Find the values of the six trigonometric functions for angle θ. **15–20. See pp. 759A–759D.**

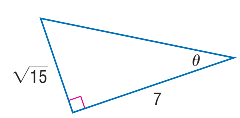
15. 

16. 

17. 

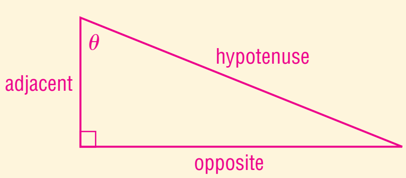
18. 

★ 19. 

★ 20. 

Answers

1. Trigonometry is the study of the relationships between the angles and sides of a right triangle.

2. 

3. Given only the measures of the angles of a right triangle, you cannot find the measures of its sides.

4. $\sin \theta = \frac{8}{17}$; $\cos \theta = \frac{15}{17}$; $\tan \theta = \frac{8}{15}$; $\csc \theta = \frac{17}{8}$; $\sec \theta = \frac{17}{15}$; $\cot \theta = \frac{15}{8}$

5. $\sin \theta = \frac{\sqrt{85}}{11}$; $\cos \theta = \frac{6}{11}$; $\tan \theta = \frac{\sqrt{85}}{6}$; $\csc \theta = \frac{11\sqrt{85}}{85}$; $\sec \theta = \frac{11}{6}$; $\cot \theta = \frac{6\sqrt{85}}{85}$

6. $\sin \theta = \frac{5}{6}$; $\cos \theta = \frac{\sqrt{11}}{6}$; $\tan \theta = \frac{5\sqrt{11}}{11}$; $\csc \theta = \frac{6}{5}$; $\sec \theta = \frac{6\sqrt{11}}{11}$; $\cot \theta = \frac{\sqrt{11}}{5}$

21. $\tan 30° = \frac{x}{10}$; $x \approx 5.8$

22. $\cos 60° = \frac{3}{x}$; $x = 6$

23. $\sin 54° = \frac{17.8}{x}$; $x \approx 22.0$

24. $\tan 17.5° = \frac{x}{23.7}$; $x \approx 7.5$

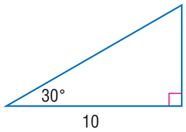
25. $\cos x° = \frac{15}{36}$; $x \approx 65$

26. $\sin x° = \frac{16}{22}$; $x \approx 47$

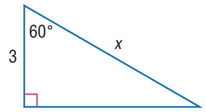
27–28. See pp. 759A–759D.

Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

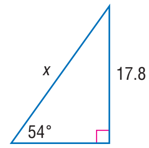
21.



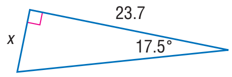
22.



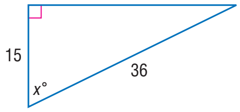
23.



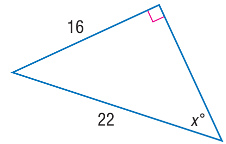
24.



25.



26.



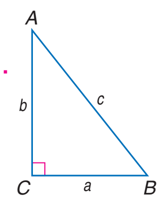
27. Using the 30°-60°-90° triangle shown on page 703, verify each value.

a. $\sin 30° = \frac{1}{2}$ **b.** $\cos 30° = \frac{\sqrt{3}}{2}$ **c.** $\sin 60° = \frac{\sqrt{3}}{2}$

28. Using the 45°-45°-90° triangle shown on page 703, verify each value.

a. $\sin 45° = \frac{\sqrt{2}}{2}$ **b.** $\cos 45° = \frac{\sqrt{2}}{2}$ **c.** $\tan 45° = 1$

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. **29–40. See pp. 759A–759D.**



29. $A = 16°, c = 14$ **30.** $B = 27°, b = 7$

31. $A = 34°, a = 10$ **32.** $B = 15°, c = 25$

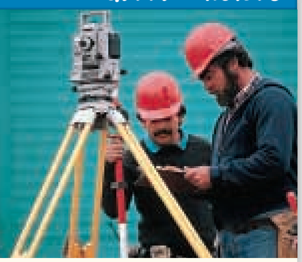
33. $B = 30°, b = 11$ **34.** $A = 45°, c = 7\sqrt{2}$

35. $B = 18°, a = \sqrt{15}$ **36.** $A = 10°, b = 15$

37. $b = 6, c = 13$ **38.** $a = 4, c = 9$

★ **39.** $\tan B = \frac{7}{8}, b = 7$ ★ **40.** $\sin A = \frac{1}{3}, a = 5$

Career Choices



Surveyor

Land surveyors manage survey parties that measure distances, directions, and angles between points, lines, and contours on Earth's surface.

Online Research For information about a career as a surveyor, visit: www.algebra2.com/careers

41. TRAVEL In a sightseeing boat near the base of the Horseshoe Falls at Niagara Falls, a passenger estimates the angle of elevation to the top of the falls to be 30°. If the Horseshoe Falls are 173 feet high, what is the distance from the boat to the base of the falls? **about 300 ft**

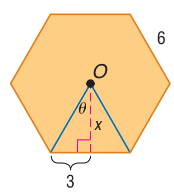
42. SURVEYING A surveyor stands 100 feet from a building and sights the top of the building at a 55° angle of elevation. Find the height of the building. **about 142.8 ft**

EXERCISE **For Exercises 43 and 44, use the following information.**
A preprogrammed workout on a treadmill consists of intervals walking at various rates and angles of incline. A 1% incline means 1 unit of vertical rise for every 100 units of horizontal run.

43. At what angle, with respect to the horizontal, is the treadmill bed when set at a 10% incline? Round to the nearest degree. **about 6°**

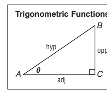
44. If the treadmill bed is 40 inches long, what is the vertical rise when set at an 8% incline? **about 3.2 in.**

★ **45. GEOMETRY** Find the area of the regular hexagon with point O as its center. (*Hint*: First find the value of x.) **93.54 units²**



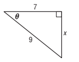
Study Guide and Intervention, p. 775 (shown) and p. 776

Trigonometric Values

Trigonometric Functions	If θ is the measure of an acute angle of a right triangle, *opp* is the measure of the leg opposite θ, *adj* is the measure of the leg adjacent to θ, and *hyp* is the measure of the hypotenuse, then the following are true.
	$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\tan\theta = \frac{\text{opp}}{\text{adj}}$ $\csc\theta = \frac{\text{hyp}}{\text{opp}}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}}$ $\cot\theta = \frac{\text{adj}}{\text{opp}}$

Example **Find the values of the six trigonometric functions for angle θ.**

Use the Pythagorean Theorem to find x, the measure of the leg opposite θ.



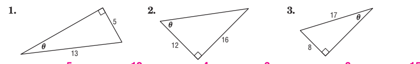
$x^2 + 7^2 = 9^2$ Pythagorean Theorem
$x^2 + 49 = 81$ Simplify.
$x^2 = 32$ Subtract 49 from each side.
$x = \sqrt{32}$ or $4\sqrt{2}$ Take the square root of each side.

Use opp $= 4\sqrt{2}$, adj $= 7$, and hyp $= 9$ to write each trigonometric ratio.

$\sin\theta = \frac{4\sqrt{2}}{9}$ $\cos\theta = \frac{7}{9}$ $\tan\theta = \frac{4\sqrt{2}}{7}$ $\csc\theta = \frac{9\sqrt{2}}{8}$ $\sec\theta = \frac{9}{7}$ $\cot\theta = \frac{7\sqrt{2}}{8}$

Exercises

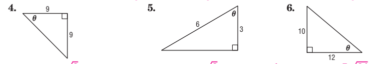
Find the values of the six trigonometric functions for angle θ.



1. $\sin\theta = \frac{5}{13}$; $\cos\theta = \frac{12}{13}$; $\tan\theta = \frac{5}{12}$; $\csc\theta = \frac{13}{5}$; $\sec\theta = \frac{13}{12}$; $\cot\theta = \frac{12}{5}$

2. $\sin\theta = \frac{4}{5}$; $\cos\theta = \frac{3}{5}$; $\tan\theta = \frac{4}{3}$; $\csc\theta = \frac{5}{4}$; $\sec\theta = \frac{5}{3}$; $\cot\theta = \frac{3}{4}$

3. $\sin\theta = \frac{8}{17}$; $\cos\theta = \frac{15}{17}$; $\tan\theta = \frac{8}{15}$; $\csc\theta = \frac{17}{8}$; $\sec\theta = \frac{17}{15}$; $\cot\theta = \frac{15}{8}$



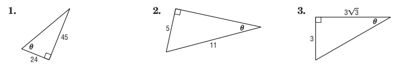
4. $\sin\theta = \frac{\sqrt{2}}{2}$; $\cos\theta = \frac{\sqrt{2}}{2}$; $\tan\theta = 1$; $\csc\theta = \sqrt{2}$; $\sec\theta = \sqrt{2}$; $\cot\theta = 1$

5. $\sin\theta = \frac{\sqrt{3}}{2}$; $\cos\theta = \frac{1}{2}$; $\tan\theta = \sqrt{3}$; $\csc\theta = \frac{2\sqrt{3}}{3}$; $\sec\theta = 2$; $\cot\theta = \frac{\sqrt{3}}{3}$

6. $\sin\theta = \frac{5\sqrt{61}}{61}$; $\cos\theta = \frac{6\sqrt{61}}{61}$; $\tan\theta = \frac{5}{6}$; $\csc\theta = \frac{\sqrt{61}}{5}$; $\sec\theta = \frac{\sqrt{61}}{6}$; $\cot\theta = \frac{6}{5}$

Skills Practice, p. 777 and Practice, p. 778 (shown)

Find the values of the six trigonometric functions for angle θ.

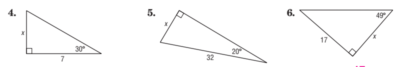


1. $\sin\theta = \frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = \frac{15}{8}$, $\csc\theta = \frac{17}{15}$, $\sec\theta = \frac{17}{8}$, $\cot\theta = \frac{8}{15}$

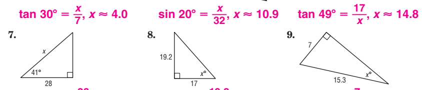
2. $\sin\theta = \frac{5}{11}$, $\cos\theta = \frac{4\sqrt{6}}{11}$, $\tan\theta = \frac{5\sqrt{6}}{24}$, $\csc\theta = \frac{11}{5}$, $\sec\theta = \frac{11\sqrt{6}}{24}$, $\cot\theta = \frac{4\sqrt{6}}{5}$

3. $\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = \frac{\sqrt{3}}{3}$, $\csc\theta = 2$, $\sec\theta = \frac{2\sqrt{3}}{3}$, $\cot\theta = \sqrt{3}$

Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

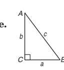


4. $\tan 30° = \frac{x}{7}$, $x \approx 4.0$ **5.** $\sin 20° = \frac{x}{32}$, $x \approx 10.9$ **6.** $\tan 49° = \frac{17}{x}$, $x \approx 14.8$



7. $\cos 41° = \frac{28}{x}$, $x \approx 37.1$ **8.** $\tan x° = \frac{19.2}{17}$, $x \approx 48$ **9.** $\sin x° = \frac{7}{15.3}$, $x \approx 27$

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



10. $A = 35°, a = 12$ — $b \approx 17.1, c \approx 20.9, B = 55°$

11. $B = 71°, b = 25$ — $a \approx 8.6, c \approx 26.4, A = 19°$

12. $B = 36°, c = 8$ — $a \approx 6.5, b \approx 4.7, A = 54°$

13. $a = 4, b = 7$ — $c \approx 8.1, A \approx 30°, B \approx 60°$

14. $A = 17°, c = 3.2$ — $a \approx 0.9, b \approx 3.1, B = 73°$

15. $b = 52, c = 95$ — $a \approx 79.5, A \approx 33°, B \approx 57°$

16. SURVEYING John stands 150 meters from a water tower and sights the top at an angle of elevation of 36°. How tall is the tower? Round to the nearest meter. **109 m**

Reading to Learn Mathematics, p. 779 ELL

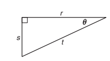
Pre-Activity **How is trigonometry used in building construction?**

Read the introduction to Lesson 13-1 at the top of page 701 in your textbook.

If a different ramp is built so that the angle shown in the figure has a tangent of $\frac{1}{14}$, will this ramp meet, exceed, or fail to meet ADA regulations? **exceed**

Reading the Lesson

1. Refer to the triangle at the right. Match each trigonometric function with the correct ratio.



i. $\frac{r}{t}$ **ii.** $\frac{r}{s}$ **iii.** $\frac{t}{r}$ **iv.** $\frac{s}{t}$ **v.** $\frac{s}{r}$ **vi.** $\frac{t}{s}$

a. $\sin\theta$ **iv** **b.** $\tan\theta$ **v** **c.** $\sec\theta$ **iii**

d. $\cot\theta$ **ii** **e.** $\cos\theta$ **i** **f.** $\csc\theta$ **vi**

2. Refer to the Key Concept box on page 703 in your textbook. Use the drawings of the 30°-60°-90° triangle and 45°-45°-90° triangle and/or the table to complete the following.

a. The tangent of 45° and the **cotangent** of 45° are equal.

b. The sine of 30° is equal to the cosine of **60°**.

c. The sine and **cosine** of 45° are equal.

d. The reciprocal of the cosecant of 60° is the **sine** of 60°.

e. The reciprocal of the cosine of 30° is the **cosecant** of 60°.

f. The reciprocal of the tangent of 60° is the **tangent** of 30°.

Helping You Remember

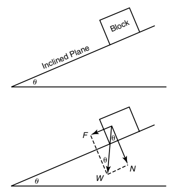
3. In studying trigonometry, it is important for you to know the relationships between the lengths of the sides of a 30°-60°-90° triangle. If you remember just one fact about this triangle, you will always be able to figure out the lengths of all the sides. What fact can you use, and why is it enough?

Sample answer: The shorter leg is half as long as the hypotenuse. You can use the Pythagorean Theorem to find the length of the longer leg.

Enrichment, p. 780

The Angle of Repose

Suppose you place a block of wood on an inclined plane, as shown at the right. If the angle, θ, at which the plane is inclined from the horizontal is very small, the block will not move. If you increase the angle, the block will eventually overcome the force of friction and start to slide down the plane.



At the instant the block begins to slide, the angle formed by the plane is called the angle of friction, or the angle of repose.

For situations in which the block and plane are smooth but unlubricated, the angle of repose depends *only* on the types of materials in the block and the plane. The angle is independent of the area of contact between the two surfaces and of the weight of the block.

The drawing at the right shows how to use vectors to find a coefficient of friction. This coefficient varies with different materials and is denoted by...

$F = W\sin\theta$ $N = W\cos$
$F = \mu N$

4 Assess

Open-Ended Assessment

Modeling Have students design a ramp, specifying the angle and how far the end of the ramp is from the place on the ground where it begins. Have them draw a picture and show steps to find the length of the ramp. Does the ramp meet ADA requirements?

Getting Ready for Lesson 13-2

PREREQUISITE SKILL Lesson 13-2 presents changing between measuring angles in radians and in degrees. Students will use their familiarity with converting units as they write the measures of angles in both radians and degrees. Exercises 59–62 should be used to determine your students' familiarity with dimensional analysis.

Answers

47. The sine and cosine ratios of acute angles of right triangles each have the longest measure of the triangle, the hypotenuse, as their denominator. A fraction whose denominator is greater than its numerator is less than 1. The tangent ratio of an acute angle of a right triangle does not involve the measure of the hypotenuse, $\frac{\text{opp}}{\text{adj}}$. If the measure of the opposite side is greater than the measure of the adjacent side, the tangent ratio is greater than 1. If the measure of the opposite side is less than the measure of the adjacent side, the tangent ratio is less than 1.

48. When construction involves right triangles, including building ramps, designing buildings, or surveying land before building, trigonometry is likely to be used. Answers should include the following.
   - If you view the ramp from the side then the vertical rise is opposite the angle that the ramp makes with the horizontal. Similarly, the horizontal run is the adjacent side. So the tangent of the angle is the ratio of the rise to the run or the slope of the ramp.
   - Given the ratio of the slope of ramp, you can find the angle of inclination by calculating $\tan^{-1}$ of this ratio.

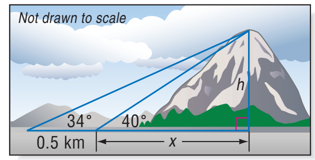
51. Band members may be more likely to like the same kinds of music.

52. This sample is random since different kinds of people go to the post office.

WebQuest

You can use the tangent ratio to determine the maximum height of a rocket. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

46. **GEOLOGY** A geologist measured a 40° angle of elevation to the top of a mountain. After moving 0.5 kilometer farther away, the angle of elevation was 34°. How high is the top of the mountain? (*Hint*: Write a system of equations in two variables.) **about 1.7 km high**



47. **CRITICAL THINKING** Explain why the sine and cosine of an acute angle are never greater than 1, but the tangent of an acute angle may be greater than 1. **See margin.**

48. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

How is trigonometry used in building construction?

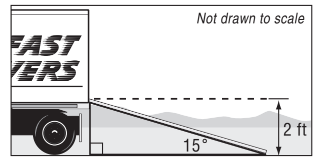
Include the following in your answer:
- an explanation as to why the ratio of vertical rise to horizontal run on an entrance ramp is the tangent of the angle the ramp makes with the horizontal, and
- an explanation of how an architect can use the tangent ratio to ensure that all the ramps he or she designs meet the ADA requirement.

Standardized Test Practice

49. If the secant of an angle θ is $\frac{25}{7}$, what is the sine of angle θ? **C**

Ⓐ $\frac{5}{25}$ Ⓑ $\frac{7}{25}$ Ⓒ $\frac{24}{25}$ Ⓓ $\frac{25}{7}$

50. **GRID IN** The tailgate of a moving truck is 2 feet above the ground. The incline of the ramp used for loading the truck is 15° as shown. Find the length of the ramp to the nearest tenth of a foot. **7.7**



Maintain Your Skills

Mixed Review

Determine whether each situation would produce a random sample. Write *yes* or *no* and explain your answer. *(Lesson 12-9)* **51–52. See margin for explanation.**

51. surveying band members to find the most popular type of music at your school **no**

52. surveying people coming into a post office to find out what color cars are most popular **yes**

Find each probability if a coin is tossed 4 times. *(Lesson 12-8)*

53. P(exactly 2 heads) $\frac{3}{8}$ 54. P(4 heads) $\frac{1}{16}$ 55. P(at least 1 heads) $\frac{15}{16}$

Solve each equation. *(Lesson 7-3)*

56. $y^4 - 64 = 0$ $\{\pm 2\sqrt{2}, \pm 2i\sqrt{2}\}$ 57. $x^5 - 5x^3 + 4x = 0$ $\{-2, -1, 0, 1, 2\}$ 58. $d + \sqrt{d} - 132 = 0$ $\{121\}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL Find each product. Include the appropriate units with your answer. *(To review **dimensional analysis**, see Lesson 5-1.)*

59. $5 \text{ gallons}\left(\frac{4 \text{ quarts}}{1 \text{ gallon}}\right)$ **20 qt** 60. $6.8 \text{ miles}\left(\frac{5280 \text{ feet}}{1 \text{ mile}}\right)$ **35,904 ft**

61. $\left(\frac{2 \text{ square meters}}{5 \text{ dollars}}\right) 30 \text{ dollars}$ **12 m²** 62. $\left(\frac{4 \text{ liters}}{5 \text{ minutes}}\right) 60 \text{ minutes}$ **48 L**

708 Chapter 13 Trigonometric Functions

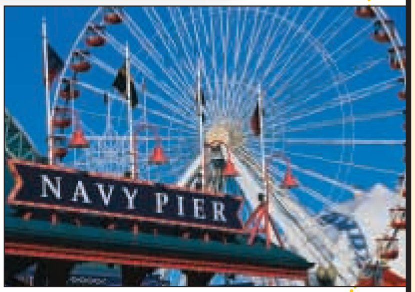
13-2 Angles and Angle Measure

What You'll Learn

- Change radian measure to degree measure and vice versa.
- Identify coterminal angles.

How can angles be used to describe circular motion?

The Ferris wheel at Navy Pier in Chicago has a 140-foot diameter and 40 gondolas equally spaced around its circumference. The average angular velocity ω of one of the gondolas is given by $\omega = \frac{\theta}{t}$, where θ is the angle through which the gondola has revolved after a specified amount of time t. For example, if a gondola revolves through an angle of 225° in 40 seconds, then its average angular velocity is 225° ÷ 40 or about 5.6° per second.



Vocabulary

- initial side
- terminal side
- standard position
- unit circle
- radian
- coterminal angles

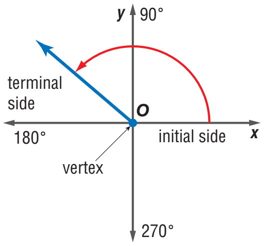
TEACHING TIP

ω is the lowercase Greek letter omega.

ANGLE MEASUREMENT

What does an angle measuring 225° look like? In Lesson 13-1, you worked only with acute angles, those measuring between 0° and 90°, but angles can have *any* real number measurement.

On a coordinate plane, an angle may be generated by the rotation of two rays that share a fixed endpoint at the origin. One ray, called the **initial side** of the angle, is fixed along the positive x-axis. The other ray, called the **terminal side** of the angle, can rotate about the center. An angle positioned so that its vertex is at the origin and its initial side is along the positive x-axis is said to be in **standard position**.



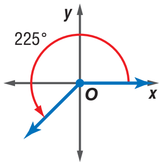
Study Tip

Reading Math

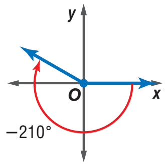
In trigonometry, an angle is sometimes referred to as an *angle of rotation*.

The measure of an angle is determined by the amount and direction of rotation from the initial side to the terminal side.

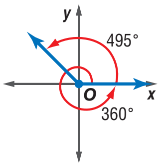
Positive Angle Measure
counterclockwise



Negative Angle Measure
clockwise



When terminal sides rotate, they may sometimes make one or more revolutions. An angle whose terminal side has made exactly one revolution has a measure of 360°.



TEACHING TIP

Why are there 360° in one revolution instead of 100° or 1000°? The answer may lie in ancient Babylon where mathematicians developed a number system based on 60. For example, there were 60 bushels in a *mana* and 60 *mana* in a *talent*. Today, we further subdivide 1 degree into 60 minutes (60′) and 1 minute into 60 seconds (60″).

13-2 Lesson Notes

1 Focus

5-Minute Check Transparency 13-2 Use as a quiz or review of Lesson 13-1.

Mathematical Background notes are available for this lesson on p. 698C.

How can angles be used to describe circular motion?

Ask students:

- What is the space between one gondola and the next around the circumference? **11 ft**
- How many degrees of the circle are between one gondola and the next? **9°**
- What is the radius of the Ferris wheel? **70 ft**

Resource Manager

Workbook and Reproducible Masters

Chapter 13 Resource Masters

- Study Guide and Intervention, pp. 781–782
- Skills Practice, p. 783
- Practice, p. 784
- Reading to Learn Mathematics, p. 785
- Enrichment, p. 786
- Assessment, p. 831

Transparencies

5-Minute Check Transparency 13-2
Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

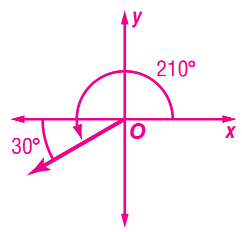
ANGLE MEASUREMENT

In-Class Example Power Point®

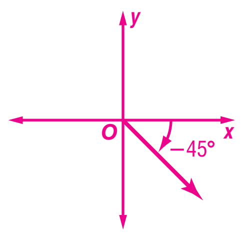
Teaching Tip Discuss with students the importance of marking the drawings of angles with the arrows as shown, and labeling the degrees.

1 Draw an angle with the given measure in standard position.

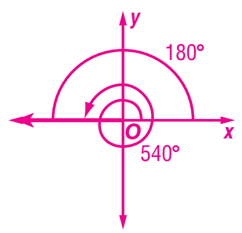
a. 210°



b. −45°



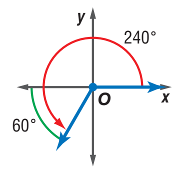
c. 540°



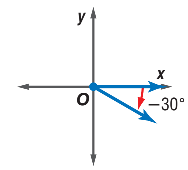
Example 1 Draw an Angle in Standard Position

Draw an angle with the given measure in standard position.

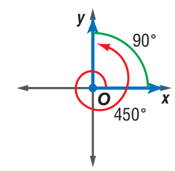
a. 240° 240° = 180° + 60°
Draw the terminal side of the angle 60° counterclockwise past the negative x-axis.



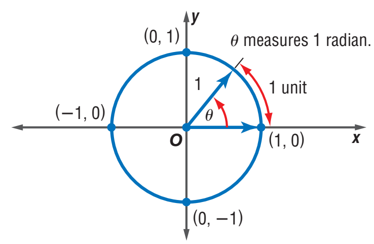
b. −30° The angle is negative.
Draw the terminal side of the angle 30° clockwise from the positive x-axis.



c. 450° 450° = 360° + 90°
Draw the terminal side of the angle 90° counterclockwise past the positive x-axis.

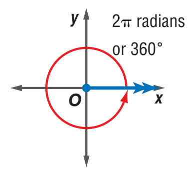


Another unit used to measure angles is a radian. The definition of a radian is based on the concept of a **unit circle**, which is a circle of radius 1 unit whose center is at the origin of a coordinate system. One **radian** is the measure of an angle θ in standard position whose rays intercept an arc of length 1 unit on the unit circle.



Study Tip

Radian Measure
As with degrees, the measure of an angle in radians is positive if its rotation is counterclockwise. The measure is negative if the rotation is clockwise.



The circumference of any circle is $2\pi r$, where r is the radius measure. So the circumference of a unit circle is $2\pi(1)$ or 2π units. Therefore, an angle representing one complete revolution of the circle measures 2π radians. This same angle measures 360°. Therefore, the following equation is true.

$$2\pi \text{ radians} = 360°$$

To change angle measures from radians to degrees or vice versa, solve the equation above in terms of both units.

$$2\pi \text{ radians} = 360°$$
$$\frac{2\pi \text{ radians}}{2\pi} = \frac{360°}{2\pi}$$
$$1 \text{ radian} = \frac{180°}{\pi}$$

1 radian is about 57 degrees.

$$2\pi \text{ radians} = 360°$$
$$\frac{2\pi \text{ radians}}{360} = \frac{360°}{360}$$
$$\frac{\pi \text{ radians}}{180} = 1°$$

1 degree is about 0.0175 radian.

These equations suggest a method for converting between radian and degree measure.

Key Concept — *Radian and Degree Measure*

- To rewrite the radian measure of an angle in degrees, multiply the number of radians by $\frac{180°}{\pi \text{ radians}}$.
- To rewrite the degree measure of an angle in radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180°}$.

Example 2 — Convert Between Degree and Radian Measure

Rewrite the degree measure in radians and the radian measure in degrees.

a. $60°$

$$60° = 60\cancel{°}\left(\frac{\pi \text{ radians}}{180\cancel{°}}\right)$$
$$= \frac{60\pi}{180} \text{ radians or } \frac{\pi}{3}$$

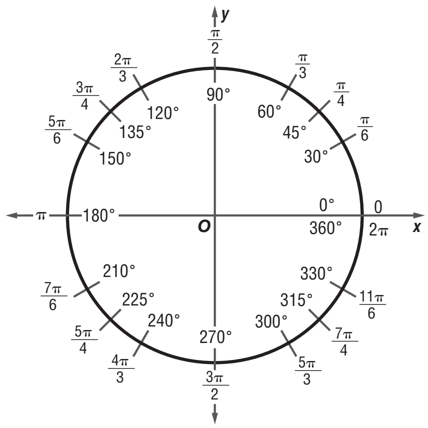
b. $-\frac{7\pi}{4}$

$$-\frac{7\pi}{4} = \left(-\frac{7\cancel{\pi}}{4}\ \cancel{\text{radians}}\right)\left(\frac{180°}{\cancel{\pi \text{ radians}}}\right)$$
$$= -\frac{1260°}{4} \text{ or } -315°$$

Study Tip

Reading Math The word *radian* is usually omitted when angles are expressed in radian measure. Thus, when no units are given for an angle measure, radian measure is implied.

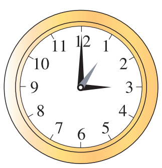
You will find it useful to learn equivalent degree and radian measures for the special angles shown in the diagram at the right. This diagram is more easily learned by memorizing the equivalent degree and radian measures for the first quadrant and for 90°. All of the other special angles are multiples of these angles.



Example 3 — Measure an Angle in Degrees and Radians

TIME **Find both the degree and radian measures of the angle through which the hour hand on a clock rotates from 1:00 P.M. to 3:00 P.M.**

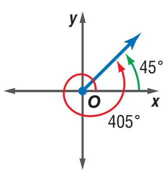
The numbers on a clock divide it into 12 equal parts with 12 equal angles. The angle from 1 to 3 on the clock represents $\frac{2}{12}$ or $\frac{1}{6}$ of a complete rotation of 360°. $\frac{1}{6}$ of 360° is 60°.



Since the rotation is clockwise, the angle through which the hour hand rotates is negative. Therefore, the angle measures $-60°$.

60° has an equivalent radian measure of $\frac{\pi}{3}$. So the equivalent radian measure of $-60°$ is $-\frac{\pi}{3}$.

COTERMINAL ANGLES If you graph a 405° angle and a 45° angle in standard position on the same coordinate plane, you will notice that the terminal side of the 405° angle is the same as the terminal side of the 45° angle. When two angles in standard position have the same terminal sides, they are called **coterminal angles**.



www.algebra2.com/extra_examples

In-Class Examples

Power Point®

2 Rewrite the degree measure in radians and the radian measure in degrees.

a. 30° $\frac{\pi}{6}$

b. $-\frac{5\pi}{3}$ $-300°$

3 Find both the degree and radian measures of the angle through which the hour hand on a clock rotates from 6 P.M. to 7 P.M. $-30°$ or $-\frac{\pi}{6}$

Tips for New Teachers

Intervention Students must be completely certain they know the definitions of the sine, cosine, and tangent ratios before they proceed with the rest of the chapter material. Have students create mnemonics, make color-coded index cards, or use other methods to help them review the ratios until they are confident of them.

Lesson 13-2 Angles and Angle Measure 711

COTERMINAL ANGLES

In-Class Example

Power Point®

4 Find one angle with positive measure and one angle with negative measure coterminal with each angle.

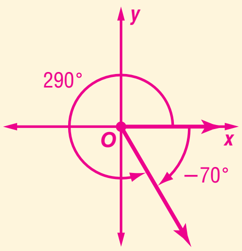
a. 210° **Sample answers: 570°, −150°**

b. $\frac{7\pi}{3}$ **Sample answers: $\frac{13\pi}{3}$, $-\frac{5\pi}{3}$**

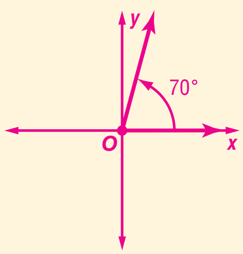
Answers

2. In a circle of radius *r* units, one radian is the measure of an angle whose rays intercept an arc length of *r* units.

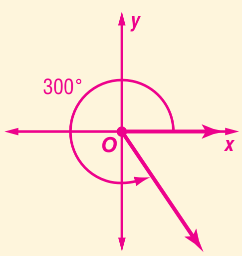
3.



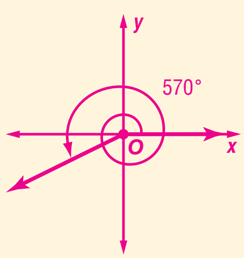
4.



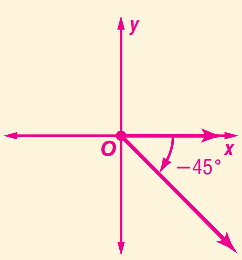
5.



6.



7.



Notice that 405° − 45° = 360°. In degree measure, coterminal angles differ by an integral multiple of 360°. You can find an angle that is coterminal to a given angle by adding or subtracting a multiple of 360°. In radian measure, a coterminal angle is found by adding or subtracting a multiple of 2π.

Example 4 Find Coterminal Angles

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

a. 240°

A positive angle is 240° + 360° or 600°.

A negative angle is 240° − 360° or −120°.

b. $\frac{9\pi}{4}$

A positive angle is $\frac{9\pi}{4} + 2\pi$ or $\frac{17\pi}{4}$. $\quad \frac{9\pi}{4} + \frac{8\pi}{4} = \frac{17\pi}{4}$

A negative angle is $\frac{9\pi}{4} - 2(2\pi)$ or $-\frac{7\pi}{4}$. $\quad \frac{9\pi}{4} + \left(-\frac{16\pi}{4}\right) = -\frac{7\pi}{4}$

Study Tip

Coterminal Angles
Notice in Example 4b that it is necessary to subtract a multiple of 2π to find a coterminal angle with negative measure.

Check for Understanding

Concept Check

1. **Name** the set of numbers to which angle measures belong. **reals**
2. **Define** the term radian. **See margin.**
3. **OPEN ENDED** Draw and label an example of an angle with negative measure in standard position. Then find an angle with positive measure that is coterminal with this angle. **See margin.**

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4–7	1
8–13	2
14–16	4
17, 18	3

Draw an angle with the given measure in standard position. 4–7. See margin.

4. 70° 5. 300° 6. 570° 7. −45°

Rewrite each degree measure in radians and each radian measure in degrees.

8. 130° **$\frac{13\pi}{18}$** 9. −10° **$-\frac{\pi}{18}$** 10. 485° **$\frac{97\pi}{36}$**

11. $\frac{3\pi}{4}$ **135°** 12. $-\frac{\pi}{6}$ **−30°** 13. $\frac{19\pi}{3}$ **1140°**

Find one angle with positive measure and one angle with negative measure coterminal with each angle. 14–16. Sample answers are given.

14. 60° **420°, −300°** 15. 425° **785°, −295°** 16. $\frac{\pi}{3}$ **$\frac{7\pi}{3}$, $-\frac{5\pi}{3}$**

Application

ASTRONOMY **For Exercises 17 and 18, use the following information.**
Earth rotates on its axis once every 24 hours.

17. How long does it take Earth to rotate through an angle of 315°? **21 h**
18. How long does it take Earth to rotate through an angle of $\frac{\pi}{6}$? **2 h**

★ indicates increased difficulty

Practice and Apply

Draw an angle with the given measure in standard position. 19–26. See pp. 759A–759D.

19. 235° 20. 270° 21. 790° 22. 380°
23. −150° 24. −50° ★ 25. π ★ 26. $-\frac{2\pi}{3}$

DAILY INTERVENTION

Differentiated Instruction

Kinesthetic Have students work with a partner so that one person models an angle with outstretched arms or with two pencils or yardsticks. The other partner then names a positive and negative angle, less than or more than a full circle, that is coterminal with the modeled angle.

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Homework Help

For Exercises	See Examples
19–26	1
27–42	2
43–54	4
55–59	3

Extra Practice

See page 857.

41. $\frac{1620}{\pi} \approx 515.7°$

42. $\frac{540}{\pi} \approx 171.9°$

Rewrite each degree measure in radians and each radian measure in degrees.

27. 120° $\frac{2\pi}{3}$
28. 60° $\frac{\pi}{3}$
29. −15° $-\frac{\pi}{12}$
30. −225° $-\frac{5\pi}{4}$
31. 660° $\frac{11\pi}{3}$
32. 570° $\frac{19\pi}{6}$
33. 158° $\frac{79\pi}{90}$
34. 260° $\frac{13\pi}{9}$
35. $\frac{5\pi}{6}$ 150°
36. $\frac{11\pi}{4}$ 495°
37. $-\frac{\pi}{4}$ −45°
38. $-\frac{\pi}{3}$ −60°
39. $\frac{29\pi}{4}$ 1305°
40. $\frac{17\pi}{6}$ 510°
★41. 9
★42. 3

Find one angle with positive measure and one angle with negative measure coterminal with each angle. 43–54. Sample answers are given.

43. 225° 585°, −135°
44. 30° 390°, −330°
45. −15° 345°, −375°
46. −140° 220°, −500°
47. 368° 8°, −352°
48. 760° 400°, −320°
49. $\frac{3\pi}{4}$ $\frac{11\pi}{4}, -\frac{5\pi}{4}$
50. $\frac{7\pi}{6}$ $\frac{19\pi}{6}, -\frac{5\pi}{6}$
51. $-\frac{5\pi}{4}$ $\frac{3\pi}{4}, -\frac{13\pi}{4}$
52. $-\frac{2\pi}{3}$ $\frac{4\pi}{3}, -\frac{8\pi}{3}$
53. $\frac{9\pi}{2}$ $\frac{13\pi}{2}, -\frac{3\pi}{2}$
54. $\frac{17\pi}{4}$ $\frac{25\pi}{4}, -\frac{7\pi}{4}$

55. **DRIVING** Some sport-utility vehicles (SUVs) use 15-inch radius wheels. When driven 40 miles per hour, determine the measure of the angle through which a point on the wheel travels every second. Round to both the nearest degree and nearest radian. 2689° per second; 47 radians per second

More About. . .

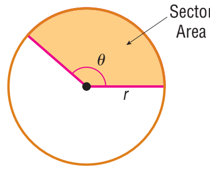
Driving

A leading U.S. automaker plans to build a hybrid sport-utility vehicle in the near future that will use an electric motor to boost fuel efficiency and reduce polluting emissions.

Source: *The Dallas Morning News*

GEOMETRY **For Exercises 56 and 57, use the following information.**

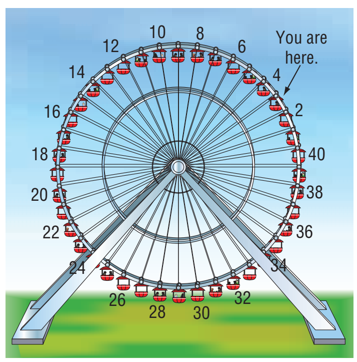
A *sector* is a region of a circle that is bounded by a central angle θ and its intercepted arc. The area A of a sector with radius r and central angle θ is given by $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.



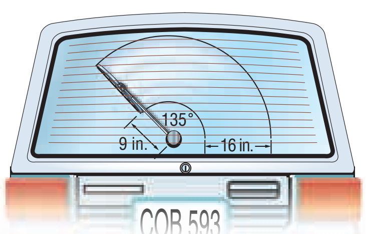
56. Find the area of a sector with a central angle of $\frac{4\pi}{3}$ radians in a circle whose radius measures 10 inches. 209.4 in^2

57. Find the area of a sector with a central angle of 150° in a circle whose radius measures 12 meters. about 188.5 m^2

58. **ENTERTAINMENT** Suppose the gondolas on the Navy Pier Ferris wheel were numbered from 1 through 40 consecutively in a counterclockwise fashion. If you were sitting in gondola number 3 and the wheel were to rotate counterclockwise through $\frac{47\pi}{10}$ radians, which gondola used to be in the position that you are in now? number 17



★59. **CARS** Use the Area of a Sector Formula in Exercises 56 and 57 to find the area swept by the rear windshield wiper of the car shown at the right. about 640.88 in^2



www.algebra2.com/self_check_quiz

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 13.
- add the diagram of radian and degree measures on p. 711 to their notebook.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective
- **Angle Measurement:** 19–42, 55–59
- **Coterminal Angles:** 43–54

Odd/Even Assignments

Exercises 19–54 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 19–23 odd, 27–39 odd, 43–57 odd, 60–81

Average: 19–59 odd, 60–81

Advanced: 20–58 even, 60–75 (optional: 76–81)

All: Practice Quiz 1 (1–10)

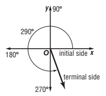
Study Guide and Intervention, p. 781 (shown) and p. 782

Angle Measurement An angle is determined by two rays. The degree measure of an angle is described by the amount and direction of rotation from the **initial side** along the positive x-axis to the **terminal side**. A counterclockwise rotation is associated with positive angle measure and a clockwise rotation is associated with negative angle measure. An angle can also be measured in **radians**.

Radian and Degree Measure	To rewrite the radian measure of an angle in degrees, multiply the number of radians by $\frac{180°}{\pi \text{ radians}}$. To rewrite the degree measure of an angle in radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180°}$.

Example 1 **Draw an angle with measure 290° in standard notation.**

The negative y-axis represents a positive rotation of 270°. To generate an angle of 290°, rotate the terminal side 20° more in the counterclockwise direction.



Example 2 **Rewrite the degree measure in radians and the radian measure in degrees.**

a. 45°

$45° = 45°\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{4}$ radians

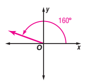
b. $\frac{5\pi}{3}$ radians

$\frac{5\pi}{3}$ radians $= \frac{5\pi}{3}\left(\frac{180°}{\pi}\right) = 300°$

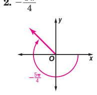
Exercises

Draw an angle with the given measure in standard position.

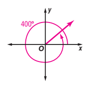
1. 160°



2. $-\frac{5\pi}{4}$



3. 400°



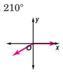
Rewrite each degree measure in radians and each radian measure in degrees.

4. 140° $\frac{7\pi}{9}$ 5. −860° $-\frac{43\pi}{9}$ 6. $-\frac{3\pi}{5}$ −108° 7. $\frac{11\pi}{3}$ 660°

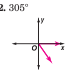
Skills Practice, p. 783 and Practice, p. 784 (shown)

Draw an angle with the given measure in standard position.

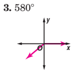
1. 210°



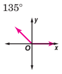
2. 305°



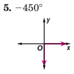
3. 580°



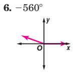
4. 135°



5. −450°



6. −560°



Rewrite each degree measure in radians and each radian measure in degrees.

7. 18° $\frac{\pi}{10}$ 8. 6° $\frac{\pi}{30}$ 9. 870° $\frac{29\pi}{6}$ 10. 347° $\frac{347\pi}{180}$

11. −72° $-\frac{2\pi}{5}$ 12. −820° $-\frac{41\pi}{9}$ 13. −250° $-\frac{25\pi}{18}$ 14. −165° $-\frac{11\pi}{12}$

15. 4π 720° 16. $\frac{5\pi}{2}$ 450° 17. $\frac{13\pi}{5}$ 468° 18. $\frac{13\pi}{30}$ 78°

19. $-\frac{9\pi}{2}$ −810° 20. $-\frac{7\pi}{12}$ −105° 21. $-\frac{3\pi}{8}$ −67.5° 22. $-\frac{3\pi}{16}$ −33.75°

Find one angle with positive measure and one angle with negative measure coterminal with each angle. 23–34. Sample answers are given.

23. 65° 425°, −295° 24. 80° 440°, −280° 25. 285° 645°, −75°

26. 110° 470°, −250° 27. −37° 323°, −397° 28. −93° 267°, −453°

29. $\frac{2\pi}{5}$ $\frac{12\pi}{5}$, $-\frac{8\pi}{5}$ 30. $\frac{5\pi}{6}$ $\frac{17\pi}{6}$, $-\frac{7\pi}{6}$ 31. $\frac{17\pi}{6}$ $\frac{29\pi}{6}$, $-\frac{7\pi}{6}$

32. $-\frac{3\pi}{2}$ $\frac{\pi}{2}$, $-\frac{7\pi}{2}$ 33. $-\frac{\pi}{4}$ $\frac{7\pi}{4}$, $-\frac{9\pi}{4}$ 34. $-\frac{5\pi}{12}$ $\frac{19\pi}{12}$, $-\frac{29\pi}{12}$

35. **TIME** Find both the degree and radian measures of the angle through which the hour hand on a clock rotates from 5 A.M. to 10 A.M. −150°; $-\frac{5\pi}{6}$

36. **ROTATION** A truck with 16-inch radius wheels is driven at 77 feet per second (52.5 miles per hour). Find the measure of the angle through which a point on the outside of the wheel travels each second. Round to the nearest degree and nearest radian. 3309°/s; 58 radians/s

Reading to Learn Mathematics, p. 785 ELL

Pre-Activity **How can angles be used to describe circular motion?**

Read the introduction to Lesson 13-2 at the top of page 709 in your textbook.

If a gondola revolves through a complete revolution in one minute, what is its angular velocity in degrees per second? 6° per second

Reading the Lesson

1. Match each degree measure with the corresponding radian measure on the right.

a. 30° v	i. $\frac{2\pi}{3}$
b. 90° ii	ii. $\frac{\pi}{2}$
c. 120° i	iii. $\frac{7\pi}{6}$
d. 135° vi	iv. π
e. 180° iv	v. $\frac{\pi}{6}$
f. 210° iii	vi. $\frac{3\pi}{4}$

2. The sine of 30° is $\frac{1}{2}$ and the sine of 150° is also $\frac{1}{2}$. Does this mean that 30° and 150° are coterminal angles? Explain your reasoning. Sample answer: No; the terminal side of a 30° angle is in Quadrant I, while the terminal side of a 150° angle is in Quadrant II.

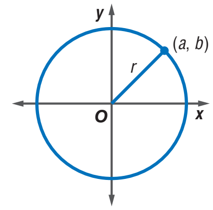
3. Describe how to find two angles that are coterminal with an angle of 155°, one with positive measure and one with negative measure. (Do not actually calculate these angles.) Sample answer: Positive angle: Add 360° to 155°. Negative angle: Subtract 360° from 155°.

4. Describe how to find two angles that are coterminal with an angle of $\frac{5\pi}{3}$, one positive and one negative. (Do not actually calculate these angles.) Sample answer: Positive angle: Add 2π to $\frac{5\pi}{3}$. Negative angle: Subtract 2π from $\frac{5\pi}{3}$.

Helping You Remember

5. How can you use what you know about the circumference of a circle to remember how to convert between radian and degree measure? Sample answer: The circumference of a circle is given by the formula $C = 2\pi r$, so the circumference of a circle with radius 1 is 2π. In degree measure, one complete circle is 360°. So 2π radians = 360° and π radians = 180°.

60. **CRITICAL THINKING** If (a, b) is on a circle that has radius r and center at the origin, prove that each of the following points is also on this circle.



a. $(a, -b)$ $a^2 + (-b)^2 = a^2 + b^2 = 1$

b. (b, a) $b^2 + a^2 = a^2 + b^2 = 1$

c. $(b, -a)$ $b^2 + (-a)^2 = a^2 + b^2 = 1$

61. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. See pp. 759A–759D.

How can angles be used to describe circular motion?

Include the following in your answer:

- an explanation of the significance of angles of more than 180° in terms of circular motion,
- an explanation of the significance of angles with negative measure in terms of circular motion, and
- an interpretation of a rate of more than 360° per minute.

Standardized Test Practice

62. **QUANTITATIVE COMPARISON** **Compare the quantity in Column A and the quantity in Column B. Then determine whether:**

Ⓐ **the quantity in Column A is greater,**

Ⓑ **the quantity in Column B is greater,**

Ⓒ **the two quantities are equal, or**

Ⓓ **the relationship cannot be determined from the information given.**

Column A	Column B
56°	$\frac{14\pi}{45}$

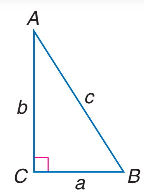
C

63. Angular velocity is defined by the equation $\omega = \frac{\theta}{t}$, where θ is usually expressed in radians and t represents time. Find the angular velocity in radians per second of a point on a bicycle tire if it completes 2 revolutions in 3 seconds. D

Ⓐ $\frac{\pi}{3}$ Ⓑ $\frac{\pi}{2}$ Ⓒ $\frac{2\pi}{3}$ Ⓓ $\frac{4\pi}{3}$

Maintain Your Skills

Mixed Review **Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.** *(Lesson 13-1)*



64. $A = 34°$, $b = 5$ 64. $a \approx 3.4$, $c \approx 6.0$, $B = 56°$

65. $B = 68°$, $b = 14.7$ 65. $A = 22°$, $a \approx 5.9$, $c \approx 15.9$

66. $B = 55°$, $c = 16$ $A = 35°$, $a \approx 9.2$, $b \approx 13.1$

67. $a = 0.4$, $b = 0.4\sqrt{3}$ $c \approx 0.8$, $A = 30°$, $B = 60°$

Find the margin of sampling error. *(Lesson 12-9)*

68. $p = 72\%$, $n = 100$ about 8.98%

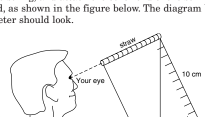
69. $p = 50\%$, $n = 200$ about 7.07%

Enrichment, p. 786

Making and Using a Hypsometer

A **hypsometer** is a device that can be used to measure the height of an object. To construct your own hypsometer, you will need a rectangular piece of heavy cardboard that is at least 7 cm by 10 cm, a straw, transparent tape, a string about 20 cm long, and a small weight that can be attached to the string.

Mark off 1-cm increments along one short side and one long side of the cardboard. Tape the straw to the other short side. Then attach the weight to one end of the string, and attach the other end of the string to one corner of the cardboard, as shown in the figure below. The diagram below shows how your hypsometer should look.



Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities. *(Lesson 12-2)*

70. permutation, 17,100,720

70. choosing an arrangement of 5 CDs from your 30 favorite CDs

71. choosing 3 different types of snack foods out of 7 at the store to take on a trip **combination, 35**

Find $[g \circ h](x)$ and $[h \circ g](x)$. *(Lesson 7-7)*

72. $[g \circ h](x) = 6x - 8$, $[h \circ g](x) = 6x - 4$

73. $[g \circ h](x) = 4x^2 - 6x + 23$, $[h \circ g](x) = 8x^2 + 34x + 44$

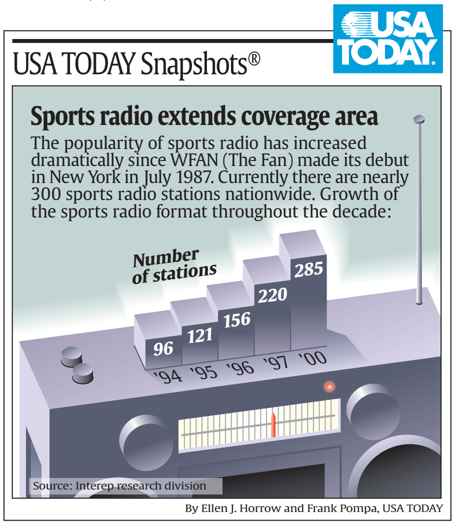
72. $g(x) = 2x$
$h(x) = 3x - 4$

73. $g(x) = 2x + 5$
$h(x) = 2x^2 - 3x + 9$

For Exercises 74 and 75, use the graph at the right.
The number of sports radio stations can be modeled by $R(x) = 7.8x^2 + 16.6x + 95.8$, where x is the number of years since 1996. *(Lesson 7-5)*

74. Use synthetic substitution to estimate the number of sports radio stations for 2006. **1041.8**

75. Evaluate $R(12)$. What does this value represent? **1418.2 or about 1418; the number of sports radio stations in 2008**



Getting Ready for the Next Lesson

PREREQUISITE SKILL Simplify each expression.
*(To review **rationalizing denominators**, see Lesson 5-6.)*

76. $\frac{2}{\sqrt{3}}$ $\frac{2\sqrt{3}}{3}$

77. $\frac{3}{\sqrt{5}}$ $\frac{3\sqrt{5}}{5}$

78. $\frac{4}{\sqrt{6}}$ $\frac{2\sqrt{6}}{3}$

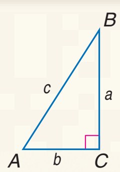
79. $\frac{5}{\sqrt{10}}$ $\frac{\sqrt{10}}{2}$

80. $\frac{\sqrt{7}}{\sqrt{2}}$ $\frac{\sqrt{14}}{2}$

81. $\frac{\sqrt{5}}{\sqrt{8}}$ $\frac{\sqrt{10}}{4}$

Practice Quiz 1 — Lessons 13-1 and 13-2

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-1)*

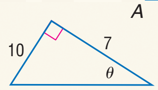


1. $A = 48°$, $b = 12$ $A = 42°$, $a \approx 13.3$, $c \approx 17.9$

2. $a = 18$, $c = 21$ $A \approx 59°$, $B \approx 31°$, $b \approx 10.8$

3. Draw an angle measuring $-60°$ in standard position. *(Lesson 13-1)* **See margin.**

4. Find the values of the six trigonometric functions for angle θ in the triangle at the right. *(Lesson 13-1)* **See margin.**



Rewrite each degree measure in radians and each radian measure in degrees. *(Lesson 13-2)*

5. $190°$ $\frac{19\pi}{18}$

6. $450°$ $\frac{5\pi}{2}$

7. $\frac{7\pi}{6}$ $210°$

8. $-\frac{11\pi}{5}$ $-396°$

Find one angle with positive measure and one angle with negative measure conterminal with each angle. *(Lesson 13-2)*

9. $-55°$ $305°$; $-415°$

10. $\frac{11\pi}{3}$ $\frac{5\pi}{3}$; $-\frac{\pi}{3}$

4 Assess

Open-Ended Assessment

Speaking Have students work in small groups to decide on an informal explanation of what a radian is and what coterminal angles are. Then have a reporter from each group share that explanation with the whole class.

Getting Ready for Lesson 13-3

PREREQUISITE SKILL Lesson 13-3 presents finding the trigonometric functions for general angles. Students will use their familiarity with rationalizing denominators as they find values of trigonometric functions. Exercises 76–81 should be used to determine your students' familiarity with rationalizing denominators.

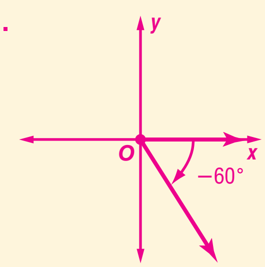
Assessment Options

Practice Quiz 1 The quiz provides students with a brief review of the concepts and skills in Lessons 13-1 and 13-2. Lesson numbers are given to the right of exercises or instruction lines so students can review concepts not yet mastered.

Quiz (Lessons 13-1 and 13-2) is available on p. 831 of the *Chapter 13 Resource Masters*.

Answers

3.



4. $\sin \theta = \frac{10\sqrt{149}}{149}$; $\cos \theta = \frac{7\sqrt{149}}{149}$; $\tan \theta = \frac{10}{7}$; $\csc \theta = \frac{\sqrt{149}}{10}$; $\sec \theta = \frac{\sqrt{149}}{7}$; $\cot \theta = \frac{7}{10}$

Online Lesson Plans

USA TODAY Education's Online site offers resources and interactive features connected to each day's newspaper. *Experience TODAY*, USA TODAY's daily lesson plan, is available on the site and delivered daily to subscribers. This plan provides instruction for integrating USA TODAY graphics and key editorial features into your mathematics classroom. Log on to www.education.usatoday.com.

Algebra Activity

A Follow-Up of Lesson 13-2

Getting Started

Objective Investigate measures in regular polygons using trigonometry.

Materials
compass
straightedge
protractor

Teach

- Have students copy the figure and use a colored pencil to show which line is the apothem.
- Make sure the students draw circles with a radius (not a diameter) of 1 inch.
- If it is available, you may want to have students use computer software to draw the inscribed regular polygons.

Assess

In **Exercises 1–3**, students should
- be able to see the pattern in the table.
- understand how the apothem changes as the number of sides in the polygon increases.

In **Exercises 4–7**, students should
- be able to develop the formula for the apothem.
- understand the effect of the length of the radius on the formula.

Algebra Activity

A Follow-Up of Lesson 13-2

Investigating Regular Polygons Using Trigonometry

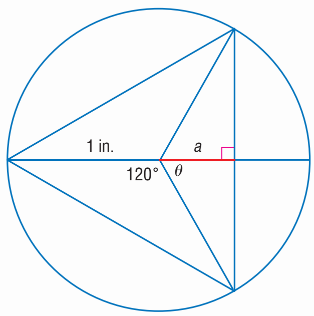
Collect the Data

- Use a compass to draw a circle with a radius of one inch. Inscribe an equilateral triangle inside of the circle. To do this, use a protractor to measure three angles of 120° at the center of the circle, since $\frac{360°}{3} = 120°$. Then connect the points where the sides of the angles intersect the circle using a straightedge.
- The **apothem** of a regular polygon is a segment that is drawn from the center of the polygon perpendicular to a side of the polygon. Use the cosine of angle θ to find the length of an apothem, labeled a in the diagram below.

Analyze the Data

1. Make a table like the one shown below and record the length of the apothem of the equilateral triangle.

Number of Sides, n	θ	a
3	60	**0.50**
4	45	**0.71**
5	**36**	**0.81**
6	**30**	**0.87**
7	**≈26**	**0.90**
8	**22.5**	**0.92**
9	**20**	**0.94**
10	**18**	**0.95**



Inscribe each regular polygon named in the table in a circle of radius one inch. Copy and complete the table.

2. What do you notice about the measure of θ as the number of sides of the inscribed polygon increases? **The measure of θ decreases.**
3. What do you notice about the values of a? **The length of the apothem increases as the number of sides increases.**

Make a Conjecture

4. Suppose you inscribe a 20-sided regular polygon inside a circle. Find the measure of angle θ. **9°**
5. Write a formula that gives the measure of angle θ for a polygon with n sides. **$\theta = 360 \div 2n$ or $\theta = 180 \div n$**
6. Write a formula that gives the length of the apothem of a regular polygon inscribed in a circle of radius one inch. **$a = \cos \theta$**
7. How would the formula you wrote in Exercise 6 change if the radius of the circle was not one inch? **See pp. 759A–759D.**

Resource Manager

Teaching Algebra with Manipulatives
- p. 23 (master for protractors)
- p. 301 (student recording sheet)

Glencoe Mathematics Classroom Manipulative Kit
- protractors
- rulers
- compasses

Study Notebook

You may wish to have students summarize this activity and what they learned from it.

13-3 Trigonometric Functions of General Angles

What You'll Learn

- Find values of trigonometric functions for general angles.
- Use reference angles to find values of trigonometric functions.

Vocabulary

- quadrantal angle
- reference angle

How can you model the position of riders on a skycoaster?

A skycoaster consists of a large arch from which two steel cables hang and are attached to riders suited together in a harness. A third cable, coming from a larger tower behind the arch, is attached with a ripcord. Riders are hoisted to the top of the larger tower, pull the ripcord, and then plunge toward Earth. They swing through the arch, reaching speeds of more than 60 miles per hour. After the first several swings of a certain skycoaster, the angle θ of the riders from the center of the arch is given by $\theta = 0.2 \cos (1.6t)$, where t is the time in seconds after leaving the bottom of their swing.

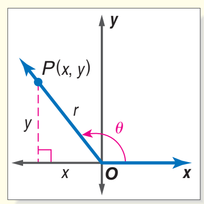
TRIGONOMETRIC FUNCTIONS AND GENERAL ANGLES In Lesson 13-1, you found values of trigonometric functions whose domains were the set of all acute angles, angles between 0 and $\frac{\pi}{2}$, of a right triangle. For $t > 0$ in the equation above, you must find the cosine of an angle greater than $\frac{\pi}{2}$. In this lesson, we will extend the domain of trigonometric functions to include angles of *any* measure.

Key Concept — *Trigonometric Functions, θ in Standard Position*

Let θ be an angle in standard position and let $P(x, y)$ be a point on the terminal side of θ. Using the Pythagorean Theorem, the distance r from the origin to P is given by $r = \sqrt{x^2 + y^2}$. The trigonometric functions of an angle in standard position may be defined as follows.

$\sin \theta = \frac{y}{r}$ $\quad$ $\cos \theta = \frac{x}{r}$ $\quad$ $\tan \theta = \frac{y}{x}, x \neq 0$

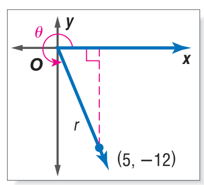
$\csc \theta = \frac{r}{y}, y \neq 0$ $\quad$ $\sec \theta = \frac{r}{x}, x \neq 0$ $\quad$ $\cot \theta = \frac{x}{y}, y \neq 0$



Example 1 *Evaluate Trigonometric Functions for a Given Point*

Find the exact values of the six trigonometric functions of θ if the terminal side of θ contains the point (5, −12).

From the coordinates given, you know that $x = 5$ and $y = -12$. Use the Pythagorean Theorem to find r.



(continued on the next page)

13-3 Lesson Notes

1 Focus

5-Minute Check Transparency 13-3 Use as a quiz or review of Lesson 13-2.

Mathematical Background notes are available for this lesson on p. 698C.

How can you model the position of riders on a skycoaster?

Ask students:

- What happens to the size of the swing (as measured by θ) as the time after the plunge increases? **The arc of the swing decreases, eventually to zero.**
- Is the t in the formula the same as the time that has elapsed since the rider plunged? **no**

Resource Manager

Workbook and Reproducible Masters

Chapter 13 Resource Masters

- Study Guide and Intervention, pp. 787–788
- Skills Practice, p. 789
- Practice, p. 790
- Reading to Learn Mathematics, p. 791
- Enrichment, p. 792

Transparencies

5-Minute Check Transparency 13-3
Real-World Transparency 13
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lessons 24 and 25
Interactive Chalkboard

2 Teach

TRIGONOMETRIC FUNCTIONS AND GENERAL ANGLES

In-Class Examples

Power Point®

1 Find the exact values of the six trigonometric functions of θ if the terminal side of θ contains the point $(8, -15)$.

$\sin\theta = -\frac{15}{17}$; $\cos\theta = \frac{8}{17}$;
$\tan\theta = -\frac{15}{8}$; $\csc\theta = -\frac{17}{15}$;
$\sec\theta = \frac{17}{8}$; $\cot\theta = -\frac{8}{15}$

Teaching Tip When finding functions for an angle, students may find it helpful to sketch an angle in standard position and drop a perpendicular to the x-axis to form a right triangle. Then they can see that, as the angle increases from 90° to 180°, the y value approaches 0, and the values of the other two sides approach each other.

2 Find the values of the six trigonometric functions for an angle in standard position that measures 180°.

$\sin\theta = 0$; $\cos\theta = -1$;
$\tan\theta = 0$; $\csc\theta$ is undefined;
$\sec\theta = -1$; $\cot\theta$ is undefined.

$r = \sqrt{x^2 + y^2}$ Pythagorean Theorem

$= \sqrt{5^2 + (-12)^2}$ Replace x with 5 and y with -12.

$= \sqrt{169}$ or 13 Simplify.

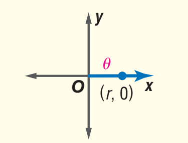
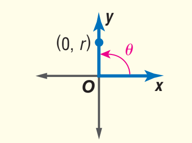
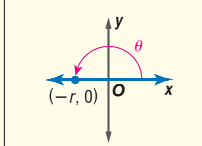
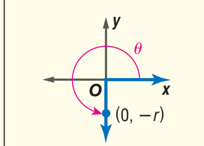
Now, use $x = 5$, $y = -12$, and $r = 13$ to write the ratios.

$\sin\theta = \frac{y}{r} = \frac{-12}{13}$ or $-\frac{12}{13}$ $\quad \cos\theta = \frac{x}{r} = \frac{5}{13}$ $\quad \tan\theta = \frac{y}{x} = \frac{-12}{5}$ or $-\frac{12}{5}$

$\csc\theta = \frac{r}{y} = \frac{13}{-12}$ or $-\frac{13}{12}$ $\quad \sec\theta = \frac{r}{x} = \frac{13}{5}$ $\quad \cot\theta = \frac{x}{y} = \frac{5}{-12}$ or $-\frac{5}{12}$

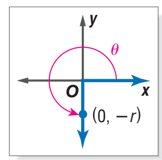
If the terminal side of angle θ lies on one of the axes, θ is called a **quadrantal angle**. The quadrantal angles are 0°, 90°, 180°, and 270°. Notice that for these angles either x or y is equal to 0. Since division by zero is undefined, two of the trigonometric values are undefined for each quadrantal angle.

Key Concept — Quadrantal Angles

$\theta = 0°$ or 0 radians	$\theta = 90°$ or $\frac{\pi}{2}$	$\theta = 180°$ or π radians	$\theta = 270°$ or $\frac{3\pi}{2}$ radians
			

Example 2 Quadrantal Angles

Find the values of the six trigonometric functions for an angle in standard position that measures 270°.

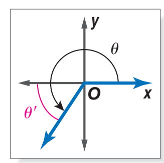


When $\theta = 270°$, $x = 0$ and $y = -r$.

$\sin\theta = \frac{y}{r} = \frac{-r}{r}$ or -1 $\quad \cos\theta = \frac{x}{r} = \frac{0}{r}$ or 0 $\quad \tan\theta = \frac{y}{x} = \frac{-r}{0}$ or undefined

$\csc\theta = \frac{r}{y} = \frac{r}{-r}$ or -1 $\quad \sec\theta = \frac{r}{x} = \frac{r}{0}$ or undefined $\quad \cot\theta = \frac{x}{y} = \frac{0}{-r}$ or 0

REFERENCE ANGLES To find the values of trigonometric functions of angles greater than 90° (or less than 0°), you need to know how to find the measures of reference angles. If θ is a nonquadrantal angle in standard position, its **reference angle**, θ', is defined as the acute angle formed by the terminal side of θ and the x-axis.



Study Tip

Reading Math
θ' is read *theta prime.*

DAILY INTERVENTION

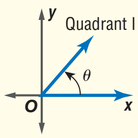
Unlocking Misconceptions

In Example 1, watch for students who think that r must be a negative number any time either x or y has a negative value.

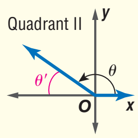
You can use the rule below to find the reference angle for any nonquadrantal angle θ where $0° < \theta < 360°$ (or $0 < \theta < 2\pi$).

Key Concept — Reference Angle Rule

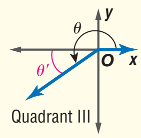
For any nonquadrantal angle θ, $0° < \theta < 360°$ (or $0 < \theta < 2\pi$), its reference angle θ' is defined as follows.



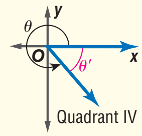
$\theta' = \theta$



$\theta' = 180° - \theta$
$(\theta' = \pi - \theta)$



$\theta' = \theta - 180°$
$(\theta' = \theta - \pi)$



$\theta' = 360° - \theta$
$(\theta' = 2\pi - \theta)$

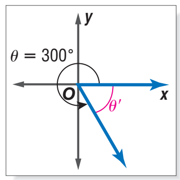
If the measure of θ is greater than 360° or less than 0°, its reference angle can be found by associating it with a coterminal angle of positive measure between 0° and 360°.

Example 3 Find the Reference Angle for a Given Angle

Sketch each angle. Then find its reference angle.

a. 300°

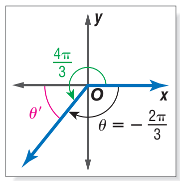
Because the terminal side of 300° lies in Quadrant IV, the reference angle is $360° - 300°$ or 60°.



b. $-\frac{2\pi}{3}$

A coterminal angle of $-\frac{2\pi}{3}$ is $2\pi - \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

Because the terminal side of this angle lies in Quadrant III, the reference angle is $\frac{4\pi}{3} - \pi$ or $\frac{\pi}{3}$.



To use the reference angle θ' to find a trigonometric value of θ, you need to know the sign of that function for an angle θ. From the function definitions, these signs are determined by x and y, since r is always positive. Thus, the sign of each trigonometric function is determined by the quadrant in which the terminal side of θ lies.

The chart below summarizes the signs of the trigonometric functions for each quadrant.

	Quadrant			
Function	**I**	**II**	**III**	**IV**
$\sin\theta$ or $\csc\theta$	+	+	−	−
$\cos\theta$ or $\sec\theta$	+	−	−	+
$\tan\theta$ or $\cot\theta$	+	−	+	−

TEACHING TIP

For example, because $\sin\theta = \frac{y}{r}$ and r is always positive, $\sin\theta$ is positive wherever $y > 0$, which is in Quadrants I and II.

www.algebra2.com/extra_examples

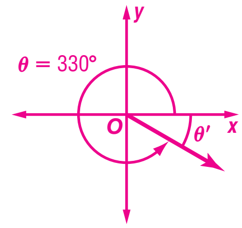
REFERENCE ANGLES

In-Class Example

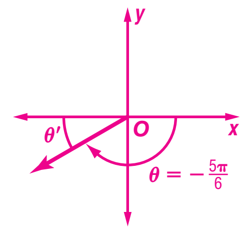
Power Point®

3 Sketch each angle. Then find its reference angle.

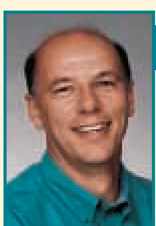
a. 330° **30°**



b. $-\frac{5\pi}{6}$ $\frac{\pi}{6}$



Teacher to Teacher



David S. Daniels — Longmeadow H.S., Longmeadow, MA

"Have students measure the distance from the tip of the minute hand on the classroom clock to the ceiling at 5 minute or 30° intervals, starting at the top of an hour. Then have students plot the data and describe the graph. Students will discover that the distance from the ceiling is a function of the number of degrees through which the minute hand has turned."

In-Class Examples

Power Point®

4 Find the exact value of each trigonometric function.

a. sin 135° $\frac{\sqrt{2}}{2}$

b. cot $\frac{7\pi}{3}$ $\frac{\sqrt{3}}{3}$

5 Suppose θ is an angle in standard position whose terminal side is in Quadrant III and $\csc\theta = -\frac{5}{3}$. Find the exact values of the remaining five trigonometric functions of θ.

$\sin\theta = -\frac{3}{5}$; $\cos\theta = -\frac{4}{5}$;

$\tan\theta = \frac{3}{4}$; $\sec\theta = -\frac{5}{4}$;

$\cot\theta = \frac{4}{3}$

Use the following steps to find the value of a trigonometric function of any angle θ.

Step 1 Find the reference angle θ'.

Step 2 Find the value of the trigonometric function for θ'.

Step 3 Using the quadrant in which the terminal side of θ lies, determine the sign of the trigonometric function value of θ.

Study Tip

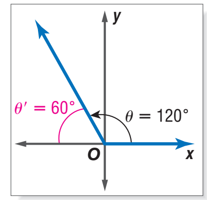
Look Back
To review **trigonometric values of angles measuring 30°, 45°, and 60°**, see Lesson 13-1.

Example 4 Use a Reference Angle to Find a Trigonometric Value

Find the exact value of each trigonometric function.

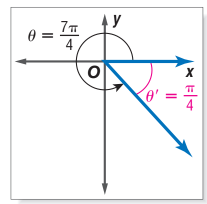
a. sin 120°

Because the terminal side of 120° lies in Quadrant II, the reference angle θ' is 180° − 120° or 60°. The sine function is positive in Quadrant II, so sin 120° = sin 60° or $\frac{\sqrt{3}}{2}$.



b. $\cot\frac{7\pi}{4}$

Because the terminal side of $\frac{7\pi}{4}$ lies in Quadrant IV, the reference angle θ' is $2\pi - \frac{7\pi}{4}$ or $\frac{\pi}{4}$. The cotangent function is negative in Quadrant IV.



$$\cot\frac{7\pi}{4} = -\cot\frac{\pi}{4}$$

$= -\cot 45°$ $\frac{\pi}{4}$ radians = 45°

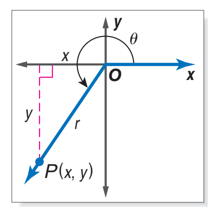
$= -1$ cot 45° = 1

If you know the quadrant that contains the terminal side of θ in standard position and the exact value of one trigonometric function of θ, you can find the values of the other trigonometric functions of θ using the function definitions.

Example 5 Quadrant and One Trigonometric Value of θ

Suppose θ is an angle in standard position whose terminal side is in the Quadrant III and $\sec\theta = -\frac{4}{3}$. Find the exact values of the remaining five trigonometric functions of θ.

Draw a diagram of this angle, labeling a point $P(x, y)$ on the terminal side of θ. Use the definition of secant to find the values of x and r.



$\sec\theta = -\frac{4}{3}$ Given

$\frac{r}{x} = -\frac{4}{3}$ Definition of secant

Since x is negative in Quadrant III and r is always positive, $x = -3$ and $r = 4$. Use these values and the Pythagorean Theorem to find y.

DAILY INTERVENTION

Differentiated Instruction

Auditory/Musical Have students work in small groups to create a jingle, song, rap, or short poem to help them remember the basic equivalences between angle measures in degrees and radians.

720 Chapter 13 Trigonometric Functions

$$x^2 + y^2 = r^2 \quad \text{Pythagorean Theorem}$$
$$(-3)^2 + y^2 = 4^2 \quad \text{Replace } x \text{ with } -3 \text{ and } r \text{ with } 4.$$
$$y^2 = 16 - 9 \quad \text{Simplify. Then subtract 9 from each side.}$$
$$y = \pm\sqrt{7} \quad \text{Simplify. Then take the square root of each side.}$$
$$y = -\sqrt{7} \quad y \text{ is negative in Quadrant III.}$$

Use $x = -3$, $y = -\sqrt{7}$, and $r = 4$ to write the remaining trigonometric ratios.

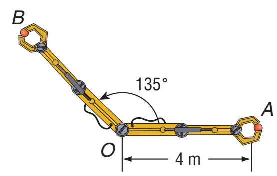
$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{7}}{4} \qquad \cos\theta = \frac{x}{r} = \frac{-3}{4} \qquad \tan\theta = \frac{y}{x} = \frac{-\sqrt{7}}{-3} \text{ or } \frac{\sqrt{7}}{3}$$

$$\csc\theta = \frac{r}{y} = -\frac{4}{\sqrt{7}} \text{ or } -\frac{4\sqrt{7}}{7} \qquad \cot\theta = \frac{x}{y} = \frac{-3}{-\sqrt{7}} \text{ or } \frac{3\sqrt{7}}{7}$$

Just as an exact point on the terminal side of an angle can be used to find trigonometric function values, trigonometric function values can be used to find the exact coordinates of a point on the terminal side of an angle.

Example 6 Find Coordinates Given a Radius and an Angle

ROBOTICS **In a robotics competition, a robotic arm 4 meters long is to pick up an object at point A and release it into a container at point B. The robot's owner programs the arm to rotate through an angle of precisely 135° to accomplish this task. What is the new position of the object relative to the pivot point O?**



With the pivot point at the origin and the angle through which the arm rotates in standard position, point A has coordinates (0, 4). The reference angle θ' for 135° is 180° − 135° or 45°.

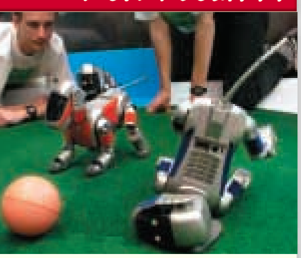
Let the position of point B have coordinates (x, y). Then, use the definitions of sine and cosine to find the value of x and y. The value of r is the length of the robotic arm, 4 meters. Because B is in Quadrant II, the cosine of 135° is negative.

$$\cos 135° = \frac{x}{r} \quad \text{cosine ratio}$$
$$-\cos 45° = \frac{x}{4} \quad 180° - 135° = 45°$$
$$-\frac{\sqrt{2}}{2} = \frac{x}{4} \quad \cos 45° = \frac{\sqrt{2}}{2}$$
$$-2\sqrt{2} = x \quad \text{Solve for } x.$$

$$\sin 135° = \frac{y}{r} \quad \text{sine ratio}$$
$$\sin 45° = \frac{y}{4} \quad 180° - 35° = 45°$$
$$\frac{\sqrt{2}}{2} = \frac{y}{4} \quad \sin 45° = \frac{\sqrt{2}}{2}$$
$$2\sqrt{2} = y \quad \text{Solve for } y.$$

The exact coordinates of B are $(-2\sqrt{2}, 2\sqrt{2})$. Since $2\sqrt{2}$ is about 2.82, the object is about 2.82 meters to the left of the pivot point and about 2.82 meters in front of the pivot point.

More About. . .



Robotics

RoboCup is an annual event in which teams from all over the world compete in a series of soccer matches in various classes according to the size and intellectual capacity of their robot. The robots are programmed to react to the ball and communicate with each other.

Source: www.robocup.org

In-Class Example

Power Point®

6 **ROBOTICS** Use the figure for Example 6 in the Student Edition to find the new position of the object relative to the pivot point for a robotic arm that is 3 meters long and that rotates through an angle of 150°.

$\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ **or about 2.60 meters to the left of *O* and 1.5 meters in front of *O***

Lesson 13-3 Trigonometric Functions of General Angles 721

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 13.
- add diagrams explaining quadrantal and reference angles.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective
- **Trigonometric Functions and General Angles:** 17–24
- **Reference Angles:** 25–52

Odd/Even Assignments
Exercises 17–52 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 17–21 odd, 25–51 odd, 56–77

Average: 17–51 odd, 55, 56–77

Advanced: 18–52 even, 53, 54, 56–71 (optional: 72–77)

Check for Understanding

Concept Check

1. False; sec 0° = $\frac{r}{r}$ or 1 and tan 0° = $\frac{0}{r}$ or 0.
2. Sample answer: 190°

1. **Determine** whether the following statement is *true* or *false*. If true, explain your reasoning. If false, give a counterexample.
The values of the secant and tangent functions for any quadrantal angle are undefined.
2. **OPEN ENDED** Give an example of an angle whose sine is negative.
3. **Explain** how the reference angle θ' is used to find the value of a trigonometric function of θ, where θ is greater than 90°. **See margin.**

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4–6	1
7–9	3
10–13	2, 4
14–16	5

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point. **4–6. See margin.**

4. $(-15, 8)$
5. $(-3, 0)$
6. $(4, 4)$

Sketch each angle. Then find its reference angle.

7. 235° **55°**
8. $\frac{7\pi}{4}$ **$\frac{\pi}{4}$**
9. −240° **60°**

7–9. See pp. 759A–759D for sketches.

Find the exact value of each trigonometric function.

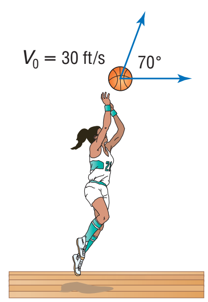
10. sin 300° **$-\frac{\sqrt{3}}{2}$**
11. cos 180° **−1**
12. $\tan \frac{5\pi}{3}$ **$-\sqrt{3}$**
13. $\sec \frac{7\pi}{6}$ **$-\frac{2\sqrt{3}}{3}$**

Suppose θ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ. **14–15. See margin.**

14. $\cos \theta = -\frac{1}{2}$, Quadrant II
15. $\cot \theta = -\frac{\sqrt{2}}{2}$, Quadrant IV

Application

16. **BASKETBALL** The maximum height H in feet that a basketball reaches after being shot is given by the formula $H = \frac{V_0^2 (\sin \theta)^2}{64}$, where V_0 represents the initial velocity in feet per second, θ represents the degree measure of the angle that the path of the basketball makes with the ground. Find the maximum height reached by a ball shot with an initial velocity of 30 feet per second at an angle of 70°. **about 12.4 ft**



★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
17–24	1
25–32	3
33–46	2, 4
47–52	5
53–55	6

Extra Practice

See page 857.

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point. **17–24. See pp. 759A–759D.**

17. $(7, 24)$
18. $(2, 1)$
19. $(5, -8)$
20. $(4, -3)$
21. $(0, -6)$
22. $(-1, 0)$
★ 23. $(\sqrt{2}, -\sqrt{2})$
★ 24. $(-\sqrt{3}, -\sqrt{6})$

25–32. See pp. 759A–759D for sketches.

Sketch each angle. Then find its reference angle.

25. 315° **45°**
26. 240° **60°**
27. −210° **30°**
28. −125° **55°**
29. $\frac{5\pi}{4}$ **$\frac{\pi}{4}$**
30. $\frac{5\pi}{6}$ **$\frac{\pi}{6}$**
31. $\frac{13\pi}{7}$ **$\frac{\pi}{7}$**
32. $-\frac{2\pi}{3}$ **$\frac{\pi}{3}$**

Answers

3. To find the value of a trigonometric function of θ, where θ is greater than 90°, find the value of the trigonometric function for θ', then use the quadrant in which the terminal side of θ lies to determine the sign of the trigonometric function value of θ.

4. $\sin \theta = \frac{8}{17}$, $\cos \theta = -\frac{15}{17}$, $\tan \theta = -\frac{8}{15}$, $\csc \theta = \frac{17}{8}$, $\sec \theta = -\frac{17}{15}$, $\cot \theta = -\frac{15}{8}$

5. $\sin \theta = 0$, $\cos \theta = -1$, $\tan \theta = 0$, $\csc \theta$ = undefined, $\sec \theta = -1$, $\cot \theta$ = undefined

6. $\sin \theta = \frac{\sqrt{2}}{2}$, $\cos \theta = \frac{\sqrt{2}}{2}$, $\tan \theta = 1$, $\csc \theta = \sqrt{2}$, $\sec \theta = \sqrt{2}$, $\cot \theta = 1$

14. $\sin \theta = \frac{\sqrt{3}}{2}$, $\tan \theta = -\sqrt{3}$, $\csc \theta = \frac{2\sqrt{3}}{3}$, $\sec \theta = -2$, $\cot \theta = -\frac{\sqrt{3}}{3}$

15. $\sin \theta = -\frac{\sqrt{6}}{3}$, $\cos \theta = \frac{\sqrt{3}}{3}$, $\tan \theta = -\sqrt{2}$, $\csc \theta = -\frac{\sqrt{6}}{2}$, $\sec \theta = \sqrt{3}$

Find the exact value of each trigonometric function.

33. $\sin 240°$ $-\frac{\sqrt{3}}{2}$
34. $\sec 120°$ -2
35. $\tan 300°$ $-\sqrt{3}$
36. $\cot 510°$ $-\sqrt{3}$
37. $\csc 5400°$
38. $\cos \frac{11\pi}{3}$ $\frac{1}{2}$
39. $\cot\left(-\frac{5\pi}{6}\right)$ $\sqrt{3}$
40. $\sin \frac{3\pi}{4}$ $\frac{\sqrt{2}}{2}$
41. $\sec \frac{3\pi}{2}$ undefined
42. $\csc \frac{17\pi}{6}$ 2
43. $\cos(-30°)$ $\frac{\sqrt{3}}{2}$
44. $\tan\left(-\frac{5\pi}{4}\right)$ -1

45. **SKYCOASTING** Mikhail and Anya visit a local amusement park to ride a skycoaster. After the first several swings, the angle the skycoaster makes with the vertical is modeled by $\theta = 0.2 \cos \pi t$, with θ measured in radians and t measured in seconds. Determine the measure of the angle for $t = 0, 0.5, 1, 1.5, 2, 2.5$, and 3 in both radians and degrees. **0.2, 0, −0.2, 0, 0.2, 0, and −0.2; or about 11.5°, 0°, −11.5°, 0°, 11.5°, 0°, and −11.5°**

46. **NAVIGATION** Ships and airplanes measure distance in nautical miles. The formula 1 nautical mile $= 6077 - 31 \cos 2\theta$ feet, where θ is the latitude in degrees, can be used to find the approximate length of a nautical mile at a certain latitude. Find the length of a nautical mile where the latitude is 60°. **6092.5 ft**

Suppose θ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ. **50–52. See pp. 759A–759D.**

47. $\cos\theta = \frac{3}{5}$, Quadrant IV
48. $\tan\theta = -\frac{1}{5}$, Quadrant II
49. $\sin\theta = \frac{1}{3}$, Quadrant II
50. $\cot\theta = \frac{1}{2}$, Quadrant III
51. $\sec\theta = -\sqrt{10}$, Quadrant III
52. $\csc\theta = -5$, Quadrant IV

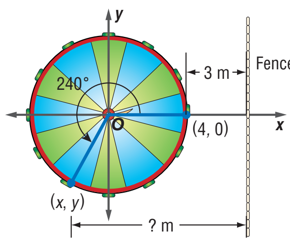
BASEBALL **For Exercises 53 and 54, use the following information.**

The formula $R = \frac{V_0^2 \sin 2\theta}{32}$ gives the distance of a baseball that is hit at an initial velocity of V_0 feet per second at an angle of θ with the ground.

53. If the ball was hit with an initial velocity of 80 feet per second at an angle of 30°, how far was it hit? **about 173.2 ft**

54. Which angle will result in the greatest distance? Explain your reasoning. **45°; $2 \times 45°$ or 90° yields the greatest value for $\sin 2\theta$.**

55. **CAROUSELS** Anthony's little brother gets on a carousel that is 8 meters in diameter. At the start of the ride, his brother is 3 meters from the fence to the ride. How far will his brother be from the fence after the carousel rotates 240°? **9 meters**



Online Research **Data Update** What is the diameter of the world's largest carousel? Visit www.algebra2.com/data_update to learn more.

CRITICAL THINKING **Suppose θ is an angle in standard position with the given conditions. State the quadrant(s) in which the terminal side of θ lies.**

56. $\sin\theta > 0$ **I, II**
57. $\sin\theta > 0, \cos\theta < 0$ **II**
58. $\tan\theta > 0, \cos\theta < 0$ **III**

37. **undefined**

47. $\sin\theta = -\frac{4}{5}$, $\tan\theta = -\frac{4}{3}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

48. $\sin\theta = \frac{\sqrt{26}}{26}$, $\cos\theta = -\frac{5\sqrt{26}}{26}$, $\csc\theta = \sqrt{26}$, $\sec\theta = -\frac{\sqrt{26}}{5}$, $\cot\theta = -5$

49. $\cos\theta = -\frac{2\sqrt{2}}{3}$, $\tan\theta = -\frac{\sqrt{2}}{4}$, $\csc\theta = 3$, $\sec\theta = -\frac{3\sqrt{2}}{4}$, $\cot\theta = -2\sqrt{2}$

More About. . .

Baseball

If a major league pitcher throws a pitch at 95 miles per hour, it takes only about 4 tenths of a second for the ball to travel the 60 feet, 6 inches from the pitcher's mound to home plate. In that time, the hitter must decide whether to swing at the ball and if so, when to swing.

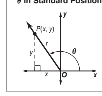
Source: www.exploratorium.edu

www.algebra2.com/self_check_quiz

Study Guide and Intervention, p. 787 (shown) and p. 788

Trigonometric Functions and General Angles

Trigonometric Functions, θ in Standard Position



Let θ be an angle in standard position and let $P(x, y)$ be a point on the terminal side of θ. By the Pythagorean Theorem, the distance r from the origin is given by $r = \sqrt{x^2 + y^2}$. The trigonometric functions of an angle in standard position may be defined as follows.

$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

$\csc\theta = \frac{r}{y}$ $\sec\theta = \frac{r}{x}$ $\cot\theta = \frac{x}{y}$

Example **Find the exact values of the six trigonometric functions of θ if the terminal side of θ contains the point $(-5, 5\sqrt{2})$.**

You know that $x = -5$ and $y = 5\sqrt{2}$. You need to find r.

$r = \sqrt{x^2 + y^2}$ Pythagorean Theorem

$= \sqrt{(-5)^2 + (5\sqrt{2})^2}$ Replace x with −5 and y with $5\sqrt{2}$.

$= \sqrt{75}$ or $5\sqrt{3}$

Now use $x = -5$, $y = 5\sqrt{2}$, and $r = 5\sqrt{3}$ to write the ratios.

$\sin\theta = \frac{y}{r} = \frac{5\sqrt{2}}{5\sqrt{3}} = \frac{\sqrt{6}}{3}$ $\cos\theta = \frac{x}{r} = \frac{-5}{5\sqrt{3}} = -\frac{\sqrt{3}}{3}$ $\tan\theta = \frac{y}{x} = \frac{5\sqrt{2}}{-5} = -\sqrt{2}$

$\csc\theta = \frac{r}{y} = \frac{5\sqrt{3}}{5\sqrt{2}} = \frac{\sqrt{6}}{2}$ $\sec\theta = \frac{r}{x} = \frac{5\sqrt{3}}{-5} = -\sqrt{3}$ $\cot\theta = \frac{x}{y} = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2}$

Exercises

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

1. (8, 4) $\sin\theta = \frac{\sqrt{5}}{5}$, $\cos\theta = \frac{2\sqrt{5}}{5}$, $\tan\theta = \frac{1}{2}$, $\csc\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$, $\cot\theta = 2$
2. $(4, 4\sqrt{3})$ $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$, $\tan\theta = \sqrt{3}$, $\csc\theta = \frac{2\sqrt{3}}{3}$, $\sec\theta = 2$, $\cot\theta = \frac{\sqrt{3}}{3}$
3. (0, −4) $\sin\theta = -1$, $\cos\theta = 0$, $\tan\theta$ undefined, $\csc\theta = -1$, $\sec\theta$ undefined, $\cot\theta = 0$
4. (6, 2) $\sin\theta = \frac{\sqrt{10}}{10}$, $\cos\theta = \frac{3\sqrt{10}}{10}$, $\tan\theta = \frac{1}{3}$, $\csc\theta = \sqrt{10}$, $\sec\theta = \frac{\sqrt{10}}{3}$, $\cot\theta = 3$

Skills Practice, p. 789 and Practice, p. 790 (shown)

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

1. (6, 8) $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\csc\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = \frac{3}{4}$
2. (−20, 21) $\sin\theta = \frac{21}{29}$, $\cos\theta = -\frac{20}{29}$, $\tan\theta = -\frac{21}{20}$, $\csc\theta = \frac{29}{21}$, $\sec\theta = -\frac{29}{20}$, $\cot\theta = -\frac{20}{21}$
3. (−2, −5) $\sin\theta = -\frac{5\sqrt{29}}{29}$, $\cos\theta = -\frac{2\sqrt{29}}{29}$, $\tan\theta = \frac{5}{2}$, $\csc\theta = -\frac{\sqrt{29}}{5}$, $\sec\theta = -\frac{\sqrt{29}}{2}$, $\cot\theta = \frac{2}{5}$

Find the reference angle for the angle with the given measure.

4. 236° **56°**
5. $\frac{13\pi}{8}$ $\frac{3\pi}{8}$
6. −210° **30°**
7. $-\frac{7\pi}{4}$ $\frac{\pi}{4}$

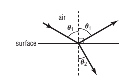
Find the exact value of each trigonometric function.

8. $\tan 135°$ -1
9. $\cot 210°$ $\sqrt{3}$
10. $\cot(-90°)$ 0
11. $\cos 405°$ $\frac{\sqrt{2}}{2}$
12. $\tan\frac{5\pi}{3}$ $-\sqrt{3}$
13. $\csc\left(-\frac{3\pi}{4}\right)$ $-\sqrt{2}$
14. $\cot 2\pi$ undefined
15. $\tan\frac{13\pi}{6}$ $\frac{\sqrt{3}}{3}$

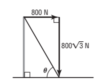
Suppose θ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

16. $\tan\theta = -\frac{12}{5}$, Quadrant IV $\sin\theta = -\frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\csc\theta = -\frac{13}{12}$, $\sec\theta = \frac{13}{5}$, $\cot\theta = -\frac{5}{12}$
17. $\sin\theta = \frac{2}{3}$, Quadrant III $\cos\theta = -\frac{\sqrt{5}}{3}$, $\tan\theta = -\frac{2\sqrt{5}}{5}$, $\csc\theta = \frac{3}{2}$, $\sec\theta = -\frac{3\sqrt{5}}{5}$, $\cot\theta = -\frac{\sqrt{5}}{2}$

18. **LIGHT** Light rays that "bounce off" a surface are *reflected* by the surface. If the surface is partially transparent, some of the light rays are bent or *refracted* as they pass from the air through the material. The angles of reflection θ_1 and of refraction θ_2 in the diagram at the right are related by the equation $\sin\theta_1 = n \sin\theta_2$. If $\theta_1 = 60°$ and $n = \sqrt{3}$, find the measure of θ_2. **30°**



19. **FORCE** A cable running from the top of a utility pole to the ground exerts a horizontal pull of 800 Newtons and a vertical pull of $800\sqrt{3}$ Newtons. What is the sine of the angle θ between the cable and the ground? What is the measure of this angle? $\frac{\sqrt{3}}{2}$; **60°**



Reading to Learn Mathematics, p. 791 ELL

Pre-Activity **How can you model the position of riders on a skycoaster?**

Read the introduction to Lesson 13-3 at the top of page 717 in your textbook.

- What does $t = 0$ represent in this application? **Sample answer: the time when the riders leave the bottom of their swing**
- Do negative values of t make sense in this application? Explain your answer. **Sample answer: No; $t = 0$ represents the starting time, so the value of t cannot be less than 0.**

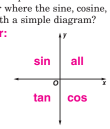
Reading the Lesson

1. Suppose θ is an angle in standard position, $P(x, y)$ is a point on the terminal side of θ, and the distance from the origin to P is r. Determine whether each of the following statements is *true* or *false*.
   a. The value of r can be found by using either the Pythagorean Theorem or the distance formula. **true**
   b. $\cos\theta = \frac{x}{r}$ **true**
   c. $\csc\theta$ is defined if $y \neq 0$. **true**
   d. $\tan\theta$ is undefined if $y = 0$. **false**
   e. $\sin\theta$ is defined for every value of θ. **true**

2. Let θ be an angle measured in degrees. Match the quadrant of θ from the first column with the description of how to find the reference angle for θ from the second column.
   a. Quadrant III **ii** — i. Subtract θ from 360°.
   b. Quadrant IV **i** — ii. Subtract 180° from θ.
   c. Quadrant II **iv** — iii. θ is its own reference angle.
   d. Quadrant I **iii** — iv. Subtract θ from 180°.

Helping You Remember

3. The chart on page 719 in your textbook summarizes the signs of the six trigonometric functions in the four quadrants. Since reciprocals always have the same sign, you only need to remember where the sine, cosine, and tangent are positive. How can you remember this with a simple diagram?
   Sample answer:



Enrichment, p. 792

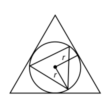
Areas of Polygons and Circles

A regular polygon has sides of equal length and angles of equal measure. A regular polygon can be inscribed in or circumscribed about a circle. For n-sided regular polygons, the following area formulas can be used.

Area of circle $A_C = \pi r^2$

Area of inscribed polygon $A_I = \frac{nr^2}{2} \times \sin\frac{360°}{n}$

Area of circumscribed polygon $A_C = nr^2 \times \tan\frac{180°}{n}$



Use a calculator to complete the chart below for a unit circle (a circle of radius 1).

	Area of ...ribed	Area of Circle	Area of	Area of Polygon

4 Assess

Open-Ended Assessment

Writing Have students choose an angle measure, draw an angle with that measure in standard position, find the reference angle, and give the values of all 6 of the trigonometric functions, in both degrees and radians. Then display some of these in the classroom.

Getting Ready for Lesson 13-4

PREREQUISITE SKILL Lesson 13-4 presents the Law of Sines which will require students to solve equations involving trigonometric functions as they apply the Law of Sines. Exercises 72–77 should be used to determine your students' familiarity with solving equations with trigonometric functions.

Answer

59. Answers should include the following.

- **The cosine of any angle is defined as $\frac{x}{r}$, where x is the x-coordinate of any point on the terminal ray of the angle and r is the distance from the origin to that point. This means that for angles with terminal sides to the left of the y-axis, the cosine is negative, and those with terminal sides to the right of the y-axis, the cosine is positive. Therefore the cosine function can be used to model real-world data that oscillate between being positive and negative.**
- **If we knew the length of the cable we could find the vertical distance from the top of the tower to the rider. Then if we knew the height of the tower we could subtract from it the vertical distance calculated previously. This will leave the height of the rider from the ground.**

59. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. **See margin.**

How can you model the position of riders on a skycoaster?

Include the following in your answer:

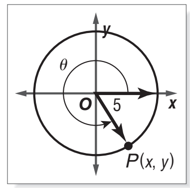
- an explanation of how you could use the cosine of the angle θ and the length of the cable from which they swing to find the horizontal position of a person on a skycoaster relative to the center of the arch, and
- an explanation of how you would use the angle θ, the height of the tower, and the length of the cable to find the height of riders from the ground.

Standardized Test Practice

60. If the cotangent of angle θ is 1, then the tangent of angle θ is **C**

Ⓐ -1. Ⓑ 0. Ⓒ 1. Ⓓ 3.

61. **SHORT RESPONSE** Find the exact coordinates of point P, which is located at the intersection of a circle of radius 5 and the terminal side of angle θ measuring $\frac{5\pi}{3}$. $\left(\frac{5}{2}, -\frac{5\sqrt{3}}{2}\right)$



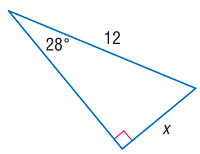
Maintain Your Skills

Mixed Review **Rewrite each degree measure in radians and each radian measure in degrees.** *(Lesson 13-2)*

62. 90° $\frac{\pi}{2}$ 63. $\frac{5\pi}{3}$ **300°** 64. 5 $\frac{900}{\pi} \approx 286.5°$

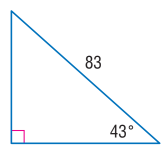
Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-1)*

65.



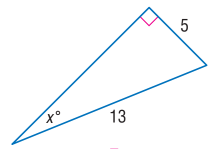
$\sin 28° = \frac{x}{12}$, 5.6

66.



$\cos 43° = \frac{x}{83}$, 60.7

67.



$\sin x° = \frac{5}{13}$, 23

68. **LITERATURE** In one of Grimm's *Fairy Tales*, Rumpelstiltskin has the ability to spin straw into gold. Suppose on the first day, he spun 5 pieces of straw into gold, and each day thereafter he spun twice as much. How many pieces of straw would he have spun into gold by the end of the week? *(Lesson 11-3)* **635**

Use Cramer's Rule to solve each system of equations. *(Lesson 4-6)*

69. $3x - 4y = 13$ **(7, 2)**
$-2x + 5y = -4$

70. $5x + 7y = 1$ **(−4, 3)**
$3x + 5y = 3$

71. $2x + 3y = -2$ **(5, −4)**
$-6x + y = -34$

Getting Ready for the Next Lesson **PREREQUISITE SKILL Solve each equation. Round to the nearest tenth.**
*(To review **solving equations with trigonometric functions**, see Lesson 13-1.)*

72. $\frac{a}{\sin 32°} = \frac{8}{\sin 65°}$ **4.7** 73. $\frac{b}{\sin 45°} = \frac{21}{\sin 100°}$ **15.1** 74. $\frac{c}{\sin 60°} = \frac{3}{\sin 75°}$ **2.7**

75. $\frac{\sin A}{14} = \frac{\sin 104°}{25}$ **32.9°** 76. $\frac{\sin B}{3} = \frac{\sin 55°}{7}$ **20.6°** 77. $\frac{\sin C}{10} = \frac{\sin 35°}{9}$ **39.6°**

724 Chapter 13 Trigonometric Functions

13-4 Law of Sines

What You'll Learn

- Solve problems by using the Law of Sines.
- Determine whether a triangle has one, two, or no solutions.

Vocabulary

- Law of Sines

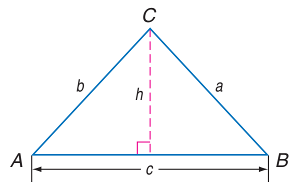
How can trigonometry be used to find the area of a triangle?

You know how to find the area of a triangle when the base and the height are known. Using this formula, the area of $\triangle ABC$ below is $\frac{1}{2}ch$. If the height h of this triangle were not known, you could still find the area given the measures of angle A and the length of side b.

$$\sin A = \frac{h}{b} \rightarrow h = b \sin A$$

By combining this equation with the area formula, you can find a new formula for the area of the triangle.

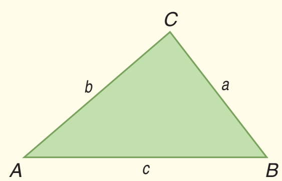
$$\text{Area} = \frac{1}{2}ch \rightarrow \text{Area} = \frac{1}{2}c(b \sin A)$$



LAW OF SINES You can find two other formulas for the area of the triangle above in a similar way. These formulas, summarized below, allow you to find the area of any triangle when you know the measures of two sides and the included angle.

Key Concept — Area of a Triangle

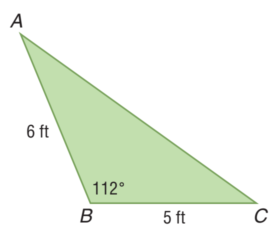
- **Words** The area of a triangle is one half the product of the lengths of two sides and the sine of their included angle.
- **Symbols** area $= \frac{1}{2}bc \sin A$
  area $= \frac{1}{2}ac \sin B$
  area $= \frac{1}{2}ab \sin C$



Example 1 Find the Area of a Triangle

Find the area of $\triangle ABC$ to the nearest tenth.

In this triangle, $a = 5$, $c = 6$, and $B = 112°$. Choose the second formula because you know the values of its variables.



Area $= \frac{1}{2}ac \sin B$	Area formula
$= \frac{1}{2}(5)(6) \sin 112°$	Replace a with 5, c with 6, and B with 112°.
≈ 13.9	Use a calculator.

To the nearest tenth, the area is 13.9 square feet.

13-4 Lesson Notes

1 Focus

5-Minute Check Transparency 13-4 Use as a quiz or review of Lesson 13-3.

Mathematical Background notes are available for this lesson on p. 698D.

How can trigonometry be used to find the area of a triangle?

Ask students:

- What kind of triangles does the height separate the triangle into? **right triangles**
- Does the size of angle C make any difference in this derivation of the area formula? **no**

Resource Manager

Workbook and Reproducible Masters

Chapter 13 Resource Masters

- Study Guide and Intervention, pp. 793–794
- Skills Practice, p. 795
- Practice, p. 796
- Reading to Learn Mathematics, p. 797
- Enrichment, p. 798
- Assessment, pp. 831, 833

Graphing Calculator and Spreadsheet Masters, p. 51
School-to-Career Masters, p. 25

Transparencies

5-Minute Check Transparency 13-4
Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

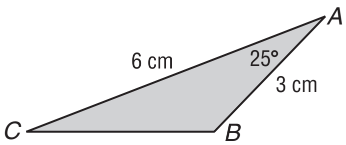
LAW OF SINES

In-Class Examples

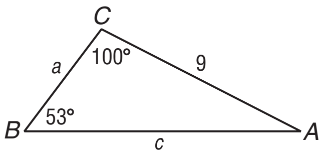
Power Point®

Teaching Tip In Example 1, ask students if they knew only the values for sides *c* and *a*, and for angle *A*, could they use the formula to find the area of the triangle? **No, to use the formula, the angle must be included in the known sides.**

1 Find the area of $\triangle ABC$ to the nearest tenth. **3.8 cm^2**



2 Solve $\triangle ABC$.



$A = 27°$, $a \approx 5.1$, $c \approx 11.1$

All of the area formulas for $\triangle ABC$ represent the area of the same triangle. So, $\frac{1}{2}bc \sin A$, $\frac{1}{2}ac \sin B$, and $\frac{1}{2}ab \sin C$ are all equal. You can use this fact to derive the **Law of Sines**.

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C \quad \text{Set area formulas equal to each other.}$$

$$\frac{\frac{1}{2}bc \sin A}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac \sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}ab \sin C}{\frac{1}{2}abc} \quad \text{Divide each expression by } \tfrac{1}{2}abc.$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{Simplify.}$$

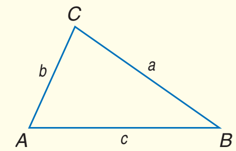
Study Tip

Alternate Representations
The Law of Sines may also be written as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Key Concept — *Law of Sines*

Let $\triangle ABC$ be any triangle with *a*, *b*, and *c* representing the measures of sides opposite angles with measurements *A*, *B*, and *C* respectively. Then,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



The Law of Sines can be used to write three different equations.

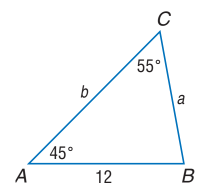
$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{or} \quad \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin C}{c}$$

In Lesson 13-1, you learned how to solve right triangles. To solve *any* triangle, you can apply the Law of Sines if you know

- the measures of two angles and any side or
- the measures of two sides and the angle opposite one of them.

Example 2 Solve a Triangle Given Two Angles and a Side

Solve $\triangle ABC$.



You are given the measures of two angles and a side. First, find the measure of the third angle.

$45° + 55° + B = 180°$ The sum of the angle measures of a triangle is 180°.

$B = 80°$ $180 - (45 + 55) = 80$

Now use the Law of Sines to find *a* and *b*. Write two equations, each with one variable.

$\frac{\sin A}{a} = \frac{\sin C}{c}$	Law of Sines	$\frac{\sin B}{b} = \frac{\sin C}{c}$
$\frac{\sin 45°}{a} = \frac{\sin 55°}{12}$	Replace *A* with 45°, *B* with 80°, *C* with 55°, and *c* with 12.	$\frac{\sin 80°}{b} = \frac{\sin 55°}{12}$
$a = \frac{12 \sin 45°}{\sin 55°}$	Solve for the variable.	$b = \frac{12 \sin 80°}{\sin 55°}$
$a \approx 10.4$	Use a calculator.	$b \approx 14.4$

Therefore, $B = 80°$, $a \approx 10.4$, and $b \approx 14.4$.

TEACHING TIP

You may wish to use the abbreviation AAS when referring to a triangle where the measure of two angles and a nonincluded side are known and ASA when the measure of two angles and their included side are known. Similarly, you can use SSA to refer to a triangle where the measures of two sides and the angle opposite one of them is known.

DAILY INTERVENTION

Unlocking Misconceptions

Students may think the Law of Sines only works for right triangles. Clarify that this formula works for any triangle, as does the Law of Cosines explored in Lesson 13-5.

726 Chapter 13 Trigonometric Functions

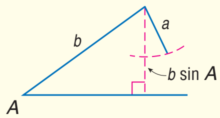
ONE, TWO, OR NO SOLUTIONS When solving a triangle, you must analyze the data you are given to determine whether there is a solution. For example, if you are given the measures of two angles and a side, as in Example 2, the triangle has a unique solution. However, if you are given the measures of two sides and the angle opposite one of them, a single solution may not exist. One of the following will be true.

- No triangle exists, and there is no solution.
- Exactly one triangle exists, and there is one solution.
- Two triangles exist, and there are two solutions.

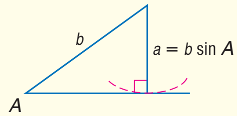
Key Concept — ***Possible Triangles Given Two Sides and One Opposite Angle***

Suppose you are given a, b, and A for a triangle.

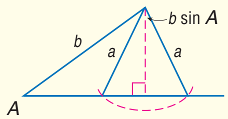
A Is Acute ($A < 90°$).



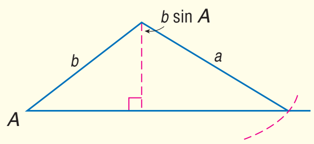
$a < b \sin A$
no solution



$a = b \sin A$
one solution

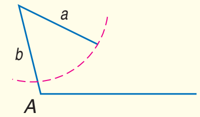


$b > a > b \sin A$
two solutions

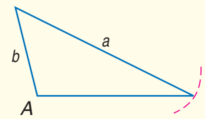


$a \geq b$
one solution

A Is Right or Obtuse ($A \geq 90°$).



$a \leq b$
no solution



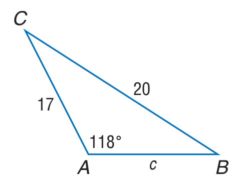
$a > b$
one solution

Example 3 One Solution

In $\triangle ABC$, $A = 118°$, $a = 20$, and $b = 17$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

Because angle A is obtuse and $a > b$, you know that one solution exists.

Make a sketch and then use the Law of Sines to find B.



$\frac{\sin B}{17} = \frac{\sin 118°}{20}$ Law of Sines

$\sin B = \frac{17 \sin 118°}{20}$ Multiply each side by 17.

$\sin B \approx 0.7505$ Use a calculator.

$B \approx 49°$ Use the $\sin^{-1}$ function.

The measure of angle C is approximately $180 - (118 + 49)$ or $13°$.

Use the Law of Sines again to find c.

$\frac{\sin 13}{c} = \frac{\sin 118°}{20}$ Law of Sines

$c = \frac{20 \sin 13°}{\sin 118°}$ or about 5.1 Use a calculator.

Therefore, $B \approx 49°$, $C \approx 13°$, and $c \approx 5.1$.

www.algebra2.com/extra_examples

ONE, TWO, OR NO SOLUTIONS

In-Class Example Power Point®

3 In $\triangle ABC$, $A = 25°$, $a = 13$, and $b = 12$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.
one; $B \approx 23°$, $C \approx 132°$; $c \approx 22.9$

In-Class Examples

Power Point®

4 In $\triangle ABC$, $A = 125°$, $a = 35$, and $b = 32$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.
no

5 In $\triangle ABC$, $A = 25°$, $a = 5$, and $b = 10$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.
two; $B \approx 58°$, $C \approx 97°$, $c \approx 11.7$; $B \approx 122°$, $C \approx 33°$, $c \approx 6.4$

Study Tip

A Is Acute
We compare $b \sin A$ to a because $b \sin A$ is the minimum distance from C to $\overline{AB}$ when A is acute.

TEACHING TIP

You can have students verify this solution using the Law of Sines.

$$\frac{\sin 50°}{5} = \frac{\sin B}{9}$$

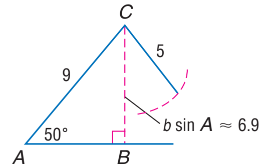
$$\sin B \approx 1.379$$

Since a sine ratio cannot exceed 1, there is no solution.

Example 4 No Solution

In $\triangle ABC$, $A = 50°$, $a = 5$, and $b = 9$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

Since angle A is acute, find $b \sin A$ and compare it with a.



$b \sin A = 9 \sin 50°$ Replace b with 9 and A with 50°.

≈ 6.9 Use a calculator.

Since $5 < 6.9$, there is no solution.

When two solutions for a triangle exist, it is called the *ambiguous case*.

Example 5 Two Solutions

In $\triangle ABC$, $A = 39°$, $a = 10$, and $b = 14$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

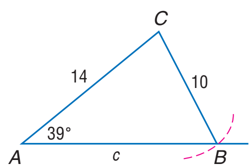
Since angle A is acute, find $b \sin A$ and compare it with a.

$b \sin A = 14 \sin 39°$ Replace b with 14 and A with 39°.

≈ 8.81 Use a calculator.

Since $14 > 10 > 8.81$, there are two solutions. Thus, there are two possible triangles to be solved.

Case 1 Acute Angle B



First, use the Law of Sines to find B.

$$\frac{\sin B}{14} = \frac{\sin 39°}{10}$$

$$\sin B = \frac{14 \sin 39°}{10}$$

$$\sin B = 0.8810$$

$$B \approx 62°$$

The measure of angle C is approximately $180 - (39 + 62)$ or $79°$.

Use the Law of Sines again to find c.

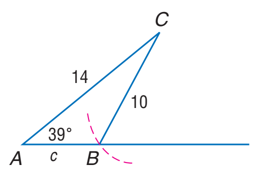
$$\frac{\sin 79°}{c} = \frac{\sin 39°}{10}$$

$$c = \frac{10 \sin 79°}{\sin 39°}$$

$$c \approx 15.6$$

Therefore, $B \approx 62°$, $C \approx 79°$, and $c \approx 15.6$.

Case 2 Obtuse Angle B



To find B, you need to find an obtuse angle whose sine is also 0.8810. To do this, subtract the angle given by your calculator, 62°, from 180°. So B is approximately $180 - 62$ or $118°$.

The measure of angle C is approximately $180 - (39 + 118)$ or $23°$.

Use the Law of Sines to find c.

$$\frac{\sin 23°}{c} = \frac{\sin 39°}{10}$$

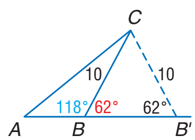
$$c = \frac{10 \sin 23°}{\sin 39°}$$

$$c \approx 6.2$$

Therefore, $B \approx 118°$, $C \approx 23°$, and $c \approx 6.2$.

Study Tip

Alternate Method
Another way to find the obtuse angle in Case 2 of Example 5 is to notice in the figure below that $\triangle CBB'$ is isosceles. Since the base angles of an isosceles triangle are always congruent and $m\angle B' = 62°$, $m\angle CBB' = 62°$. Also, $\angle ABC$ and $m\angle CBB'$ are supplementary. Therefore, $m\angle ABC = 180° - 62°$ or $118°$.



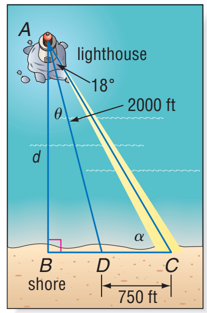
DAILY INTERVENTION

Differentiated Instruction

Intrapersonal Have students write a journal entry about which example they found the most challenging and why. Ask them to include any questions they still have about the lesson.

Example 6 Use the Law of Sines to Solve a Problem

LIGHTHOUSES A lighthouse is located on a rock at a certain distance from a straight shore. The light revolves counterclockwise at a steady rate of one revolution per minute. As the beam revolves, it strikes a point on the shore that is 2000 feet from the lighthouse. Three seconds later, the light strikes a point 750 feet further down the shore. To the nearest foot, how far is the lighthouse from the shore?



Because the lighthouse makes one revolution every 60 seconds, the angle through which the light revolves in 3 seconds is $\frac{3}{60}(360°)$ or 18°.

Use the Law of Sines to find the measure of angle α.

$$\frac{\sin \alpha}{2000} = \frac{\sin 18°}{750}$$ Law of Sines

$$\sin \alpha = \frac{2000 \sin 18°}{750}$$ Multiply each side by 2000.

$$\sin \alpha \approx 0.8240$$ Use a calculator.

$$\alpha \approx 55°$$ Use the $\sin^{-1}$ function.

Use this angle measure to find the measure of angle θ. Since $\triangle ABC$ is a right triangle, the measures of angle α and $\angle BAC$ are complementary.

$$\alpha + m\angle BAC = 90°$$ Angles α and $\angle BAC$ are complementary.

$$55° + (\theta + 18°) \approx 90°$$ $\alpha \approx 55°$ and $m\angle BAC = \theta + 18°$

$$\theta + 73° \approx 90°$$ Simplify.

$$\theta \approx 17°$$ Solve for θ.

To find the distance from the lighthouse to the shore, solve $\triangle ABD$ for d.

$$\cos \theta = \frac{AB}{AD}$$ Cosine ratio

$$\cos 17° \approx \frac{d}{2000}$$ $\theta = 17°$ and $AD = 2000$

$$d \approx 2000 \cos 17°$$ Solve for d.

$$d \approx 1913$$ Use a calculator.

The distance from the lighthouse to the shore, to the nearest foot, is 1913 feet. This answer is reasonable since 1913 is less than 2000.

More About. . .

Lighthouses

Standing 208 feet tall, the Cape Hatteras Lighthouse in North Carolina is the tallest lighthouse in the United States.

Source: www.oldcapehatteraslighthouse.com

TEACHING TIP

Point out to students that the Greek letter *alpha*, α, can be used to denote the measure of an angle.

Check for Understanding

Concept Check

1. **Determine** whether the following statement is *sometimes*, *always* or *never* true. Explain your reasoning.
   If given the measure of two sides of a triangle and the angle opposite one of them, you will be able to find a unique solution.
   1. Sometimes; only if when A is acute, $a = b \sin A$ or $a > b$ and if when A is obtuse, $a > b$.

2. **OPEN ENDED** Give an example of a triangle that has two solutions by listing measures for A, a, and b, where a and b are in centimeters. Then draw both cases using a ruler and protractor. **Sample answer: $A = 42°$, $a = 2.6$ cm, $b = 3.2$ cm; See margin for drawings.**

In-Class Example

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6 **LIGHTHOUSES** Refer to Example 6 in the Student Edition. Suppose a different lighthouse has a beam that revolves at the same rate (one revolution per minute) but the beam strikes a point on the shore that is 1840 feet from the lighthouse. Two seconds later, the light strikes a point 500 feet farther down the shore. To the nearest foot, how far is the lighthouse from the shore? **1625 ft**

3 Practice/Apply

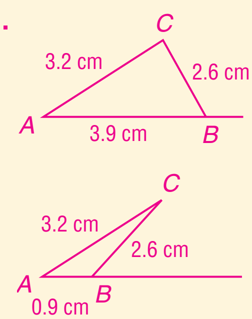
Study Notebook

Have students—

- *add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 13.*
- *include any other item(s) that they find helpful in mastering the skills in this lesson.*

Answer

2.



Lesson 13-4 Law of Sines 729

DAILY INTERVENTION

FIND THE ERROR Have students sketch a triangle and label sides a, b, and angle A on the sketch. Make sure students label angle A opposite side a.

About the Exercises…

Organization by Objective

- **Law of Sines:** 14–27, 38–41
- **One, Two, or No Solutions:** 28–37

Odd/Even Assignments

Exercises 14–37 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 15–37 odd, 42–58

Average: 15–37 odd, 42–58

Advanced: 14–40 even, 41–54 (optional: 55–58)

3. **FIND THE ERROR** Dulce and Gabe are finding the area of $\triangle ABC$ for $A = 64°$, $a = 15$ meters, and $b = 8$ meters using the sine function.

Dulce	Gabe
$\text{Area} = \frac{1}{2}(15)(8)\sin 64°$ $\approx 53.9 \text{ m}^2$	There is not enough information to find the area of $\triangle ABC$.

Who is correct? Explain your reasoning. **Gabe; see margin for explanation.**

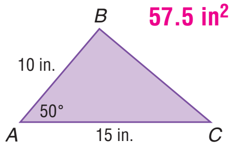
Guided Practice

GUIDED PRACTICE KEY

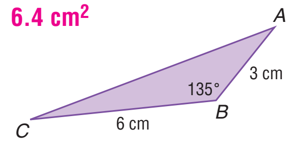
Exercises	Examples
4, 5	1
6, 7	2
8–12	3–5
13	6

Find the area of $\triangle ABC$ to the nearest tenth.

4. **57.5 in²**

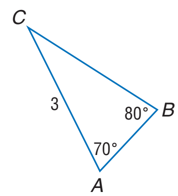


5. **6.4 cm²**

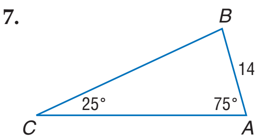


Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. **6–8. See margin.**

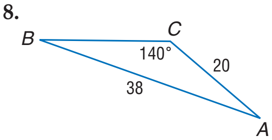
6.



7.



8.



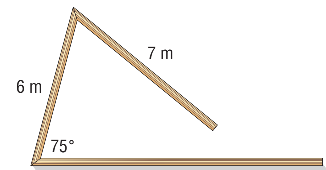
Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

10. two; $B = 42°$, $C = 108°$, $c = 5.7$; $B = 138°$, $C = 12°$, $c = 1.2$

9. $A = 123°$, $a = 12$, $b = 23$ **no**
10. $A = 30°$, $a = 3$, $b = 4$
11. $A = 55°$, $a = 10$, $b = 5$ **one; $B \approx 24°$, $C \approx 101°$, $c \approx 12.0$**
12. $A = 145°$, $a = 18$, $b = 10$ **one; $B \approx 19°$, $C \approx 16°$, $c \approx 8.7$**

Application

13. **WOODWORKING** Latisha is constructing a triangular brace from three beams of wood. She is to join the 6-meter beam to the 7-meter beam so that angle opposite the 7-meter beam measures 75°. To what length should Latisha cut the third beam in order to form the triangular brace? Round to the nearest tenth. **5.5 m**



★ indicates increased difficulty

Practice and Apply

Homework Help

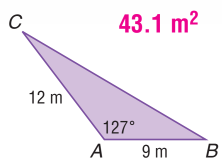
For Exercises	See Examples
14–19	1
20–37	2–5
38–41	6

Extra Practice

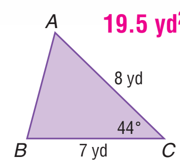
See page 858.

Find the area of $\triangle ABC$ to the nearest tenth.

14. **43.1 m²**



15. **19.5 yd²**

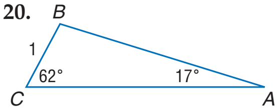


16. $B = 85°$, $c = 23$ ft, $a = 50$ ft **572.8 ft²**
17. $A = 60°$, $b = 12$ cm, $c = 12$ cm **62.4 cm²**
18. $C = 136°$, $a = 3$ m, $b = 4$ m **4.2 m²**
19. $B = 32°$, $a = 11$ mi, $c = 5$ mi **14.6 mi²**

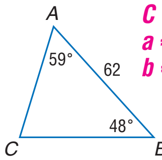
Answers

3. The information given is of two sides and an angle, but the angle is not between the two sides, therefore the area formula involving sine cannot be used.
6. $C = 30°$, $a \approx 2.9$, $c \approx 1.5$
7. $B = 80°$, $a \approx 32.0$, $b \approx 32.6$
8. $B \approx 20°$, $A \approx 20°$, $a \approx 20.2$
28. no
29. one; $B \approx 36°$, $C \approx 45°$, $c \approx 1.8$
30. two; $B \approx 72°$, $C \approx 75°$, $c \approx 3.5$; $B \approx 108°$, $C \approx 39°$, $c \approx 2.3$
31. no
32. one; $B = 90°$, $C = 60°$, $c \approx 24.2$
33. one; $B \approx 18°$, $C \approx 101°$, $c \approx 25.8$
34. two; $B \approx 56°$, $C \approx 72°$, $c \approx 229.3$; $B \approx 124°$, $C \approx 4°$, $c \approx 16.8$
35. two; $B \approx 85°$, $C \approx 15°$, $c \approx 2.4$; $B \approx 95°$, $C \approx 5°$, $c \approx 0.8$
36. one; $B \approx 23°$, $C \approx 129°$, $c \approx 14.1$
37. two; $B \approx 65°$, $C \approx 68°$, $c \approx 84.9$; $B \approx 115°$, $C \approx 18°$, $c \approx 28.3$

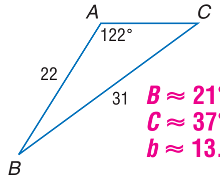
Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

20.


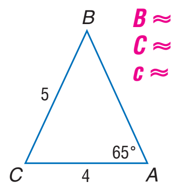
$B = 101°$, $c \approx 3.0$, $b \approx 3.4$

21.


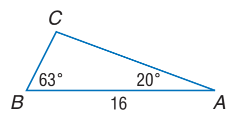
$C = 73°$, $a \approx 55.6$, $b \approx 48.2$

22.


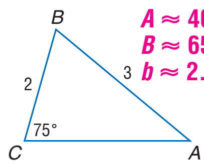
$B \approx 21°$, $C \approx 37°$, $b \approx 13.1$

23.


$B \approx 47°$, $C \approx 68°$, $c \approx 5.1$

24.


$C = 97°$, $a \approx 5.5$, $b \approx 14.4$

25.


$A \approx 40°$, $B \approx 65°$, $b \approx 2.8$

26. $A = 50°, a = 2.5, c = 3$
$C \approx 67°$, $B \approx 63°$, $b \approx 2.9$

27. $B = 18°, C = 142°, b = 20$
$A = 20°$, $a \approx 22.1$, $c \approx 39.8$

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. 28–37. See margin.

28. $A = 124°, a = 1, b = 2$

29. $A = 99°, a = 2.5, b = 1.5$

30. $A = 33°, a = 2, b = 3.5$

31. $A = 68°, a = 3, b = 5$

32. $A = 30°, a = 14, b = 28$

33. $A = 61°, a = 23, b = 8$

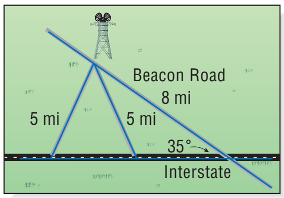
34. $A = 52°, a = 190, b = 200$

35. $A = 80°, a = 9, b = 9.1$

36. $A = 28°, a = 8.5, b = 7.2$

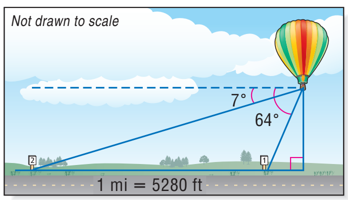
37. $A = 47°, a = 67, b = 83$

38. RADIO A radio station providing local tourist information has its transmitter on Beacon Road, 8 miles from where it intersects with the interstate highway. If the radio station has a range of 5 miles, between what two distances from the intersection can cars on the interstate tune in to hear this information? 4.6 and 8.5 mi

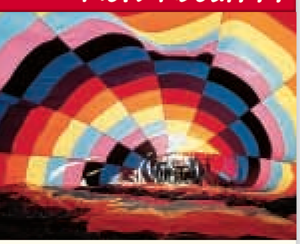


39. FORESTRY Two forest rangers, 12 miles from each other on a straight service road, both sight an illegal bonfire away from the road. Using their radios to communicate with each other, they determine that the fire is between them. The first ranger's line of sight to the fire makes an angle of 38° with the road, and the second ranger's line of sight to the fire makes a 63° angle with the road. How far is the fire from each ranger? 7.5 mi from Ranger B, 10.9 mi from Ranger A

40. BALLOONING As a hot-air balloon crosses over a straight portion of interstate highway, its pilot eyes two consecutive mileposts on the same side of the balloon. When viewing the mileposts the angles of depression are 64° and 7°. How high is the balloon to the nearest foot? 690 ft



More About. . .



Ballooning

Hot-air balloons range in size from approximately 54,000 cubic feet to over 250,000 cubic feet.

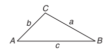
Source: www.unicorn-ballon.com

www.algebra2.com/self_check_quiz

Study Guide and Intervention, p. 793 (shown) and p. 794

Law of Sines The area of any triangle is one half the product of the lengths of two sides and the sine of the included angle.

Area of a Triangle	area $= \frac{1}{2}bc \sin A$ area $= \frac{1}{2}ac \sin B$ area $= \frac{1}{2}ab \sin C$



You can use the Law of Sines to solve any triangle if you know the measures of two angles and any side, or the measures of two sides and the angle opposite one of them.

Law of Sines	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Example 1 **Find the area of $\triangle ABC$ if $a = 10$, $b = 14$, and $C = 40°$.**

Area $= \frac{1}{2}ab \sin C$ — Area formula
$= \frac{1}{2}(10)(14)\sin 40°$ — Replace a, b, and C.
≈ 44.9951 — Use a calculator.

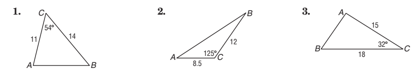
The area of the triangle is approximately 45 square units.

Example 2 **If $a = 12$, $b = 9$, and $A = 28°$, find B.**

$\frac{\sin A}{a} = \frac{\sin B}{b}$ — Law of Sines
$\frac{\sin 28°}{12} = \frac{\sin B}{9}$ — Replace A, a, and b.
$\sin B = \frac{9 \sin 28°}{12}$ — Solve for $\sin B$.
$\sin B \approx 0.3521$ — Use a calculator.
$B \approx 20.62°$ — Use the $\sin^{-1}$ function.

Exercises

Find the area of $\triangle ABC$ to the nearest tenth.



1. 62.3 units²
2. 41.8 units²
3. 71.5 units²

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

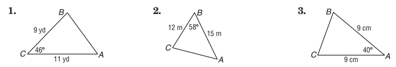
4. $B = 42°, C = 68°, a = 10$ — $A = 70°$, $b \approx 7.1$, $c \approx 9.9$

5. $A = 40°, B = 14°, a = 52$ — $b \approx 19.6$, $c \approx 65.4$, $C = 126°$

6. $A = 15°, B = 50°, b = 36$ — $C = 115°$, $a \approx 12.2$, $c \approx 42.6$

Skills Practice, p. 795 and Practice, p. 796 (shown)

Find the area of $\triangle ABC$ to the nearest tenth.



1. 35.6 yd²
2. 76.3 m²
3. 26.0 cm²

4. $C = 32°, a = 12.6$ m, $b = 8.9$ m — 29.7 m²

5. $B = 27°, a = 14.9$ cm, $c = 18.6$ cm — 62.9 cm²

6. $A = 17.4°, b = 12$ km, $c = 14$ km — 25.1 km²

7. $A = 34°, b = 19.4$ ft, $c = 8.6$ ft — 46.6 ft²

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

8. $A = 50°, B = 30°, c = 9$ — $C = 100°$, $a \approx 7.0$, $b \approx 4.6$

9. $A = 56°, B = 38°, a = 12$ — $C = 86°$, $b \approx 8.9$, $c \approx 14.4$

10. $A = 80°, C = 14°, a = 40$ — $B = 86°$, $b \approx 40.5$, $c \approx 9.8$

11. $B = 47°, C = 112°, b = 13$ — $A = 21°$, $a \approx 6.4$, $c \approx 16.5$

12. $A = 72°, a = 8, c = 6$ — $B \approx 62°$, $C \approx 46°$, $b \approx 7.5$

13. $A = 25°, C = 107°, b = 12$ — $B = 48°$, $a \approx 6.8$, $c \approx 15.4$

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

14. $A = 29°, a = 6, b = 13$ — no solution

15. $A = 70°, a = 25, b = 20$ — one solution; $B \approx 49°$, $C \approx 61°$, $c \approx 23.3$

16. $A = 113°, a = 21, b = 25$ — no solution

17. $A = 110°, a = 20, b = 8$ — one solution; $B \approx 22°$, $C \approx 48°$, $c \approx 15.8$

18. $A = 66°, a = 12, b = 7$ — one solution; $B \approx 32°$, $C \approx 82°$, $c \approx 13.0$

19. $A = 54°, a = 5, b = 8$ — no solution

20. $A = 45°, a = 15, b = 18$ — two solutions; $B \approx 58°$, $C \approx 77°$, $c \approx 20.7$; $B \approx 122°$, $C \approx 13°$, $c \approx 4.8$

21. $A = 60°, a = 4\sqrt{3}, b = 8$ — one solution; $B = 90°$, $C = 30°$, $c \approx 23.3$

22. WILDLIFE Sarah Phillips, an officer for the Department of Fisheries and Wildlife, checks boaters on a lake to make sure they do not disturb two osprey nesting sites. She leaves a dock and heads due north in her boat to the first nesting site. From here, she turns 5° north of due west and travels an additional 2.14 miles to the second nesting site. She then travels 6.7 miles directly back to the dock. How far from the dock is the first osprey nesting site? Round to the nearest tenth. 6.2 mi

Reading to Learn Mathematics, p. 797 ELL

Pre-Activity **How can trigonometry be used to find the area of a triangle?**

Read the introduction to Lesson 13-4 at the top of page 725 in your textbook.

What happens when the formula Area $= \frac{1}{2}ab \sin C$ is applied to a right triangle in which C is the right angle? Sample answer: The formula gives Area $= \frac{1}{2}ab \sin 90° = \frac{1}{2}ab \cdot 1 = \frac{1}{2}ab$, which is the same as the result from using the formula Area $= \frac{1}{2}$(base)(height).

Reading the Lesson

1. In each case below, the measures of three parts of a triangle are given. For each case, write the formula you would use to find the area of the triangle. Show the formulas with specific values substituted, but do not actually calculate the area. If there is not enough information provided to find the area of the triangle by using the area formulas on page 725 in your textbook and without finding other parts of the triangle first, explain why.

a. $A = 48°, b = 9, c = 5$ — $\frac{1}{2}(9)(5) \sin 48°$

b. $a = 15, b = 15, C = 120°$ — $\frac{1}{2}(15)(15) \sin 120°$

c. $b = 16, c = 10, B = 120°$ — Not enough information; B is not the included angle between the two given sides.

2. Tell whether the equation must be true based on the Law of Sines. Write *yes* or *no*.

a. $\frac{\sin A}{b} = \frac{\sin B}{a}$ no

b. $\frac{b}{\sin B} = \frac{c}{\sin C}$ yes

c. $a \sin C = c \sin A$ yes

d. $b = \frac{a \sin A}{\sin B}$ no

3. Determine whether $\triangle ABC$ has *no solution*, *one solution*, or *two solutions*. Do not try to solve the triangle.

a. $a = 20, A = 30°, B = 70°$ — one solution

b. $A = 55°, b = 5, a = 3$ ($b \sin A \approx 4.1$) — no solution

c. $c = 12, A = 100°, a = 30$ — one solution

d. $C = 27°, b = 23.5, c = 17.5$ ($b \sin C \approx 10.7$) — two solutions

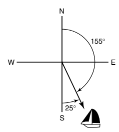
Helping You Remember

4. Suppose that you are taking a quiz and cannot remember whether the formula for the area of a triangle is Area $= \frac{1}{2}ab \cos C$ or Area $= \frac{1}{2}ab \sin C$. How can you quickly remember which of these is correct? Sample answer: The formula has to work when C is a right angle. The formula cannot contain $\cos C$ because $\cos 90° = 0$ and this would make the area of a right triangle be 0.

Enrichment, p. 798

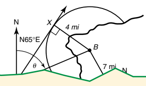
Navigation

The bearing of a boat is an angle showing the direction the boat is heading. Often, the angle is measured from north, but it can be measured from any of the four compass directions. At the right, the bearing of the boat is 155°. Or, it can be described as 25° east of south (S25°E).



Example **A boat A sights the lighthouse B in the direction N65°E and the spire of a church C in the direction S75°E. According to the map, B is 7 miles from C in the direction N30°W. In order for A to avoid running aground, find the bearing it should keep to pass B at 4 miles distance.**

In $\triangle ABC$, $\angle\alpha = 180° - 65° - 75°$ or $40°$
$\angle C = 180° - 30° - (180° - 75°)$
$= 45°$
$a = 7$ miles



4 Assess

Open-Ended Assessment

Modeling Have students work in small groups to build models with coffee stirrer sticks (or similar objects) to illustrate and explain the various number of solutions that are possible for triangles.

Assessment Options

Quiz (Lessons 13-3 and 13-4) is available on p. 831 of the *Chapter 13 Resource Masters*.

Mid-Chapter Test (Lessons 13-1 through 13-4) is available on p. 833 of the *Chapter 13 Resource Masters*.

Getting Ready for Lesson 13-5

PREREQUISITE SKILL Lesson 13-5 presents the Law of Cosines. Students will use their familiarity with solving equations involving trigonometric functions as they apply the Law of Cosines. Exercises 55–58 should be used to determine your students' familiarity with solving equations with trigonometric functions.

★ **41. NAVIGATION** Two fishing boats, A and B, are anchored 4500 feet apart in open water. A plane flies at a constant speed in a straight path directly over the two boats, maintaining a constant altitude. At one point during the flight, the angle of depression to A is 85°, and the angle of depression to B is 25°. Ten seconds later the plane has passed over A and spots B at a 35° angle of depression. How fast is the plane flying? **107 mph**

42. CRITICAL THINKING Given $\triangle ABC$, if $a = 20$ and $B = 47°$, then determine all possible values of b so that the triangle has

a. two solutions. **$14.63 < b < 20$**

b. one solution. **$b = 14.63$ or $b \geq 20$**

c. no solutions. **$b < 14.63$**

43. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. **See margin.**

How can trigonometry be used to find the area of a triangle?

Include the following in your answer:

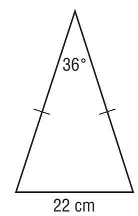
- the conditions that would indicate that trigonometry is needed to find the area of a triangle,
- an example of a real-world situation in which you would need trigonometry to find the area of a triangle, and
- a derivation of one of the other two area formulas.

Standardized Test Practice

44. Which of the following is the perimeter of the triangle shown? **D**

(A) 49.0 cm (B) 66.0 cm

(C) 91.4 cm (D) 93.2 cm



45. SHORT RESPONSE The longest side of a triangle is 67 inches. Two angles have measures of 47° and 55°. Solve the triangle. **$B = 78°$, $a \approx 50.1$, $c \approx 56.1$**

Maintain Your Skills

Mixed Review

Find the exact value of each trigonometric function. *(Lesson 13-3)*

46. $\cos 30°$ **$\frac{\sqrt{3}}{3}$**

47. $\cot\left(\frac{\pi}{3}\right)$ **$\frac{\sqrt{3}}{3}$**

48. $\csc\left(\frac{\pi}{4}\right)$ **$\sqrt{2}$**

Find one angle with positive measure and one angle with negative measure coterminal with each angle. *(Lesson 13-2)*

49. 300° **660°, −60°**

50. 47° **407°, −313°**

51. $\frac{5\pi}{6}$ **$\frac{17\pi}{6}, -\frac{7\pi}{6}$**

Two cards are drawn from a deck of cards. Find each probability. *(Lesson 12-5)*

52. P(both 5s or both spades) **$\frac{3}{68}$**

53. P(both 7s or both red) **$\frac{55}{221}$**

54. AERONAUTICS A rocket rises 20 feet in the first second, 60 feet in the second second, and 100 feet in the third second. If it continues at this rate, how many feet will it rise in the 20th second? *(Lesson 11-1)* **780 ft**

Getting Ready for the Next Lesson

PREREQUISITE SKILL Solve each equation. Round to the nearest tenth.
*(To review **solving equations with trigonometric functions**, see Lesson 13-1.)*

55. $a^2 = 3^2 + 5^2 - 2(3)(5)\cos 85°$ **5.6**

56. $c^2 = 12^2 + 10^2 - 2(12)(10)\cos 40°$ **7.8**

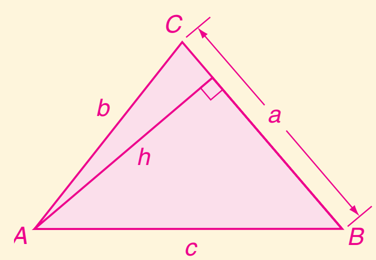
57. $7^2 = 11^2 + 9^2 - 2(11)(9)\cos B°$ **39.4°**

58. $13^2 = 8^2 + 6^2 - 2(8)(6)\cos A°$ **136.0°**

Answer

43. Answers should include the following.

- **If the height of the triangle is not given, but the measure of two sides and their included angle are given, then the formula for the area of a triangle using the sine function should be used.**
- **You might use this formula to find the area of a triangular piece of land, since it might be easier to measure two sides and use surveying equipment to measure the included angle than to measure the perpendicular distance from one vertex to its opposite side.**
- **The area of $\triangle ABC$ is $\frac{1}{2}ah$.**
  $\sin B = \frac{h}{c}$ or $c = c \sin B$
  Area $= \frac{1}{2}ah$ or Area $= \frac{1}{2}a(c \sin B)$



13-5 Law of Cosines

What You'll Learn

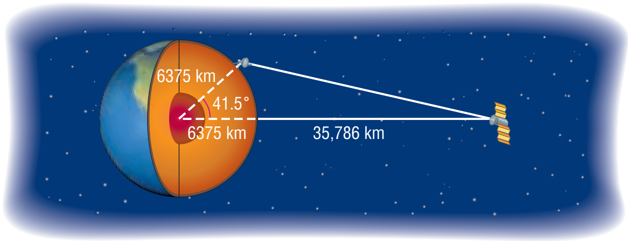
- Solve problems by using the Law of Cosines.
- Determine whether a triangle can be solved by first using the Law of Sines or the Law of Cosines.

Vocabulary

- Law of Cosines

How can you determine the angle at which to install a satellite dish?

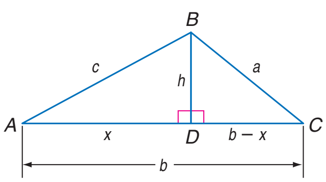
The GE-3 satellite is in a *geosynchronous orbit* about Earth, meaning that it circles Earth once each day. As a result, the satellite appears to remain stationary over one point on the equator. A receiving dish for the satellite can be directed at one spot in the sky. The satellite orbits 35,786 kilometers above the equator at 87°W longitude. The city of Valparaiso, Indiana, is located at approximately 87°W longitude and 41.5°N latitude.



Knowing the radius of Earth to be about 6375 kilometers, a satellite dish installer can use trigonometry to determine the angle at which to direct the receiver.

LAW OF COSINES Problems such as this, in which you know the measures of two sides and the included angle of a triangle, cannot be solved using the Law of Sines. You can solve problems such as this by using the **Law of Cosines**.

To derive the Law of Cosines, consider $\triangle ABC$. What relationship exists between a, b, c, and A?

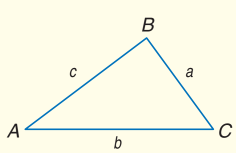


$a^2 = (b - x)^2 + h^2$	Use the Pythagorean Theorem for $\triangle DBC$.
$= b^2 - 2bx + x^2 + h^2$	Expand $(b - x)^2$.
$= b^2 - 2bx + c^2$	In $\triangle ADB$, $c^2 = x^2 + h^2$.
$= b^2 - 2b(c \cos A) + c^2$	$\cos A = \frac{x}{c}$, so $x = c \cos A$.
$= b^2 + c^2 - 2bc \cos A$	Commutative Property

Key Concept — *Law of Cosines*

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides, and opposite angles with measures A, B, and C, respectively. Then the following equations are true.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$



13-5 Lesson Notes

1 Focus

5-Minute Check Transparency 13-5 Use as a quiz or review of Lesson 13-4.

Mathematical Background notes are available for this lesson on p. 698D.

How can you determine the angle at which to install a satellite dish?

Ask students:

- Why does the satellite appear to remain stationary over one point on the equator? **because the satellite and Earth are turning at the same rate**
- What is the zero point for measuring longitude? **The zero meridian of longitude passes through Greenwich, England.**

Resource Manager

Workbook and Reproducible Masters

Chapter 13 Resource Masters

- Study Guide and Intervention, pp. 799–800
- Skills Practice, p. 801
- Practice, p. 802
- Reading to Learn Mathematics, p. 803
- Enrichment, p. 804

School-to-Career Masters, p. 26

Transparencies

5-Minute Check Transparency 13-5
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lesson 26
Interactive Chalkboard

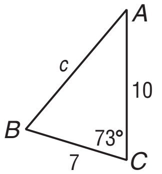
2 Teach

LAW OF COSINES

In-Class Examples

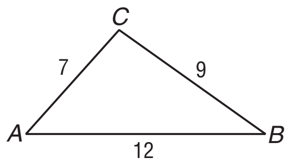
Power Point®

1 Solve $\triangle ABC$.



$A \approx 40°$; $B \approx 67°$; $c \approx 10.4$

2 Solve $\triangle ABC$.



$A \approx 48°$, $B \approx 35°$, $C \approx 97°$

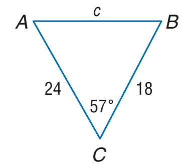
TEACHING TIP

You may wish to use the abbreviation SAS when referring to a triangle where the measures of the sides and the included angle are known and SSS when the measure of three sides are known.

You can apply the Law of Cosines to a triangle if you know
- the measures of two sides and the included angle, or
- the measures of three sides.

Example 1 Solve a Triangle Given Two Sides and Included Angle

Solve $\triangle ABC$.



You are given the measure of two sides and the included angle. Begin by using the Law of Cosines to determine c.

$c^2 = a^2 + b^2 - 2ab \cos C$	Law of Cosines
$c^2 = 18^2 + 24^2 - 2(18)(24) \cos 57°$	$a = 18$, $b = 24$, and $C = 57°$
$c^2 \approx 429.4$	Simplify using a calculator.
$c \approx 20.7$	Take the square root of each side.

Next, you can use the Law of Sines to find the measure of angle A.

$\frac{\sin A}{a} = \frac{\sin C}{c}$	Law of Sines
$\frac{\sin A}{18} \approx \frac{\sin 57°}{20.7}$	$a = 18$, $C = 57°$, and $c \approx 20.7$
$\sin A \approx \frac{18 \sin 57°}{20.7}$	Multiply each side by 18.
$\sin A \approx 0.7293$	Use a calculator.
$A \approx 47°$	Use the $\sin^{-1}$ function.

The measure of the angle B is approximately $180° - (57° + 47°)$ or $76°$. Therefore, $c \approx 20.7$, $A \approx 47°$, and $B \approx 76°$.

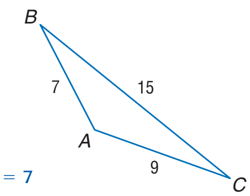
Study Tip

Alternate Method

After finding the measure of c in Example 1, the Law of Cosines could be used again to find a second angle.

Example 2 Solve a Triangle Given Three Sides

Solve $\triangle ABC$.



You are given the measures of three sides. Use the Law of Cosines to find the measure of the largest angle first, angle A.

$a^2 = b^2 + c^2 - 2bc \cos A$	Law of Cosines
$15^2 = 9^2 + 7^2 - 2(9)(7) \cos A$	$a = 15$, $b = 9$, and $c = 7$
$15^2 - 9^2 - 7^2 = -2(9)(7) \cos A$	Subtract 9^2 and 7^2 from each side.
$\frac{15^2 - 9^2 - 7^2}{-2(9)(7)} = \cos A$	Divide each side by $-2(9)(7)$.
$-0.7540 \approx \cos A$	Use a calculator.
$139° \approx A$	Use the $\cos^{-1}$ function.

You can use the Law of Sines to find the measure of angle B.

$\frac{\sin B}{b} = \frac{\sin A}{a}$	Law of Sines
$\frac{\sin B}{9} \approx \frac{\sin 139°}{15}$	$b = 9$, $A \approx 139°$, and $a = 15$
$\sin B \approx \frac{9 \sin 139°}{15}$	Multiply each side by 9.
$\sin B \approx 0.3936$	Use a calculator.
$B \approx 23°$	Use the $\sin^{-1}$ function.

The measure of the angle C is approximately $180° - (139° + 23°)$ or $18°$. Therefore, $A \approx 139°$, $B \approx 23°$, and $C \approx 18°$.

Study Tip

Sides and Angles

When solving triangles, remember that the angle with the greatest measure is always opposite the longest side. The angle with the least measure is always opposite the shortest side.

734 Chapter 13 Trigonometric Functions

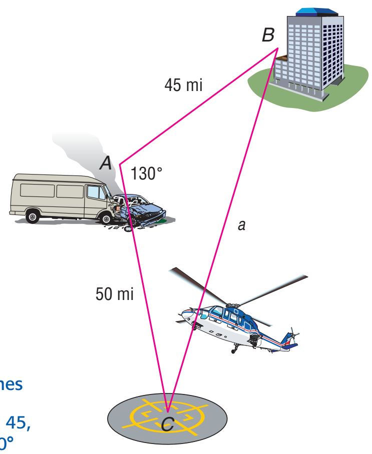
CHOOSE THE METHOD To solve a triangle that is *oblique*, or having no right angle, you need to know the measure of at least one side and any two other parts. If the triangle has a solution, then you must decide whether to begin solving by using the Law of Sines or by using the Law of Cosines. Use the chart below to help you choose.

Concept Summary — *Solving an Oblique Triangle*

Given	Begin by Using
two angles and any side	Law of Sines
two sides and an angle opposite one of them	Law of Sines
two sides and their included angle	Law of Cosines
three sides	Law of Cosines

Example 3 *Apply the Law of Cosines*

EMERGENCY MEDICINE **A medical rescue helicopter has flown from its home base at point C to pick up an accident victim at point B and then from there to the hospital at point A. The pilot needs to know how far he is now from his home base so he can decide whether to refuel before returning. How far is the hospital from the helicopter's base?**



You are given the measures of two sides and their included angle, so use the Law of Cosines to find a.

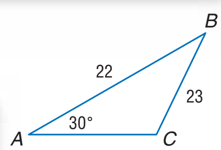
$a^2 = b^2 + c^2 - 2bc \cos A$	Law of Cosines
$a^2 = 50^2 + 45^2 - 2(50)(45) \cos 130°$	$b = 50$, $c = 45$, and $A = 130°$
$a^2 \approx 7417.5$	Use a calculator to simplify.
$a \approx 86.1$	Take the square root of each side.

The distance between the hospital and the helicopter base is approximately 86.1 miles.

Maintain Your Skills

Concept Check

1. **FIND THE ERROR** Mateo and Amy are deciding which method, the Law of Sines or the Law of Cosines, should be used first to solve $\triangle ABC$.



Mateo	Amy
Begin by using the Law of Sines, since you are given two sides and an angle opposite one of them.	Begin by using the Law of Cosines, since you are given two sides and their included angle.

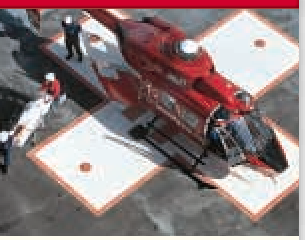
Who is correct? Explain your reasoning.

www.algebra2.com/extra_examples

TEACHING TIP

Remind students that when given two sides and an angle opposite one of them, they must first decide whether the triangle has one solution, two solutions, or no solution.

More About...



Emergency Medicine

Medical evacuation (Medevac) helicopters provide quick transportation from areas that are difficult to reach by any other means. These helicopters can cover long distances and are primary emergency vehicles in locations where there are few hospitals.

Source: The Helicopter Education Center

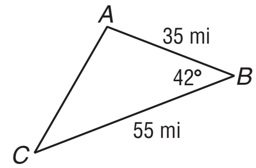
1. Mateo; the angle given is not between the two sides; therefore the Law of Sines should be used.

CHOOSE THE METHOD

In-Class Example

Power Point®

3 **EMERGENCY MEDICINE** Refer to Example 3 in the Student Edition. A helicopter flies 55 miles from its base at point C to an accident at point B and then 35 miles to the hospital at point A. Angle B equals 42°. How far will the helicopter have to fly to return to its base from the hospital?



about 37.27 mi

3 *Practice/Apply*

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 13.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

DAILY INTERVENTION

FIND THE ERROR Watch for students who think the angle is included. Review the definition of included angles with them.

DAILY INTERVENTION

Differentiated Instruction ELL

Verbal/Linguistic Have students discuss in small groups how to choose which method to use when solving a triangle. Then have them compare their approaches and develop a brief explanation to help others decide. Then have each group share their conclusions with the class.

About the Exercises...

Organization by Objective
- **Law of Cosines:** 10–33
- **Choose the Method:** 10–27

Odd/Even Assignments
Exercises 10–27 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 11–25 odd, 31, 34–37, 41–54

Average: 11–27 odd, 31, 33–37, 41–54 (optional: 38–40)

Advanced: 10–26 even, 28–30, 32–48 (optional: 49–54)

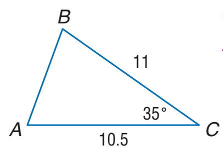
All: Practice Quiz 2 (1–5)

2. **Explain** how to solve a triangle by using the Law of Cosines if the lengths of
   a. three sides are known.
   b. two sides and the measure of the angle between them are known.
   See margin.
3. **OPEN ENDED** Give an example of a triangle that can be solved by first using the Law of Cosines. See margin.

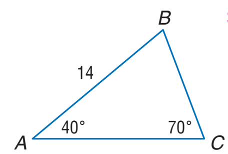
Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4–7	1, 2
8, 9	3

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

4. 

cosines; $A \approx 76°$, $B \approx 69°$, $c \approx 6.5$

5. 

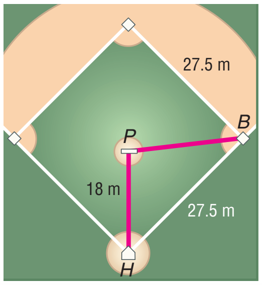
sines; $B = 70°$, $a \approx 9.6$, $b = 14$

6. $A = 42°, b = 57, a = 63$ sines; $C \approx 101°$, $B \approx 37°$, $c \approx 92.5$

7. $a = 5, b = 12, c = 13$ cosines; $A = 23°$, $B \approx 67°$, $C \approx 90°$

Application

BASEBALL For Exercises 8 and 9, use the following information.
In Australian baseball, the bases lie at the vertices of a square 27.5 meters on a side and the pitcher's mound is 18 meters from home plate.



8. Find the distance from the pitcher's mound to first base. 19.5 m
9. Find the angle between home plate, the pitcher's mound, and first base. 94.3°

★ indicates increased difficulty

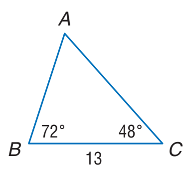
Practice and Apply

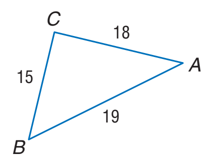
Homework Help

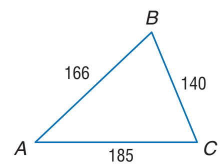
For Exercises	See Examples
10–27	1, 2
28–33	3

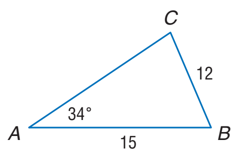
Extra Practice
See page 858.

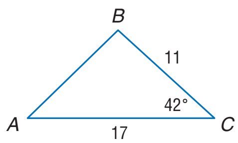
Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. 10–27. See margin.

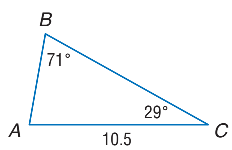
10. 

11. 

12. 

13. 

14. 

15. 

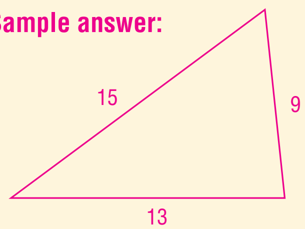
16. $a = 20, c = 24, B = 47°$
17. $a = 345, b = 648, c = 442$
18. $A = 36°, a = 10, b = 19$
19. $A = 25°, B = 78°, a = 13.7$
20. $a = 21.5, b = 16.7, c = 10.3$
21. $a = 16, b = 24, c = 41$
22. $a = 8, b = 24, c = 18$
23. $B = 19°, a = 51, c = 61$
24. $A = 56°, B = 22°, a = 12.2$
25. $a = 4, b = 8, c = 5$
26. $a = 21.5, b = 13, C = 38°$
★27. $A = 40°, b = 7, c = 6$

Answers

2a. Use the Law of Cosines to find the measure of one angle. Then use the Law of Sines or the Law of Cosines to find the measure of a second angle. Finally, subtract the sum of these two angles from 180° to find the measure of the third angle.

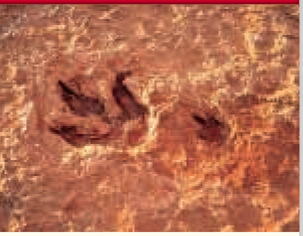
2b. Use the Law of Cosines to find the measure of the third side. Then use the Law of Sines or the Law of Cosines to find the measure of a second angle. Finally, subtract the sum of these two angles from 180° to find the measure of the third angle.

3. Sample answer:



10. sines; $A = 60°$, $b \approx 14.3$, $c \approx 11.2$
11. cosines; $A \approx 48°$, $B \approx 62°$, $C \approx 70°$
12. cosines; $A \approx 46°$, $B \approx 74°$, $C \approx 59.6$
13. sines; $B \approx 102°$, $C \approx 44°$, $b \approx 21.0$
14. cosines; $A \approx 56.8°$, $B \approx 82°$, $c \approx 11.5$
15. sines; $A = 80°$, $a \approx 10.9$, $c \approx 5.4$
16. cosines; $A \approx 55°$, $C \approx 78°$, $b \approx 17.9$
17. cosines; $A \approx 30°$, $B \approx 110°$, $C \approx 40°$
18. no
19. sines; $C \approx 77°$, $b \approx 31.7$, $c \approx 31.6$
20. cosines; $A \approx 103°$, $B \approx 49°$, $C \approx 28°$
21. no
22. cosines; $A \approx 15°$, $B \approx 131°$, $C \approx 34°$
23. cosines; $A \approx 52°$, $C \approx 109°$, $b \approx 21.0$
24. sines; $C = 102°$, $b \approx 5.5$, $c \approx 14.4$
25. cosines; $A \approx 24°$, $B \approx 125°$, $C \approx 31°$
26. cosines; $A \approx 107°$, $B \approx 35°$, $c \approx 13.8$
27. cosines; $B \approx 82°$, $C \approx 58°$, $a \approx 4.5$

More About. . .

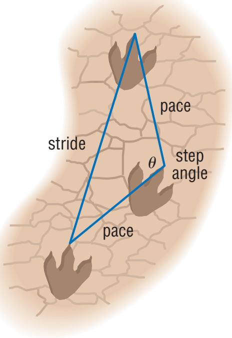


Dinosaurs

At digs such as the one at the Glen Rose formation in Texas, anthropologists study the footprints made by dinosaurs millions of years ago. *Locomoter* parameters, such as pace and stride, taken from these prints can be used to describe how a dinosaur once moved.

Source: Mid-America Paleontology Society

DINOSAURS For Exercises 28–30, use the diagram at the right.



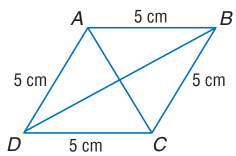
28. An anthropologist examining the footprints made by a bipedal (two-footed) dinosaur finds that the dinosaur's average pace was about 1.60 meters and average stride was about 3.15 meters. Find the step angle θ for this dinosaur. **about 159.7°**

29. Find the step angle θ made by the hindfeet of a herbivorous dinosaur whose pace averages 1.78 meters and stride averages 2.73 meters. **about 100.1°**

30. An efficient walker has a step angle that approaches 180°, meaning that the animal minimizes "zig-zag" motion while maximizing forward motion. What can you tell about the motion of each dinosaur from its step angle?

30. Since the step angle for the carnivore is closer to 180°, it appears as though the carnivore made more forward progress with each step than the herbivore did.

31. GEOMETRY In rhombus $ABCD$, the measure of $\angle ADC$ is 52°. Find the measures of diagonals $\overline{AC}$ and $\overline{DB}$ to the nearest tenth. **4.4 cm, 9.0 cm**



32. SURVEYING Two sides of a triangular plot of land have lengths of 425 feet and 550 feet. The measure of the angle between those sides is 44.5°. Find the perimeter and area of the plot. **about 1362 ft; about 81,919 ft^2**

★ **33. AVIATION** A pilot typically flies a route from Bloomington to Rockford, covering a distance of 117 miles. In order to avoid a storm, the pilot first flies from Bloomington to Peoria, a distance of 42 miles, then turns the plane and flies 108 miles on to Rockford. Through what angle did the pilot turn the plane over Peoria? **87.4°**

34. CRITICAL THINKING Explain how the Pythagorean Theorem is a special case of the Law of Cosines.

34. Since cos 90° = 0, $a^2 = b^2 + c^2 - 2bc$ cos A becomes $a^2 = b^2 + c^2$.

35. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson. **See pp. 759A–759D.**

How can you determine the angle at which to install a satellite dish?

Include the following in your answer:

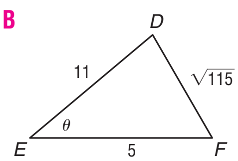
- a description of the conditions under which you can use the Law of Cosines to solve a triangle, and
- given the latitude of a point on Earth's surface, an explanation of how can you determine the angle at which to install a satellite dish at the same longitude.

Standardized Test Practice

A B C D

36. In $\triangle DEF$, what is the value of θ to the nearest degree? **B**

Ⓐ 26° Ⓑ 74°
Ⓒ 80° Ⓓ 141°

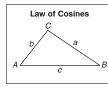


www.algebra2.com/self_check_quiz

Study Guide and Intervention, p. 799 (shown) and p. 800

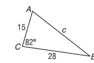
Law of Cosines

Law of Cosines	Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of the sides, and opposite angles with measures A, B, and C, respectively. Then the following equations are true. $a^2 = b^2 + c^2 - 2bc \cos A$ $b^2 = a^2 + c^2 - 2ac \cos B$ $c^2 = a^2 + b^2 - 2ab \cos C$



You can use the Law of Cosines to solve any triangle if you know the measures of two sides and the included angle, or the measures of three sides.

Example Solve $\triangle ABC$.



You are given the measures of two sides and the included angle. Begin by using the Law of Cosines to determine c.

$c^2 = a^2 + b^2 - 2ab \cos C$
$c^2 = 28^2 + 15^2 - 2(28)(15)\cos 82°$
$c^2 \approx 892.09$
$c \approx 29.9$

Next you can use the Law of Sines to find the measure of angle A.

$\frac{\sin A}{a} = \frac{\sin C}{c}$
$\frac{\sin A}{28} \approx \frac{\sin 82°}{29.9}$
$\sin A \approx 0.9273$
$A \approx 68°$

The measure of B is about $180° - (82° + 68°)$ or about 30°.

Exercises

Solve each triangle described below. Round measures of sides to the nearest tenth and angles to the nearest degree.

1. $a = 14, c = 20, B = 38°$ **$b \approx 12.4, A \approx 44°, C \approx 98°$**

2. $A = 60°, c = 17, b = 12$ **$a \approx 15.1, B \approx 43°, C \approx 77°$**

3. $a = 4, b = 6, c = 3$ **$A \approx 36°, B \approx 118°, C \approx 26°$**

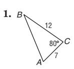
4. $A = 103°, b = 31, c = 52$ **$a \approx 66, B \approx 27°, C \approx 50°$**

5. $a = 15, b = 26, C = 132°$ **$c \approx 38, A \approx 17°, B \approx 31°$**

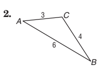
6. $a = 31, b = 52, c = 43$ **$A \approx 36°, B \approx 88°, C \approx 56°$**

Skills Practice, p. 801 and Practice, p. 802 (shown)

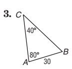
Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

1. 

cosines; $c \approx 12.8$, $A \approx 67°, B \approx 33°$

2. 

cosines; $A \approx 36°$, $B \approx 26°, C \approx 117°$

3. 

sines; $B = 60°$, $a \approx 46.0, b \approx 40.4$

4. $a = 16, b = 20, C = 54°$ **cosines; $A \approx 51°, B \approx 75°, c \approx 16.7$**

5. $B = 71°, c = 6, a = 11$ **cosines; $A \approx 77°, C \approx 32°, b \approx 10.7$**

6. $A = 37°, a = 20, b = 18$ **sines; $B \approx 33°, C \approx 110°, c \approx 31.2$**

7. $C = 35°, a = 18, b = 24$ **cosines; $A \approx 48°, B \approx 97°, c \approx 13.9$**

8. $a = 8, b = 6, c = 9$ **cosines; $A \approx 61°, B \approx 41°, C \approx 79°$**

9. $A = 23°, b = 10, c = 12$ **cosines; $B \approx 54°, C \approx 103°, a \approx 4.8$**

10. $a = 4, b = 5, c = 8$ **cosines; $A \approx 24°, B \approx 31°, C \approx 125°$**

11. $B = 46.6°, C = 112°, b = 13$ **sines; $A = 21°, a \approx 6.5, c \approx 16.6$**

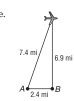
12. $A = 46.3°, a = 35, b = 30$ **sines; $B \approx 38°, C \approx 95°, c \approx 48.2$**

13. $a = 16.4, b = 21.1, c = 18.5$ **cosines; $A \approx 48°, B \approx 74°, C \approx 57°$**

14. $C = 43.5°, b = 8, c = 6$ **sines; $A \approx 70°, B \approx 67°, a \approx 8.2$**

15. $A = 78.3°, b = 7, c = 11$ **cosines; $B \approx 36°, C \approx 66°, a \approx 11.8$**

16. SATELLITES Two radar stations 2.4 miles apart are tracking an airplane. The straight-line distance between Station A and the plane is 7.4 miles. The straight-line distance between Station B and the plane is 6.9 miles. What is the angle of elevation from Station A to the plane? Round to the nearest degree. **69°**



17. DRAFTING Marion is using a computer-aided drafting program to produce a drawing for a client. She begins a triangle by drawing a segment 4.2 inches long from point A to point B. From B, she moves 42° degrees counterclockwise from the segment connecting A and B and draws a second segment that is 6.4 inches long, ending at point C. To the nearest tenth, how long is the segment from C to A? **9.9 in.**

Reading to Learn Mathematics, p. 803 ELL

Pre-Activity **How can you determine the angle at which to install a satellite dish?**

Read the introduction to Lesson 13-5 at the top of page 733 in your textbook.

One side of the triangle in the figure is not labeled with a length. What does the length of this side represent? Is this length greater than or less than the distance from the satellite to the equator?

the distance from the satellite to Valparaiso; greater than

Reading the Lesson

1. Each of the following equations can be changed into a correct statement of the Law of Cosines by making one change. In each case, indicate what change should be made to make the statement correct.

a. $b^2 = a^2 + c^2 + 2ac \cos B$ **Change the second + to −.**

b. $a^2 = b^2 + c^2 - 2bc \sin A$ **Change sin A to cos A.**

c. $c = a^2 + b^2 - 2ab \cos C$ **Change c to c^2.**

d. $a^2 = b^2 - c^2 - 2bc \cos A$ **Change the first − to +.**

2. Suppose that you are asked to solve $\triangle ABC$ given the following information about the sides and angles of the triangle. In each case, indicate whether you would begin by using the *Law of Sines* or the *Law of Cosines*.

a. $a = 8, b = 7, c = 6$ **Law of Cosines**

b. $b = 9.5, A = 72°, B = 39°$ **Law of Sines**

c. $C = 123°, b = 22.95, a = 34.35$ **Law of Cosines**

Helping You Remember

3. It is often easier to remember a complicated procedure if you can break it down into small steps. Describe in your own words how to use the Law of Cosines to find the length of one side of a triangle if you know the lengths of the other two sides and the measure of the included angle. Use numbered steps. (You may use mathematical terms, but do not use any mathematical symbols.)

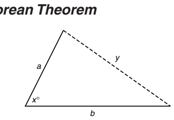
Sample answer: 1. Square each of the lengths of the two known sides. 2. Add these squares. 3. Find the cosine of the included angle. 4. Multiply this cosine by two times the product of the lengths of the two known sides. 5. Subtract the product from the sum. 6. Take the positive square root of the result.

Enrichment, p. 804

The Law of Cosines and the Pythagorean Theorem

The law of cosines bears strong similarities to the Pythagorean theorem. According to the law of cosines, if two sides of a triangle have lengths a and b and if the angle between them has a measure of $x°$, then the length, y, of the third side of the triangle can be found by using the equation

$y^2 = a^2 + b^2 - 2ab \cos x°$.



Answer the following questions to clarify the relationship between the law of cosines and the Pythagorean theorem.

1. If the value of $x°$ becomes less and less, what number is $\cos x°$ close to? **1**

2. If the value of $x°$ is very close to zero but then increases, what happens to $\cos x°$ as $x°$ approaches 90°? **decreases, approaches 0**

4 Assess

Open-Ended Assessment

Writing Have students sketch and label some of the parts of three triangles—one that they would choose to solve using the Law of Cosines, one that they would choose to solve using the Law of Sines, and one that they would use both laws to solve.

Assessment Options

Practice Quiz 2 The quiz provides students with a brief review of the concepts and skills in Lessons 13-3 through 13-5. Lesson numbers are given to the right of exercises or instruction lines so students can review concepts not yet mastered.

Getting Ready for Lesson 13-6

PREREQUISITE SKILL Lesson 13-6 presents the unit circle approach to trigonometric functions. Students will use their familiarity with coterminal angles as they work with the unit circle. Exercises 49–54 should be used to determine your students' familiarity with finding coterminal angles.

Answers

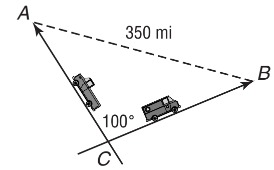
40. By finding the measure of angle C in one step using the Law of Cosines, only the given information was used. By finding this angle measure using the Law of Cosines and then the Law of Sines, a calculated value that was not exact was introduced.

43. $\sin\theta = \frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = \frac{12}{5}$, $\csc\theta = \frac{13}{12}$, $\sec\theta = \frac{13}{5}$, $\cot\theta = \frac{5}{12}$

44. $\sin\theta = \frac{7\sqrt{65}}{65}$, $\cos\theta = \frac{4\sqrt{65}}{65}$, $\tan\theta = \frac{7}{4}$, $\csc\theta = \frac{\sqrt{65}}{7}$, $\sec\theta = \frac{\sqrt{65}}{4}$, $\cot\theta = \frac{4}{7}$

45. $\sin\theta = \frac{\sqrt{6}}{4}$, $\cos\theta = \frac{\sqrt{10}}{4}$, $\tan\theta = \frac{\sqrt{15}}{5}$, $\csc\theta = \frac{2\sqrt{6}}{3}$, $\sec\theta = \frac{2\sqrt{10}}{5}$, $\cot\theta = \frac{\sqrt{15}}{3}$

37. Two trucks, A and B, start from the intersection C of two straight roads at the same time. Truck A is traveling twice as fast as truck B and after 4 hours, the two trucks are 350 miles apart. Find the approximate speed of truck B in miles per hour. **B**



Ⓐ 35 Ⓑ 37 Ⓒ 57 Ⓓ 73

Extending the Lesson

ERROR IN MEASUREMENT **For Exercises 38–40, use the following information.** Consider $\triangle ABC$, in which $a = 17$, $b = 8$, and $c = 20$.

38. Find the measure of angle C in one step using the Law of Cosines. Round to the nearest tenth. **100.0°**

39. Find the measure of angle C in two steps using the Law of Cosines and then the Law of Sines. Round to the nearest tenth. **Sample answer: 100.2°**

40. Explain why your answers for Exercises 38 and 39 are different. Which answer gives you the better approximation for the measure of angle C? **See margin for explanation; 100.0°.**

Check for Understanding

Mixed Review

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-4)*

41. $A = 55°$, $a = 8$, $b = 7$ **one; $B = 46°$, $C = 79°$, $c = 9.6$**

42. $A = 70°$, $a = 7$, $b = 10$ **no solution**

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point. *(Lesson 13-3)* **43–45. See margin.**

43. $(5, 12)$ **44.** $(4, 7)$ **45.** $(\sqrt{10}, \sqrt{6})$

Solve each equation or inequality. *(Lesson 10-5)*

46. $e^x + 5 = 9$ **1.3863**

47. $4e^x - 3 > -1$ **$\{x \mid x > -0.6931\}$**

48. $\ln(x + 3) = 2$ **4.3891**

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find one angle with positive measure and one angle with negative measure coterminal with each angle.**
*(To review **coterminal angles**, see Lesson 13-2.)*

49. 45° **405, −315°**

50. 30° **390°, −330°**

51. 180° **540°, −180°**

52. $\frac{\pi}{2}$ $\frac{5\pi}{2}, -\frac{3\pi}{2}$

53. $\frac{7\pi}{6}$ $\frac{19\pi}{6}, -\frac{5\pi}{6}$

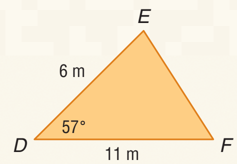
54. $\frac{4\pi}{3}$ $\frac{10\pi}{3}, -\frac{2\pi}{3}$

Practice Quiz 2 — *Lessons 13-3 through 13-5*

1. Find the exact value of the six trigonometric functions of θ if the terminal side of θ in standard position contains the point $(-2, 3)$. *(Lesson 13-3)* **See margin.**

2. Find the exact value of $\csc \frac{5\pi}{3}$. *(Lesson 13-3)* $-\frac{2\sqrt{3}}{3}$

3. Find the area of $\triangle DEF$ to the nearest tenth. *(Lesson 13-4)* **27.7 m²**



4. Determine whether $\triangle ABC$, with $A = 22°$, $a = 15$, and $b = 18$, has *no* solution, *one* solution, or *two* solutions. Then solve the triangle, if possible. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-4)* **See margin.**

5. Determine whether $\triangle ABC$, with $b = 11$, $c = 14$, and $A = 78°$, should be solved by beginning with the Law of Sines or Law of Cosines. Then solve the triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-5)* **cosines; $c \approx 15.9$, $A \approx 59°$, $B \approx 43°$**

Answer (Practice Quiz 2)

1. $\sin\theta = \frac{3\sqrt{13}}{13}$; $\cos\theta = -\frac{2\sqrt{13}}{13}$; $\tan\theta = -\frac{3}{2}$; $\csc\theta = \frac{\sqrt{13}}{3}$; $\sec\theta = -\frac{\sqrt{13}}{2}$; $\cot\theta = -\frac{2}{3}$

4. two; $B \approx 27°$; $C \approx 131°$; $c \approx 30.2$; $B \approx 153°$; $C \approx 5°$; $c \approx 3.5$

738 Chapter 13 Trigonometric Functions

13-6 Circular Functions

What You'll Learn

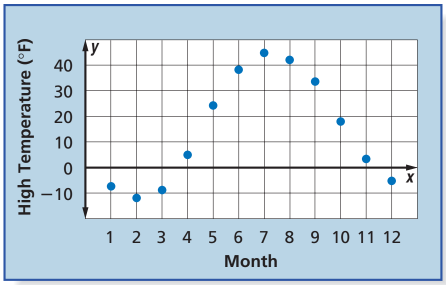
- Define and use the trigonometric functions based on the unit circle.
- Find the exact values of trigonometric functions of angles.

Vocabulary

- circular function
- periodic
- period

How can you model annual temperature fluctuations?

The average high temperatures, in degrees Fahrenheit, for Barrow, Alaska, are given in the table at the right. With January assigned a value of 1, February a value of 2, March a value of 3, and so on, these data can be graphed as shown below. This pattern of temperature fluctuations repeats after a period of 12 months.

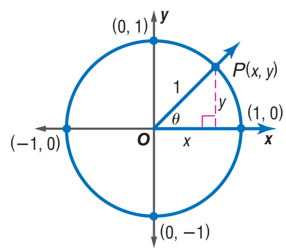


BARROW, ALASKA

MONTH	HIGH TEMP. (°F)
Jan	-7.4
Feb	-11.8
March	-9.0
April	4.7
May	24.2
June	38.3
July	45.0
Aug	42.3
Sept	33.8
Oct	18.1
Nov	3.5
Dec	-5.2

Source: www.met.utah.edu

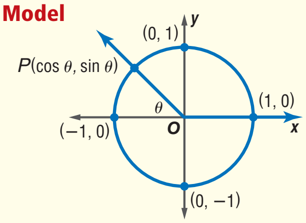
UNIT CIRCLE DEFINITIONS From your work with reference angles, you know that the values of trigonometric functions also repeat. For example, sin 30° and sin 150° have the same value, $\frac{1}{2}$. In this lesson, we will further generalize the trigonometric functions by defining them in terms of the unit circle.



Consider an angle θ in standard position. The terminal side of the angle intersects the unit circle at a unique point, $P(x, y)$. Recall that $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$. Since $P(x, y)$ is on the unit circle, $r = 1$. Therefore, $\sin \theta = y$ and $\cos \theta = x$.

Key Concept — *Definition of Sine and Cosine*

- **Words** If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$. Therefore, the coordinates of P can be written as $P(\cos \theta, \sin \theta)$.
- **Model**



13-6 Lesson Notes

1 Focus

5-Minute Check Transparency 13-6 Use as a quiz or review of Lesson 13-5.

Mathematical Background notes are available for this lesson on p. 698D.

How can you model annual temperature fluctuations?

Ask students:

- Describe the change in the average high temperature from February to July. **It is increasing.**
- Describe the change in the average high temperature from August to December. **It is decreasing.**
- Does the pattern of change in temperature seem to fall along a straight line? **no**
- What curve does the pattern of change in temperature seem to suggest? **a sine or cosine curve**

Resource Manager

Workbook and Reproducible Masters

Chapter 13 Resource Masters

- Study Guide and Intervention, pp. 805–806
- Skills Practice, p. 807
- Practice, p. 808
- Reading to Learn Mathematics, p. 809
- Enrichment, p. 810
- Assessment, p. 832

Teaching Algebra With Manipulatives Masters, p. 302

Transparencies

5-Minute Check Transparency 13-6
Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

UNIT CIRCLE DEFINITIONS

In-Class Example Power Point®

1 Given an angle θ in standard position, if $P\left(\frac{\sqrt{7}}{4}, \frac{3}{4}\right)$ lies on the terminal side of θ and on the unit circle, find $\sin \theta$ and $\cos \theta$.

$\sin \theta = \frac{3}{4}$, $\cos \theta = \frac{\sqrt{7}}{4}$

Study Tip

Reading Math

To help you remember that $x = \cos \theta$ and $y = \sin \theta$, notice that alphabetically x comes before y and cosine comes before sine.

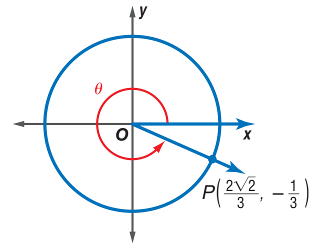
Since there is exactly one point $P(x, y)$ for any angle θ, the relations $\cos \theta = x$ and $\sin \theta = y$ are functions of θ. Because they are both defined using a unit circle, they are often called **circular functions**.

Example 1 Find Sine and Cosine Given Point on Unit Circle

Given an angle θ in standard position, if $P\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$ lies on the terminal side and on the unit circle, find $\sin \theta$ and $\cos \theta$.

$P\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right) = P(\cos \theta, \sin \theta)$,

so $\sin \theta = -\frac{1}{3}$ and $\cos \theta = \frac{2\sqrt{2}}{3}$.



In the Investigation below, you will explore the behavior of the sine and cosine functions on the unit circle.

Graphing Calculator Investigation

Sine and Cosine on the Unit Circle

Press MODE on a TI-83 Plus and highlight Degree and Par. Then use the following range values to set up a viewing window: TMIN = 0, TMAX = 360, TSTEP = 15, XMIN = −2.4, XMAX = 2.35, XSCL = 0.5, YMIN = −1.5, YMAX = 1.55, YSCL = 0.5.

Press Y= to define the unit circle with $X_{1T} = \cos T$ and $Y_{1T} = \sin T$. Press GRAPH. Use the TRACE function to move around the circle.

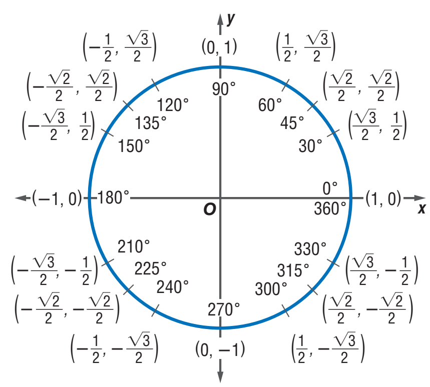
Think and Discuss

1. What does T represent? What does the x value represent? What does the y value represent? **the angle θ; $\cos \theta$; $\sin \theta$**
2. Determine the sine and cosine of the angles whose terminal sides lie at 0°, 90°, 180°, and 270°.
3. How do the values of sine change as you move around the unit circle? How do the values of cosine change?

2. $\sin 0° = 0$, $\cos 0° = 1$, $\sin 90° = 1$, $\cos 90° = 0$, $\sin 180° = 0$, $\cos 180° = -1$, $\sin 270° = -1$, $\cos 270° = 0$

3. For $0° \le \theta \le 90°$, $\sin \theta$ increases from 0 to 1. For $90° \le \theta \le 270°$, $\sin \theta$ decreases from 1 to −1. For $270° \le \theta \le 360°$, $\sin \theta$ increases from −1 to 0. For $0° \le \theta \le 180°$, $\cos \theta$ decreases from 1 to −1. For $180° \le \theta \le 270°$, $\cos \theta$ increases from −1 to 1.

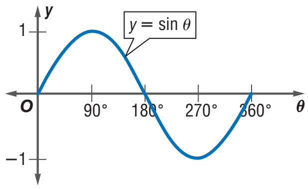
The exact values of the sine and cosine functions for specific angles are summarized using the definition of sine and cosine on the unit circle below.

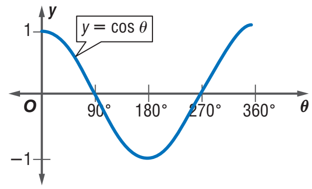


Graphing Calculator Investigation

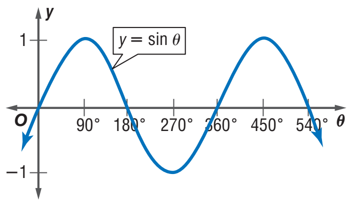
Sine and Cosine on the Unit Circle Follow the given steps for the TI-83 calculator also. Point out that, after the mode and window have been defined, from the Y= screen, pressing the X,T,θ,n key will automatically enter T. After pressing TRACE, use ▶ to move around the circle counterclockwise. Call attention to the information that appears on the screen as the cursor moves around the circle.

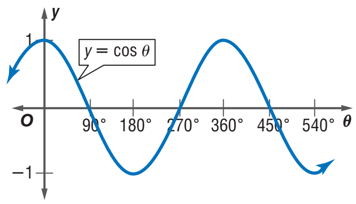
This same information is presented on the graphs of the sine and cosine functions below, where the horizontal axis shows the values of θ and the vertical axis shows the values of $\sin\theta$ or $\cos\theta$.





PERIODIC FUNCTIONS Notice in the graph above that the values of sine for the coterminal angles 0° and 360° are both 0. The values of cosine for these angles are both 1. Every 360° or 2π radians, the sine and cosine functions repeat their values. So, we can say that the sine and cosine functions are **periodic**, each having a **period** of 360° or 2π radians.





Key Concept — *Periodic Function*

A function is called periodic if there is a number a such that $f(x) = f(x + a)$ for all x in the domain of the function. The least positive value of a for which $f(x) = f(x + a)$ is called the period of the function.

For the sine and cosine functions, $\cos(x + 360°) = \cos x$, and $\sin(x + 360°) = \sin x$. In radian measure, $\cos(x + 2\pi) = \cos x$, and $\sin(x + 2\pi) = \sin x$. Therefore, the period of the sine and cosine functions is 360° or 2π.

Example 2 Find the Value of a Trigonometric Function

Find the exact value of each function.

a. $\cos 675°$

$$\begin{aligned}\cos 675° &= \cos(315° + 360°)\\ &= \cos 315°\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

b. $\sin\left(-\frac{5\pi}{6}\right)$

$$\begin{aligned}\sin\left(-\frac{5\pi}{6}\right) &= \sin\left(-\frac{5\pi}{6} + 2\pi\right)\\ &= \sin\frac{7\pi}{6}\\ &= -\frac{1}{2}\end{aligned}$$

When you look at the graph of a periodic function, you will see a repeating pattern: a shape that repeats over and over as you move to the right on the x-axis. The period is the distance along the x-axis from the beginning of the pattern to the point at which it begins again.

www.algebra2.com/extra_examples

PERIODIC FUNCTIONS

In-Class Example Power Point®

2 Find the exact value of each function.

a. $\cos 690°$ $\frac{\sqrt{3}}{2}$

b. $\sin\left(-\frac{3\pi}{4}\right)$ $-\frac{\sqrt{2}}{2}$

Lesson 13-6 Circular Functions 741

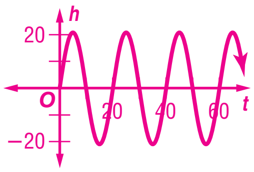
In-Class Example

Power Point®

3 **FERRIS WHEEL** On another Ferris wheel, the diameter is 42 feet, and it travels at a rate of 3 revolutions per minute.

a. Identify the period of this function. **20 seconds**

b. Make a graph in which the horizontal axis represents the time t in seconds and the vertical axis represents the height h in feet in relation to the starting point.



Answer

1. The terminal side of the angle θ in standard position must intersect the unit circle at $P(x, y)$.

Many real-world situations have characteristics that can be described with periodic functions.

Example 3 Find the Value of a Trigonometric Function

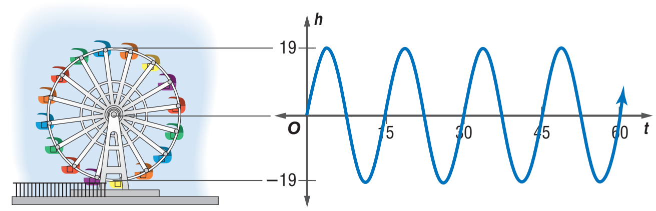
FERRIS WHEEL **As you ride a Ferris wheel, the height that you are above the ground varies periodically as a function of time. Consider the height of the center of the wheel to be the starting point. A particular wheel has a diameter of 38 feet and travels at a rate of 4 revolutions per minute.**

a. Identify the period of this function.

Since the wheel makes 4 complete counterclockwise rotations every minute, the period is the time it takes to complete one rotation, which is $\frac{1}{4}$ of a minute or 15 seconds.

b. Make a graph in which the horizontal axis represents the time t in seconds and the vertical axis represents the height h in feet in relation to the starting point.

Since the diameter of the wheel is 38 feet, the wheel reaches a maximum height of $\frac{38}{2}$ or 19 feet above the starting point and a minimum of 19 feet below the starting point.



Check for Understanding

Concept Check

1. **State** the conditions under which $\cos \theta = x$ and $\sin \theta = y$. **See margin.**
2. **OPEN ENDED** Give an example of a situation that could be described by a periodic function. Then state the period of the function. **2. Sample answer: the motion of the minute hand on a clock; 60 s**
3. **Compare and contrast** the graphs of the sine and cosine functions on page 741. **Sample answer: The graphs have the same shape, but cross the x-axis at different points.**

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4, 5	1
6, 7	2
8–10	3

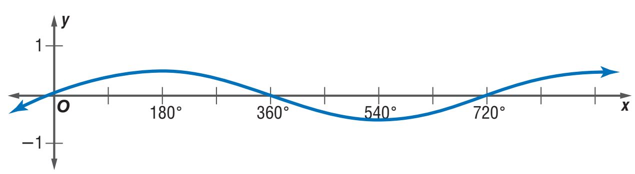
If the given point P is located on the unit circle, find $\sin \theta$ and $\cos \theta$.

4. $P\left(\frac{5}{13}, -\frac{12}{13}\right)$ **$\sin \theta = -\frac{12}{13}$; $\cos \theta = \frac{5}{13}$**
5. $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **$\sin \theta = \frac{\sqrt{2}}{2}$; $\cos \theta = \frac{\sqrt{2}}{2}$**

Find the exact value of each function.

6. $\sin -240°$ **$\frac{\sqrt{3}}{2}$**
7. $\cos \frac{10\pi}{3}$ **$-\frac{1}{2}$**

8. Determine the period of the function that is graphed below. **720°**



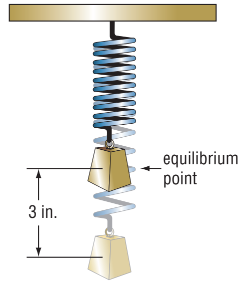
DAILY INTERVENTION

Differentiated Instruction

Naturalist Have students research various kinds of circular calendars, such as those used by the Mayans, to predict the weather and determine the best time for planting crops.

Application **PHYSICS** **For Exercises 9 and 10, use the following information.**

The motion of a weight on a spring varies periodically as a function of time. Suppose you pull the weight down 3 inches from its equilibrium point and then release it. It bounces above the equilibrium point and then returns below the equilibrium point in 2 seconds.



9. Find the period of this function. **2 s**

10. Graph the height of the spring as a function of time. **See margin.**

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
11–16	1
17–28	2
29–42	3

Extra Practice

See page 858.

11. $\sin\theta = \frac{4}{5}$; $\cos\theta = -\frac{3}{5}$

12. $\sin\theta = -\frac{5}{13}$; $\cos\theta = -\frac{12}{13}$

13. $\sin\theta = \frac{15}{17}$; $\cos\theta = \frac{8}{17}$

14. $\sin\theta = -\frac{1}{2}$; $\cos\theta = \frac{\sqrt{3}}{2}$

15. $\sin\theta = \frac{\sqrt{3}}{2}$; $\cos\theta = -\frac{1}{2}$

16. $\sin\theta = 0.8$; $\cos\theta = 0.6$

The given point P is located on the unit circle. Find $\sin\theta$ and $\cos\theta$.

11. $P\left(-\frac{3}{5}, \frac{4}{5}\right)$
12. $P\left(-\frac{12}{13}, -\frac{5}{13}\right)$
13. $P\left(\frac{8}{17}, \frac{15}{17}\right)$
14. $P\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
15. $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
16. $P(0.6, 0.8)$

Find the exact value of each function.

17. $\sin 690°$ $-\frac{1}{2}$
18. $\cos 750°$ $\frac{\sqrt{3}}{2}$
19. $\cos 5\pi$ -1
20. $\sin\left(\frac{14\pi}{6}\right)$ $\frac{\sqrt{3}}{2}$
21. $\sin\left(-\frac{3\pi}{2}\right)$ 1
22. $\cos(-225°)$ $-\frac{\sqrt{2}}{2}$

★ 23. $\frac{\cos 60° + \sin 30°}{4}$ $\frac{1}{4}$

★ 24. $3(\sin 60°)(\cos 30°)$ $\frac{9}{4}$

★ 25. $\sin 30° - \sin 60°$ $\frac{1-\sqrt{3}}{2}$

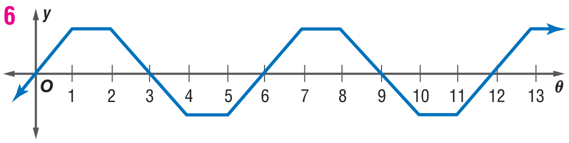
★ 26. $\frac{4\cos 330° + 2\sin 60°}{3}$ $\sqrt{3}$

★ 27. $12(\sin 150°)(\cos 150°)$ $-3\sqrt{3}$

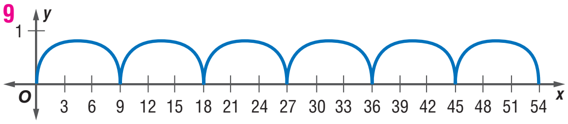
★ 28. $(\sin 30°)^2 + (\cos 30°)^2$ 1

Determine the period of each function.

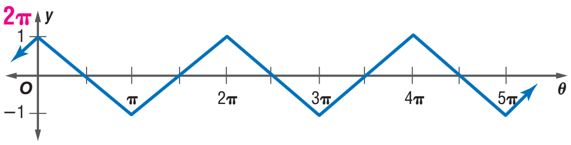
29. **6**



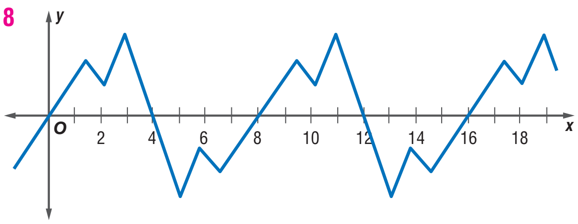
30. **9**



31. **2π**



32. **8**



www.algebra2.com/self_check_quiz

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 13.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective

- **Unit Circle Definitions:** 11–16
- **Periodic Functions:** 17–42

Odd/Even Assignments

Exercises 11–32 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

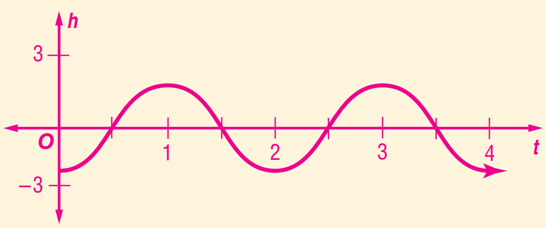
Basic: 11–21 odd, 29, 31, 33, 34, 43–69

Average: 11–31 odd, 33–35, 43–69

Advanced: 12–32 even, 33, 34, 36–63 (optional: 64–69)

Answer

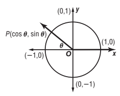
10.



Study Guide and Intervention, p. 805 (shown) and p. 806

Unit Circle Definitions

Definition of Sine and Cosine	If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos\theta = x$ and $\sin\theta = y$. Therefore, the coordinates of P can be written as $P(\cos\theta, \sin\theta)$.



Example **Given an angle θ in standard position, if $P\left(-\frac{5}{6}, \frac{\sqrt{11}}{6}\right)$ lies on the terminal side and on the unit circle, find $\sin\theta$ and $\cos\theta$.**

$P\left(-\frac{5}{6}, \frac{\sqrt{11}}{6}\right) = P(\cos\theta, \sin\theta)$, so $\sin\theta = \frac{\sqrt{11}}{6}$ and $\cos\theta = -\frac{5}{6}$.

Exercises

If θ is an angle in standard position and if the given point P is located on the terminal side of θ and on the unit circle, find $\sin\theta$ and $\cos\theta$.

1. $P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **$\sin\theta = \frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$**
2. $P(0, -1)$ **$\sin\theta = -1$, $\cos\theta = 0$**
3. $P\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$ **$\sin\theta = \frac{\sqrt{5}}{3}$, $\cos\theta = -\frac{2}{3}$**
4. $P\left(-\frac{4}{5}, -\frac{3}{5}\right)$ **$\sin\theta = -\frac{3}{5}$, $\cos\theta = -\frac{4}{5}$**
5. $P\left(\frac{1}{6}, -\frac{\sqrt{35}}{6}\right)$ **$\sin\theta = -\frac{\sqrt{35}}{6}$, $\cos\theta = \frac{1}{6}$**
6. $P\left(\frac{\sqrt{7}}{4}, \frac{3}{4}\right)$ **$\sin\theta = \frac{3}{4}$, $\cos\theta = \frac{\sqrt{7}}{4}$**
7. P is on the terminal side of $\theta = 45°$. **$\sin\theta = \frac{\sqrt{2}}{2}$, $\cos\theta = \frac{\sqrt{2}}{2}$**
8. P is on the terminal side of $\theta = 120°$. **$\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$**
9. P is on the terminal side of $\theta = 240°$. **$\sin\theta = -\frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$**
10. P is on the terminal side of $\theta = 330°$. **$\sin\theta = -\frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$**

Skills Practice, p. 807 and Practice, p. 808 (shown)

The given point P is located on the unit circle. Find $\sin\theta$ and $\cos\theta$.

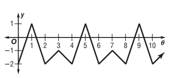
1. $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **$\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$**
2. $P\left(\frac{20}{29}, -\frac{21}{29}\right)$ **$\sin\theta = -\frac{21}{29}$, $\cos\theta = \frac{20}{29}$**
3. $P(0.8, 0.6)$ **$\sin\theta = 0.6$, $\cos\theta = 0.8$**
4. $P(0, -1)$ **$\sin\theta = -1$, $\cos\theta = 0$**
5. $P\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **$\sin\theta = -\frac{\sqrt{2}}{2}$, $\cos\theta = -\frac{\sqrt{2}}{2}$**
6. $P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **$\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$**

Find the exact value of each function.

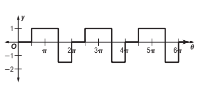
7. $\cos\frac{7\pi}{4}$ **$\frac{\sqrt{2}}{2}$**
8. $\sin(-30°)$ **$-\frac{1}{2}$**
9. $\sin\left(-\frac{2\pi}{3}\right)$ **$-\frac{\sqrt{3}}{2}$**
10. $\cos(-330°)$ **$\frac{\sqrt{3}}{2}$**
11. $\cos 600°$ **$-\frac{1}{2}$**
12. $\sin\frac{9\pi}{2}$ **1**
13. $\cos 7\pi$ **-1**
14. $\cos\left(-\frac{11\pi}{4}\right)$ **$-\frac{\sqrt{2}}{2}$**
15. $\sin(-225°)$ **$\frac{\sqrt{2}}{2}$**
16. $\sin 585°$ **$-\frac{\sqrt{2}}{2}$**
17. $\cos\left(-\frac{10\pi}{3}\right)$ **$-\frac{1}{2}$**
18. $\sin 840°$ **$\frac{\sqrt{3}}{2}$**

Determine the period of each function.

19. **4**



20. **2π**



21. **FERRIS WHEELS** A Ferris wheel with a diameter of 100 feet completes 2.5 revolutions per minute. What is the period of the function that describes the height of a seat on the outside edge of the Ferris Wheel as a function of time? **24 s**

Reading to Learn Mathematics, p. 809 ELL

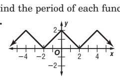
Pre-Activity **How can you model annual temperature fluctuations?**

Read the introduction to Lesson 13-6 at the top of page 739 in your textbook.

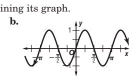
- If the graph in your textbook is continued, what month will $x = 17$ represent? **May of the following year**
- About what do you expect the average high temperature to be for that month? **24.2°F**
- Will this be exactly the average high temperature for that month? Explain your answer. **Sample answer: No; temperatures vary from year to year.**

Reading the Lesson

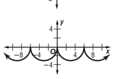
1. Use the unit circle on page 740 in your textbook to find the exact values of each expression.
   a. $\cos 45°$ **$\frac{\sqrt{2}}{2}$** b. $\sin 150°$ **$\frac{1}{2}$** c. $\sin 240°$ **$-\frac{\sqrt{3}}{2}$**
   d. $\sin 315°$ **$-\frac{\sqrt{2}}{2}$** e. $\cos 270°$ **0** f. $\sin 210°$ **$-\frac{1}{2}$**
   g. $\cos 0°$ **1** h. $\sin 180°$ **0** i. $\cos 330°$ **$\frac{\sqrt{3}}{2}$**
2. Tell whether each function is periodic. Write *yes* or *no*.
   a. $y = 2x$ **no** b. $y = x^2$ **no** c. $y = \cos x$ **yes** d. $y = |x|$ **no**
3. Find the period of each function by examining its graph.
   a. **4**



   b. **π**



   c. **6**



Helping You Remember

4. What is an easy way to remember the periods of the sine and cosine functions in radian measure? **Sample answer: The period of both functions is 2π, which is the circumference of the unit circle.**

More About. . .

Guitar

Most guitars have six strings. The frequency at which one of these strings vibrates is controlled by the length of the string, the amount of tension on the string, the weight of the string, and springiness of the strings' material.

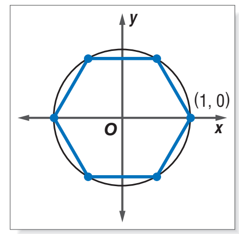
Source: www.howstuffworks.com

GUITAR **For Exercises 33 and 34, use the following information.**
When a guitar string is plucked, it is displaced from a fixed point in the middle of the string and vibrates back and forth, producing a musical tone. The exact tone depends on the frequency, or number of cycles per second, that the string vibrates. To produce an A, the frequency is 440 cycles per second, or 440 hertz (Hz).

33. Find the period of this function. **$\frac{1}{440}$ s**

34. Graph the height of the fixed point on the string from its resting position as a function of time. Let the maximum distance above the resting position have a value of 1 unit and the minimum distance below this position have a value of 1 unit. **See pp. 759A–759D.**

35. **GEOMETRY** A regular hexagon is inscribed in a unit circle centered at the origin. If one vertex of the hexagon is at (1, 0), find the exact coordinates of the remaining vertices. **$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $(-1, 0)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$**



36. **BIOLOGY** In a certain area of forested land, the population of rabbits R increases and decreases periodically throughout the year. If the population can be modeled by $R = 425 + 200\sin\left[\frac{\pi}{365}(d - 60)\right]$, where d represents the dth day of the year, describe what happens to the population throughout the year. **See margin.**

SLOPE **For Exercises 37–42, use the following information.**
Suppose the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$.

37. What is the slope of $\overline{OP}$? **$\frac{y}{x}$**

38. Which of the six trigonometric functions is equal to the slope of $\overline{OP}$? **$\tan\theta$**

39. What is the slope of any line perpendicular to $\overline{OP}$? **$-\frac{x}{y}$**

40. Which of the six trigonometric functions is equal to the slope of any line perpendicular to $\overline{OP}$? **$-\cot\theta$**

41. Find the slope of $\overline{OP}$ when $\theta = 60°$. **$\sqrt{3}$**

42. If $\theta = 60°$, find the slope of the line tangent to circle O at point P. **$-\frac{\sqrt{3}}{3}$**

43. **CRITICAL THINKING** Determine the domain and range of the functions $y = \sin\theta$ and $y = \cos\theta$. **sine: D = {all reals}, R = $\{-1 \le y \le 1\}$; cosine: D = {all reals}, R = $\{-1 \le y \le 1\}$**

44. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

How can you model annual temperature fluctuations?

Include the following in your answer:

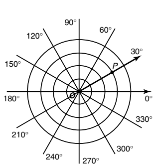
- a description of how the sine and cosine functions are similar to annual temperature fluctuations, and
- if the formula for the temperature T in degrees Fahrenheit of a city t months into the year is given by $T = 50 + 25\sin\left(\frac{\pi}{6}t\right)$, explain how to find the average temperature and the maximum and minimum predicted over the year.

Enrichment, p. 810

Polar Coordinates

Consider an angle in standard position with its vertex at a point O called the *pole.* Its initial side is on a coordinated axis called the *polar axis.* A point P on the terminal side of the angle is named by the *polar coordinates* (r, θ) where r is the directed distance of the point from O and θ is the measure of the angle.

Graphs in this system may be drawn on polar coordinate paper such as the kind shown at the right.



The polar coordinates of a point are not unique. For example, (3, 30°) names point P as well as (3, 390°). Another name for P is (−3, 210°). Can you see why? The coordinates of the pole are $(0, \theta)$ where θ may be any angle.

Example **Draw the graph of the function $r = \cos\theta$. M**[illegible]**a table of convenient values for θ** [illegible] **points**

Answer

36. The population is around 425 near the 60th day of the year. It rises to around 625 in May/June. It falls to around 425 again by August/September. It continues to drop to around 225 in November/December.

Standardized Test Practice

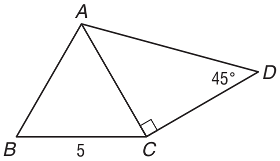
45. If $\triangle ABC$ is an equilateral triangle, what is the length of $\overline{AD}$, in units? **A**

(A) $5\sqrt{2}$

(B) 5

(C) $10\sqrt{2}$

(D) 10



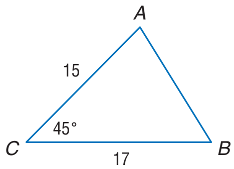
46. **SHORT RESPONSE** What is the exact value of tan 1830°? $\frac{\sqrt{3}}{3}$

Maintain Your Skills

Mixed Review

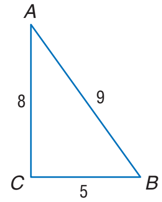
Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-5)*

47.



cosines; $c \approx 12.4$, $B \approx 59°$, $A \approx 76°$

48.



cosines; $A \approx 34°$, $B \approx 62°$, $C \approx 84°$

Find the area of $\triangle ABC$. Round to the nearest tenth. *(Lesson 13-4)*

49. $a = 11$ in., $c = 5$ in., $B = 79°$ **27 in²**

50. $b = 4$ m, $c = 7$ m, $A = 63°$ **12.5 m²**

BULBS **For Exercises 51–56, use the following information.**
The lifetimes of 10,000 light bulbs are normally distributed. The mean lifetime is 300 days, and the standard deviation is 40 days. *(Lesson 12-7)*

51. How many light bulbs will last 260 and 340 days? **6800**

52. How many light bulbs will last between 220 and 380 days? **9500**

53. How many light bulbs will last fewer than 300 days? **5000**

54. How many light bulbs will last more than 300 days? **5000**

55. How many light bulbs will last more than 380 days? **250**

56. How many light bulbs will last fewer than 180 days? **50**

Find the sum of each infinite geometric series, if it exists. *(Lesson 11-5)*

57. $a_1 = 3, r = 1.2$ **does not exist**

58. $16, 4, 1, \frac{1}{4}, \ldots$ $\frac{64}{3}$

59. $\sum_{n=1}^{\infty} 13(-0.625)^{n-1}$ **8**

Use synthetic division to find each quotient. *(Lesson 5-3)*

60. $(4x^2 - 13x + 10) \div (x - 2)$ $4x - 5$

61. $(2x^2 + 21x + 54) \div (x + 6)$ $2x + 9$

62. $(5y^3 + y^2 - 7) \div (y + 1)$ $5y^2 - 4y + 4 - \frac{11}{y+1}$

63. $(2y^2 + y - 16) \div (y - 3)$ $2y + 7 + \frac{5}{y-3}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each value of θ. Round to the nearest degree.**
*(To review **finding angle measures**, see Lesson 13-1.)*

64. $\sin \theta = 0.3420$ **20°**

65. $\cos \theta = -0.3420$ **110°**

66. $\tan \theta = 3.2709$ **73°**

67. $\tan \theta = 5.6713$ **80°**

68. $\sin \theta = 0.8290$ **56°**

69. $\cos \theta = 0.0175$ **89°**

4 Assess

Open-Ended Assessment

Modeling Have students use a geoboard to make a model of a regular octagon inscribed in a unit circle, similar to that shown in Exercise 35, and find the exact coordinates of the vertices.

Assessment Options

Quiz (Lessons 13-5 and 13-6) is available on p. 832 of the *Chapter 13 Resource Masters*.

Getting Ready for Lesson 13-7

PREREQUISITE SKILL Lesson 13-7 presents inverse trigonometric functions. Students will use their familiarity with finding angle measures as they solve equations with inverse trigonometric functions. Exercises 64–69 should be used to determine your students' familiarity with finding angle measures.

Answer

44. Answers should include the following.

- Over the course of one period both the sine and cosine function attain their maximum value once and their minimum value once. From the maximum to the minimum the functions decrease slowly at first, then decrease more quickly and return to a slow rate of change as they come into the minimum. Similarly, the functions rise slowly from their minimum. They begin to increase more rapidly as they pass the halfway point, and then begin to rise more slowly as they increase into the maximum. Annual temperature fluctuations behave in exactly the same manner.
- The maximum value of the sine function is 1 so the maximum temperature would be $50 + 25(1)$ or 75° F. Similarly, the minimum value would be $50 + 25(-1)$ or 25° F. The average temperature over this time period occurs when the sine function takes on a value of 0. In this case that would be 50° F.

13-7 Lesson Notes

1 Focus

5-Minute Check Transparency 13-7 Use as a quiz or review of Lesson 13-6.

Mathematical Background notes are available for this lesson on p. 698D.

How are inverse trigonometric functions used in road design?

Ask students:

- Where else have you seen banking on curves used? **Sample answers: skating, skateboarding, and other sports**
- The force of gravity is 32 feet per second per second. Where does gravity appear in the formula? **in the denominator**

13-7 Inverse Trigonometric Functions

What You'll Learn

- Solve equations by using inverse trigonometric functions.
- Find values of expressions involving trigonometric functions.

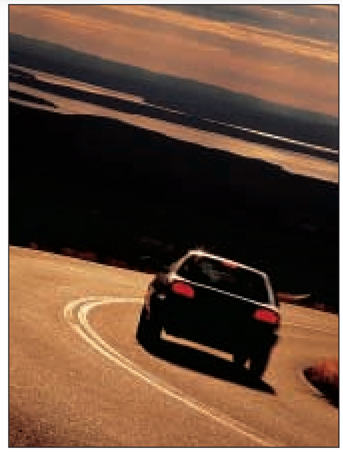
Vocabulary

- principal values
- Arcsine function
- Arccosine function
- Arctangent function

How are inverse trigonometric functions used in road design?

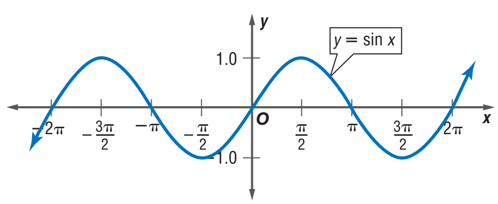
When a car travels a curve on a horizontal road, the friction between the tires and the road keeps the car on the road. Above a certain speed, however, the force of friction will not be great enough to hold the car in the curve. For this reason, civil engineers design banked curves.

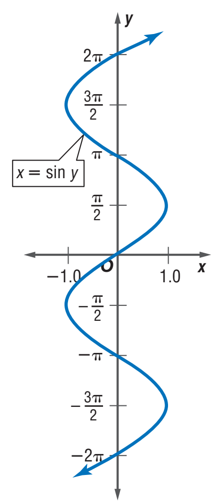
The proper banking angle θ for a car making a turn of radius r feet at a velocity v in feet per second is given by the equation $\tan \theta = \frac{v^2}{32r}$. In order to determine the appropriate value of θ for a specific curve, you need to know the radius of the curve, the maximum allowable velocity of cars making the curve, and how to determine the angle θ given the value of its tangent.



SOLVE EQUATIONS USING INVERSES Sometimes the value of a trigonometric function for an angle is known and it is necessary to find the measure of the angle. The concept of inverse functions can be applied to find the inverse of trigonometric functions.

In Lesson 8-8, you learned that the inverse of a function is the relation in which all the values of x and y are reversed. The graphs of $y = \sin x$ and its inverse, $x = \sin y$, are shown below.





Notice that the inverse is not a function, since it fails the vertical line test. None of the inverses of the trigonometric functions are functions.

We must restrict the domain of trigonometric functions so that their inverses are functions. The values in these restricted domains are called **principal values**. Capital letters are used to distinguish trigonometric functions with restricted domains from the usual trigonometric functions.

Resource Manager

Workbook and Reproducible Masters

Chapter 13 Resource Masters

- Study Guide and Intervention, pp. 811–812
- Skills Practice, p. 813
- Practice, p. 814
- Reading to Learn Mathematics, p. 815
- Enrichment, p. 816
- Assessment, p. 832

Transparencies

5-Minute Check Transparency 13-7
Answer Key Transparencies

Technology

Interactive Chalkboard

Key Concept — Principal Values of Sine, Cosine, and Tangent

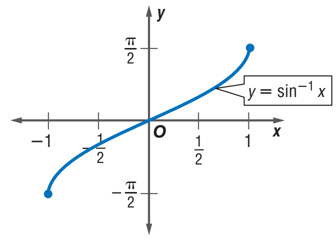
$y = \text{Sin } x$ if and only if $y = \sin x$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

$y = \text{Cos } x$ if and only if $y = \cos x$ and $0 \leq x \leq \pi$.

$y = \text{Tan } x$ if and only if $y = \tan x$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The inverse of the Sine function is called the **Arcsine function** and is symbolized by **Sin^{-1}** or **Arcsin**. The Arcsine function has the following characteristics.

- Its domain is the set of real numbers from -1 to 1.
- Its range is the set of angle measures from $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
- $\text{Sin } x = y$ if and only if $\text{Sin}^{-1} y = x$.
- $[\text{Sin}^{-1} \circ \text{Sin}](x) = [\text{Sin} \circ \text{Sin}^{-1}](x) = x$.



Study Tip

Look Back
To review **composition of functions**, see Lesson 8-7.

The definitions of the Arccosine and Arctangent functions are similar to the definition of the Arcsine function.

Concept Summary — Inverse Sine, Cosine, and Tangent

- Given $y = \text{Sin } x$, the inverse Sine function is defined by $y = \text{Sin}^{-1} x$ or $y = \text{Arcsin } x$.
- Given $y = \text{Cos } x$, the inverse Cosine function is defined by $y = \text{Cos}^{-1} x$ or $y = \text{Arccos } x$.
- Given $y = \text{Tan } x$, the inverse Tangent function is defined by $y = \text{Tan}^{-1} x$ or $y = \text{Arctan } x$.

The expressions in each row of the table below are equivalent. You can use these expressions to rewrite and solve trigonometric equations.

$y = \text{Sin } x$	$x = \text{Sin}^{-1} y$	$x = \text{Arcsin } y$
$y = \text{Cos } x$	$x = \text{Cos}^{-1} y$	$x = \text{Arccos } y$
$y = \text{Tan } x$	$x = \text{Tan}^{-1} y$	$x = \text{Arctan } y$

Example 1 — Solve an Equation

Solve $\text{Sin } x = \frac{\sqrt{3}}{2}$ by finding the value of x to the nearest degree.

If $\text{Sin } x = \frac{\sqrt{3}}{2}$, then x is the least value whose sine is $\frac{\sqrt{3}}{2}$. So, $x = \text{Arcsin } \frac{\sqrt{3}}{2}$.

Use a calculator to find x.

KEYSTROKES: [2nd] [SIN^{-1}] [2nd] [√] 3 [)] [÷] 2 [)] [ENTER] 60

Therefore, $x = 60°$.

www.algebra2.com/extra_examples

2 Teach

SOLVE EQUATIONS USING INVERSES

In-Class Example Power Point®

Teaching Tip It may help students to understand the nature of inverse functions if they read arcsin x and $\sin^{-1} x$ as "the angle whose sine is x."

1 Solve $\text{Sin } x = \frac{\sqrt{2}}{2}$ by finding the value of x to the nearest degree. $x = 45°$

Lesson 13-7 Inverse Trigonometric Functions 747

In-Class Example

Power Point®

2 **DRAWBRIDGE** For the drawbridge shown in Example 2 in the Student Edition, what is the minimum angle θ, to the nearest degree, to which each leaf should open so that a ship that is 100 feet wide will fit? **52°**

TRIGONOMETRIC VALUES

In-Class Example

Power Point®

Teaching Tip Remind students to first put their calculators in Radian mode.

3 Find each value. Write angle measures in radians. Round to the nearest hundredth.

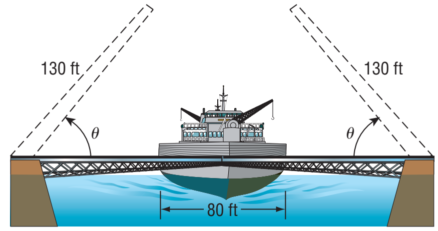
a. $\text{Arcsin } \frac{\sqrt{2}}{2}$ **about 0.79 radians**

b. $\tan\left(\text{Cos}^{-1} \frac{4}{5}\right)$ **0.75 radians**

Many application problems involve finding the inverse of a trigonometric function.

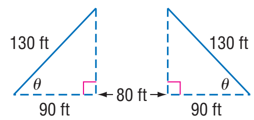
Example 2 Apply an Inverse to Solve a Problem

DRAWBRIDGE **Each leaf of a certain double-leaf drawbridge is 130 feet long. If an 80-foot wide ship needs to pass through the bridge, what is the minimum angle θ, to the nearest degree, which each leaf of the bridge should open so that the ship will fit?**



When the two parts of the bridge are in their lowered position, the bridge spans 130 + 130 or 260 feet. In order for the ship to fit, the distance between the leaves must be at least 80 feet.

This leaves a horizontal distance of $\frac{260 - 80}{2}$ or 90 feet from the pivot point of each leaf to the ship as shown in the diagram at the right.



To find the measure of angle θ, use the cosine ratio for right triangles.

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ Cosine ratio

$\cos\theta = \frac{90}{130}$ Replace *adj* with 90 and *hyp* with 130.

$\theta = \cos^{-1}\left(\frac{90}{130}\right)$ Inverse cosine function

$\theta \approx 46.2°$ Use a calculator.

Thus, the minimum angle through which each leaf of the bridge should open is 47°.

More About. . .

Drawbridges

Bascule bridges have spans (leaves) that pivot upward utilizing gears, motors, and counterweights.

Source: www.multnomah.lib.or.us

TRIGONOMETRIC VALUES You can use a calculator to find the values of trigonometric expressions.

Example 3 Find a Trigonometric Value

Find each value. Write angle measures in radians. Round to the nearest hundredth.

a. $\text{ArcSin } \frac{\sqrt{3}}{2}$

KEYSTROKES: [2nd] [SIN⁻¹] [2nd] [√] 3 [)] [÷] 2 [)] [ENTER] 1.047197551

Therefore, $\text{ArcSin } \frac{\sqrt{3}}{2} \approx 1.05$ radians.

b. $\tan\left(\text{Cos}^{-1} \frac{6}{7}\right)$

KEYSTROKES: [TAN] [2nd] [COS⁻¹] 6 [÷] 7 [)] [ENTER] .6009252126

Therefore, $\tan\left(\text{Cos}^{-1} \frac{6}{7}\right) \approx 0.60$.

Study Tip

Angle Measure

Remember that when evaluating an inverse trigonometric function the result is an angle measure.

DAILY INTERVENTION

Differentiated Instruction

Visual/Spatial Ask students to find Arcsin 2. If they use a calculator, suggest that they study the graph of $y = \sin x$ to explain why an error message was the result. **The graph of $y = \sin x$ has no y values greater than 1 or less than −1.**

Check for Understanding

Concept Check

1. **Explain** how you know when the domain of a trigonometric function is restricted. **Restricted domains are denoted with a capital letter.**
2. **OPEN ENDED** Write an equation giving the value of the Cosine function for an angle measure in its domain. Then, write your equation in the form of an inverse function.
3. Describe how $y = \text{Cos } x$ and $y = \text{Arccos } x$ are related. **They are inverses of each other.**

2. Sample answer: $\text{Cos } 45° = \frac{\sqrt{2}}{2}$; $\text{Cos}^{-1} \frac{\sqrt{2}}{2} = 45°$

Guided Practice

GUIDED PRACTICE KEY

Exercises	Examples
4–7	1
8–13	3
14	2

Write each equation in the form of an inverse function.

4. $\tan \theta = x$ **$\theta = \text{Arctan } x$**
5. $\cos \alpha = 0.5$ **$\alpha = \text{Arccos } 0.5$**

Solve each equation by finding the value of x to the nearest degree.

6. $x = \text{Cos}^{-1} \frac{\sqrt{2}}{2}$ **45°**
7. $\text{Arctan } 0 = x$ **0°**

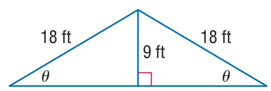
Find each value. Write angle measures in radians. Round to the nearest hundredth.

8. $\text{Tan}^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
9. $\text{Cos}^{-1}(-1)$ **$\pi \approx 3.14$**
10. $\cos\left(\text{Cos}^{-1} \frac{2}{9}\right)$ **0.22**
11. $\sin\left(\text{Sin}^{-1} \frac{3}{4}\right)$ **0.75**
12. $\sin\left(\text{Cos}^{-1} \frac{3}{4}\right)$ **0.66**
13. $\tan\left(\text{Sin}^{-1} \frac{1}{2}\right)$ **0.58**

8. $-\frac{\pi}{6} \approx -0.52$

Application

14. **ARCHITECTURE** The support for a roof is shaped like two right triangles as shown at the right. Find θ. **30°**



Practice and Apply

Homework Help

For Exercises	See Examples
15–26	1
27–42	3
43–48	2

Extra Practice
See page 859.

Write each equation in the form of an inverse function. **15–20. See margin.**

15. $\alpha = \sin \beta$
16. $\tan a = b$
17. $\cos y = x$
18. $\sin 30° = \frac{1}{2}$
19. $\cos 45° = y$
20. $-\frac{4}{3} = \tan x$

Solve each equation by finding the value of x to the nearest degree.

21. $x = \text{Cos}^{-1} \frac{1}{2}$ **60°**
22. $\text{Sin}^{-1} \frac{1}{2} = x$ **30°**
23. $\text{Arctan } 1 = x$ **45°**
24. $x = \text{Arctan } \frac{\sqrt{3}}{3}$ **30°**
25. $x = \text{Sin}^{-1} \frac{1}{\sqrt{2}}$ **45°**
26. $x = \text{Cos}^{-1} 0$ **90°**

Find each value. Write angle measures in radians. Round to the nearest hundredth.

27. $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$ **2.09**
28. $\text{Sin}^{-1} \frac{\pi}{2}$
29. $\text{Arctan } \frac{\sqrt{3}}{3}$ **0.52**
30. $\text{Arccos } \frac{\sqrt{3}}{2}$ **0.52**
31. $\sin\left(\text{Sin}^{-1} \frac{1}{2}\right)$ **0.5**
32. $\cot\left(\text{Sin}^{-1} \frac{5}{6}\right)$ **0.66**
33. $\tan\left(\text{Cos}^{-1} \frac{6}{7}\right)$ **0.60**
34. $\sin\left(\text{Arctan } \frac{\sqrt{3}}{3}\right)$ **0.5**
35. $\cos\left(\text{Arcsin } \frac{3}{5}\right)$ **0.8**
36. $\cot\left(\text{Sin}^{-1} \frac{7}{9}\right)$ **0.81**
37. $\cos\left(\text{Tan}^{-1} \sqrt{3}\right)$ **0.5**
38. $\tan\left(\text{Arctan } 3\right)$ **3**
39. $\cos\left[\text{Arccos}\left(-\frac{1}{2}\right)\right]$ **−0.5**
40. $\text{Sin}^{-1}\left(\tan \frac{\pi}{4}\right)$ **1.57**
41. $\cos\left(\text{Cos}^{-1} \frac{\sqrt{2}}{2} - \frac{\pi}{2}\right)$
42. $\cos^{-1}\left(\text{Sin}^{-1} 90\right)$ **does not exist**
43. $\sin\left(2\ \text{Cos}^{-1} \frac{3}{5}\right)$ **0.96**
44. $\sin\left(2\ \text{Sin}^{-1} \frac{1}{2}\right)$ **0.87**

28. does not exist

41. 0.71

www.algebra2.com/self_check_quiz

3 Practice/Apply

Study Notebook

Have students—
- complete the definitions/examples for the remaining terms on their Vocabulary Builder worksheets for Chapter 13.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective
- **Solve Equations Using Inverses:** 15–26, 45–48
- **Trigonometric Values:** 27–44

Odd/Even Assignments

Exercises 15–44 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 15–45 odd, 49–54, 58–66

Average: 15–47 odd, 49–54, 58–66 (optional: 55–57)

Advanced: 16–48 even, 49–66

Answers

15. $\beta = \text{Arcsin } \alpha$
16. $a = \text{Arctan } b$
17. $y = \text{Arccos } x$
18. $30° = \text{Arcsin } \frac{1}{2}$
19. $\text{Arccos } y = 45°$
20. $\text{Arctan}\left(-\frac{4}{3}\right) = x$

Study Guide and Intervention, p. 811 (shown) and p. 812

Solve Equations Using Inverses If the domains of trigonometric functions are restricted to their **principal values**, then their inverses are also functions.

Principal Values of Sine, Cosine, and Tangent	$y = \text{Sin } x$ if and only if $y = \sin x$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. $y = \text{Cos } x$ if and only if $y = \cos x$ and $0 \le x \le \pi$. $y = \text{Tan } x$ if and only if $y = \tan x$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
Inverse Sine, Cosine, and Tangent	Given $y = \text{Sin } x$, the inverse Sine function is defined by $y = \text{Sin}^{-1} x$ or $y = \text{Arcsin } x$. Given $y = \text{Cos } x$, the inverse Cosine function is defined by $y = \text{Cos}^{-1} x$ or $y = \text{Arccos } x$. Given $y = \text{Tan } x$, the inverse Tangent function is given by $y = \text{Tan}^{-1} x$ or $y = \text{Arctan } x$.

Example 1 **Solve $x = \text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)$.**

If $x = \text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)$, then $\text{Sin } x = \frac{\sqrt{3}}{2}$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

The only x that satisfies both criteria is $x = \frac{\pi}{3}$ or $60°$.

Example 2 **Solve $\text{Arctan}\left(-\frac{\sqrt{3}}{3}\right) = x$.**

If $x = \text{Arctan}\left(-\frac{\sqrt{3}}{3}\right)$, then $\text{Tan } x = -\frac{\sqrt{3}}{3}$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

The only x that satisfies both criteria is $-\frac{\pi}{6}$ or $-30°$.

Exercises

Solve each equation by finding the value of x to the nearest degree.

1. $\text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right) = x$ **150°**
2. $x = \text{Sin}^{-1} \frac{\sqrt{3}}{2}$ **60°**
3. $x = \text{Arccos}(-0.8)$ **143°**
4. $x = \text{Arctan } \sqrt{3}$ **60°**
5. $x = \text{Arccos}\left(-\frac{\sqrt{2}}{2}\right)$ **135°**
6. $x = \text{Tan}^{-1}(-1)$ **−45°**
7. $\text{Sin}^{-1} 0.45 = x$ **27°**
8. $x = \text{Arcsin}\left(-\frac{\sqrt{3}}{2}\right)$ **−60°**
9. $x = \text{Arccos}\left(-\frac{1}{2}\right)$ **120°**
10. $\text{Cos}^{-1}(-0.2) = x$ **102°**
11. $x = \text{Tan}^{-1}(-\sqrt{3})$ **−60°**
12. $x = \text{Arcsin } 0.3$ **17°**
13. $x = \text{Tan}^{-1}(15)$ **86°**
14. $x = \text{Cos}^{-1} 1$ **0**
15. $\text{Arctan}^{-1}(-3) = x$ **−72°**
16. $x = \text{Sin}^{-1}(-0.9)$ **−64°**
17. $\text{Arccos}^{-1} 0.15$ **81°**
18. $x = \text{Tan}^{-1} 0.2$ **11°**

Skills Practice, p. 813 and Practice, p. 814 (shown)

Write each equation in the form of an inverse function.

1. $\beta = \cos \alpha$ **$\alpha = \cos^{-1} \beta$**
2. $\tan \beta = \alpha$ **$\beta = \tan^{-1} \alpha$**
3. $y = \tan 120°$ **$120° = \tan^{-1} y$**
4. $-\frac{1}{2} = \cos x$ **$x = \cos^{-1}\left(-\frac{1}{2}\right)$**
5. $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ **$\sin^{-1} \frac{\sqrt{3}}{2} = \frac{2\pi}{3}$**
6. $\cos \frac{\pi}{3} = \frac{1}{2}$ **$\cos^{-1} \frac{1}{2} = \frac{\pi}{3}$**

Solve each equation by finding the value of x to the nearest degree.

7. $\text{Arcsin } 1 = x$ **90°**
8. $\text{Cos}^{-1} \frac{\sqrt{3}}{2} = x$ **30°**
9. $x = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ **−30°**
10. $x = \text{Arccos } \frac{\sqrt{2}}{2}$ **45°**
11. $x = \text{Arctan}(-\sqrt{3})$ **−60°**
12. $\text{Sin}^{-1}\left(-\frac{1}{2}\right) = x$ **−30°**

Find each value. Write angle measures in radians. Round to the nearest hundredth.

13. $\text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **2.62 radians**
14. $\text{Sin}^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ **−0.79 radians**
15. $\text{Arctan}\left(-\frac{\sqrt{3}}{3}\right)$ **−0.52 radians**
16. $\tan\left(\text{Cos}^{-1} \frac{1}{2}\right)$ **1.73**
17. $\cos\left[\text{Sin}^{-1}\left(-\frac{3}{5}\right)\right]$ **0.8**
18. $\cos[\text{Arctan}(-1)]$ **0.71**
19. $\tan\left(\sin^{-1} \frac{12}{13}\right)$ **2.4**
20. $\sin\left(\text{Arctan } \frac{\sqrt{3}}{3}\right)$ **0.5**
21. $\text{Cos}^{-1}\left(\tan \frac{3\pi}{4}\right)$ **3.14 radians**
22. $\text{Sin}^{-1}\left(\cos \frac{\pi}{3}\right)$ **0.52 radians**
23. $\sin\left(2 \text{ Cos}^{-1} \frac{15}{17}\right)$ **0.83**
24. $\cos\left(2 \text{ Sin}^{-1} \frac{\sqrt{3}}{2}\right)$ **−0.5**

25. **PULLEYS** The equation $x = \cos^{-1} 0.95$ describes the angle through which pulley A moves, and $y = \cos^{-1} 0.17$ describes the angle through which pulley B moves. Both angles are greater than 270° and less than 360°. Which pulley moves through a greater angle? **pulley *A***

26. **FLYWHEELS** The equation $y = \text{Arctan } 1$ describes the counterclockwise angle through which a flywheel rotates in 1 millisecond. Through how many degrees has the flywheel rotated after 25 milliseconds? **1125°**

Reading to Learn Mathematics, p. 815 ELL

Pre-Activity **How are inverse trigonometric functions used in road design?**

Read the introduction to Lesson 13-7 at the top of page 746 in your textbook.

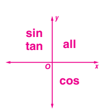
Suppose you are given specific values for v and r. What feature of your graphing calculator could you use to find the approximate measure of the banking angle θ? **Sample answer: the TABLE feature**

Reading the Lesson

1. Indicate whether each statement is *true* or *false*.
   a. The domain of the function $y = \sin x$ is the set of all real numbers. **true**
   b. The domain of the function $y = \text{Cos } x$ is $0 \le x \le \pi$. **true**
   c. The range of the function $y = \text{Tan } x$ is $-1 \le y \le 1$. **false**
   d. The domain of the function $y = \text{Cos}^{-1} x$ is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. **false**
   e. The domain of the function $y = \text{Tan}^{-1} x$ is the set of all real numbers. **true**
   f. The range of the function $y = \text{Arcsin } x$ is $0 \le x \le \pi$. **false**
2. Answer each question in your own words.
   a. What is the difference between the functions $y = \sin x$ and the function $y = \text{Sin } x$? **Sample answer: The domain of $y = \sin x$ is the set of all real numbers, while the domain of $y = \text{Sin } x$ is restricted to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.**
   b. Why is it necessary to restrict the domains of the trigonometric functions in order to define their inverses? **Sample answer: Only one-to-one functions have inverses. None of the six basic trigonometric functions is one-to-one, but related one-to-one functions can be formed if the domains are restricted in certain ways.**

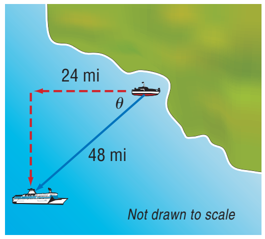
Helping You Remember

3. What is a good way to remember the domains of the functions $y = \text{Sin } x$, $y = \text{Cos } x$, and $y = \text{Tan } x$, which are also the range of the functions $y = \text{Arcsin } x$, $y = \text{Arccos } x$, and $y = \text{Arctan } x$? (You may want to draw a diagram.) **Sample answer: Each restricted domain must include an interval of numbers for which the function values are positive and one for which they are negative.**

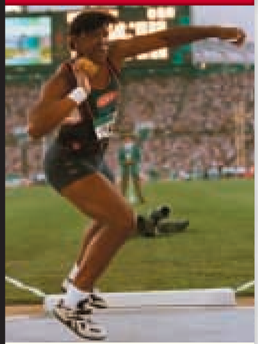


47. No; with this point on the terminal side of the throwing angle θ, the measure of θ is found by solving the equation $\tan \theta = \frac{17}{18}$. Thus $\theta = \tan^{-1} \frac{17}{18}$ or about 43.3°, which is greater than the 40° requirement.

45. **TRAVEL** The cruise ship *Reno* sailed due west 24 miles before turning south. When the *Reno* became disabled and radioed for help, the rescue boat found that the fastest route to her covered a distance of 48 miles. The cosine of the angle at which the rescue boat should sail is 0.5. Find the angle θ, to the nearest tenth of a degree, at which the rescue boat should travel to aid the *Reno*. **60° south of west**



More About. . .



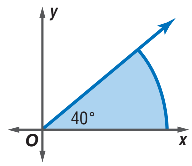
Track and Field

The shot is a metal sphere that can be made out of solid iron. Shot putters stand inside a seven-foot circle and must "put" the shot from the shoulder with one hand.

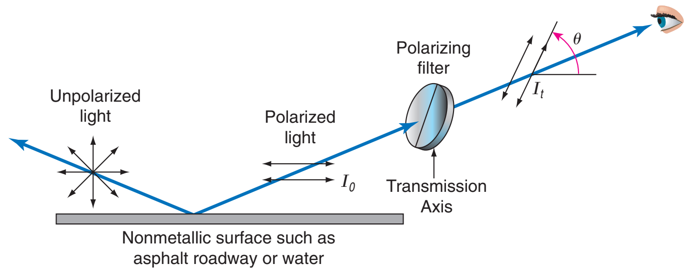
Source: www.coolrunning.com.au

46. **FOUNTAINS** Architects who design fountains know that both the height and distance that a water jet will project is dependent on the angle θ at which the water is aimed. For a given angle θ, the ratio of the maximum height H of the parabolic arc to the horizontal distance D it travels is given by $\frac{H}{D} = \frac{1}{4} \tan \theta$. Find the value of θ, to the nearest degree, that will cause the arc to go twice as high as it travels horizontally. **83°**

47. **TRACK AND FIELD** When a shot put is thrown, it must land in a 40° sector. Consider a coordinate system in which the vertex of the sector is at the origin and one side lies along the x-axis. If an athlete puts the shot so that it lands at a point with coordinates (18, 17), did the shot land in the required region? Explain your reasoning.

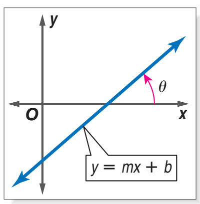


48. **OPTICS** You may have polarized sunglasses that eliminate glare by polarizing the light. When light is polarized, all of the waves are traveling in parallel planes. Suppose horizontally-polarized light with intensity I_0 strikes a polarizing filter with its axis at an angle of θ with the horizontal. The intensity of the transmitted light I_t and θ are related by the equation $\cos \theta = \sqrt{\frac{I_t}{I_0}}$. If one fourth of the polarized light is transmitted through the lens, what angle does the transmission axis of the filter make with the horizontal? **60°**



CRITICAL THINKING **For Exercises 49–51, use the following information.**

If the graph of the line $y = mx + b$ intersects the x-axis such that an angle of θ is formed with the positive x-axis, then $\tan \theta = m$.



49. Find the acute angle that the graph of $3x + 5y = 7$ makes with the positive x-axis to the nearest degree. **31°**

50. Determine the obtuse angle formed at the intersection of the graphs of $2x + 5y = 8$ and $6x - y = -8$. State the measure of the angle to the nearest degree. **102°**

51. Explain why this relationship, $\tan \theta = m$, holds true. **See margin.**

Enrichment, p. 816

Snell's Law

Snell's Law describes what happens to a ray of light that passes from air into water or some other substance. In the figure, the ray starts at the left and makes an angle of incidence θ with the surface.

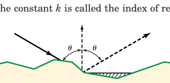
Part of the ray is reflected, creating an angle of reflection θ. The rest of the ray is bent, or refracted, as it passes through the other medium. This creates angle θ'.

The angle of incidence equals the angle of reflection.

The angles of incidence and refraction are related by Snell's Law:

$\sin \theta = k \sin \theta'$

The constant k is called the index of refraction.



k	Substance
1.33	Water
1.36	Ethyl alcohol

52. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. **See margin.**

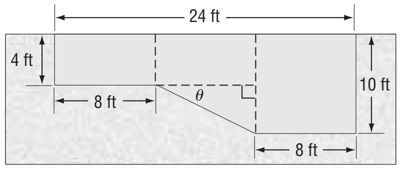
How are inverse trigonometric functions used in road design?

Include the following in your answer:

- a few sentences describing how to determine the banking angle for a road, and
- a description of what would have to be done to a road if the speed limit were increased and the banking angle was not changed.

Standardized Test Practice

53. **GRID IN** Find the angle of depression θ between the shallow end and the deep end of the swimming pool to the nearest degree. **37°**



Side View of Swimming Pool

54. If $\sin \theta = \frac{2}{3}$ and $-90° \leq \theta \leq 90°$, then $\cos 2\theta =$ **D**

Ⓐ $-\frac{1}{9}$. Ⓑ $-\frac{1}{3}$. Ⓒ $\frac{1}{3}$. Ⓓ $\frac{1}{9}$. Ⓔ 1.

Graphing Calculator

ADDITION OF TRIGONOMETRIC INVERSES Consider the function $y = \mathrm{Sin}^{-1}\, x + \mathrm{Cos}^{-1}\, x$.

55. Copy and complete the table below by evaluating y for each value of x.

x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
y	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$

56. $\mathrm{Sin}^{-1}\, x + \mathrm{Cos}^{-1}\, x = \frac{\pi}{2}$ for all values of x

56. Make a conjecture about the function $y = \mathrm{Sin}^{-1}\, x + \mathrm{Cos}^{-1}\, x$.

57. Considering only positive values of x, provide an explanation of why your conjecture might be true. **See margin.**

Maintain Your Skills

Mixed Review

Find the exact value of each function. *(Lesson 13-6)*

58. $\sin -660°$ **$\frac{\sqrt{3}}{2}$**

59. $\cos 25\pi$ **-1**

60. $(\sin 135°)^2 + (\cos -675°)^2$ **1**

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *(Lesson 13-5)*

61. $a = 3.1, b = 5.8, A = 30°$ **sines; $B \approx 69°$, $C \approx 81°$, $c \approx 6.1$**

62. $a = 9, b = 40, c = 41$ **cosines; $A = 13°$, $B = 77°$, $C = 90°$**

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function. *(Lesson 7-4)*

63. $f(x) = 5x^2 + 6x - 17$ **46, 39**

64. $f(x) = -3x^2 + 2x - 1$ **−22, −57**

65. $f(x) = 4x^2 - 10x + 5$ **11, 109**

66. **PHYSICS** A toy rocket is fired upward from the top of a 200-foot tower at a velocity of 80 feet per second. The height of the rocket t seconds after firing is given by the formula $h(t) = -16t^2 + 80t + 200$. Find the time at which the rocket reaches its maximum height of 300 feet. *(Lesson 6-5)* **2.5 s**

4 Assess

Open-Ended Assessment

Speaking Have students write several problems of their own using both types of notation for inverse trigonometric functions. Then ask them to read their problems aloud and explain how to solve each one.

Assessment Options

Quiz (Lesson 13-7) is available on p. 832 of the *Chapter 13 Resource Masters*.

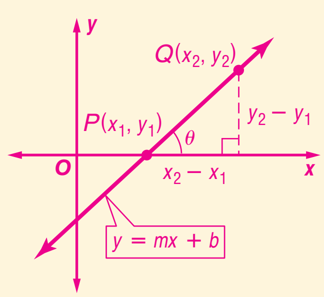
Answers

52. Trigonometry is used to determine proper banking angles. Answers should include the following.

- **Knowing the velocity of the cars to be traveling on a road and the radius of the curve to be built, then the banking angle can be determined. First find the ratio of the square of the velocity to the product of the acceleration due to gravity and the radius of the curve. Then determine the angle that had this ratio as its tangent. This will be the banking angle for the turn.**
- **If the speed limit were increased and the banking angle remained the same, then in order to maintain a safe road the curvature would have to be decreased. That is, the radius of the curve would also have to increase, which would make the road less curved.**

57. From a right triangle perspective, if an acute angle θ has a given sine, say x, then the complementary angle $\frac{\pi}{2} - \theta$ has that same value as its cosine. This can be verified by looking at a right triangle. Therefore, the sum of the angle whose sine is x and the angle whose cosine is x should be $\frac{\pi}{2}$.

51. Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the line $y = mx + b$. Then $m = \frac{y_2 - y_1}{x_2 - x_1}$. The tangent of the angle θ the line makes with the positive x-axis is equal to the ratio $\frac{\text{opp}}{\text{adj}}$ or $\frac{y_2 - y_1}{x_2 - x_1}$. Thus $\tan \theta = m$.



Chapter 13 Study Guide and Review

Vocabulary and Concept Check

- This alphabetical list of vocabulary terms in Chapter 13 includes a page reference where each term was introduced.
- **Assessment** A vocabulary test/review for Chapter 13 is available on p. 830 of the *Chapter 13 Resource Masters*.

Lesson-by-Lesson Review

For each lesson,

- the main ideas are summarized,
- additional examples review concepts, and
- practice exercises are provided.

Vocabulary PuzzleMaker

ELL The Vocabulary PuzzleMaker software improves students' mathematics vocabulary using four puzzle formats—crossword, scramble, word search using a word list, and word search using clues. Students can work on a computer screen or from a printed handout.

MindJogger Videoquizzes

ELL MindJogger Videoquizzes provide an alternative review of concepts presented in this chapter. Students work in teams in a game show format to gain points for correct answers. The questions are presented in three rounds.

Round 1 Concepts (5 questions)
Round 2 Skills (4 questions)
Round 3 Problem Solving (4 questions)

Chapter 13 Study Guide and Review

Vocabulary and Concept Check

angle of depression (p. 705)	initial side (p. 709)	secant (p. 701)
angle of elevation (p. 705)	law of cosines (p. 733)	sine (p. 701)
arccosine function (p. 747)	law of sines (p. 726)	solve a right triangle (p. 704)
arcsine function (p. 747)	period (p. 741)	standard position (p. 709)
arctangent function (p. 747)	periodic (p. 741)	tangent (p. 701)
circular function (p. 740)	principal values (p. 746)	terminal side (p. 709)
cosecant (p. 701)	quadrantal angles (p. 718)	trigonometric functions (p. 701)
cosine (p. 701)	radian (p. 710)	trigonometry (p. 701)
cotangent (p. 701)	reference angle (p. 718)	unit circle (p. 710)
coterminal angles (p. 712)		

State whether each sentence is *true* or *false*. If false, replace the underlined word(s) or number to make a true sentence.

1. When two angles in standard position have the same terminal side, they are called <u>quadrantal</u> angles. **false, coterminal**
2. The <u>Law of Sines</u> is used to solve a triangle when the measure of two angles and the measure of any side are known. **true**
3. <u>Trigonometric</u> functions can be defined by using a unit circle. **true**
4. For all values of θ, $\underline{\csc \theta} = \frac{1}{\cos \theta}$. **false; sec θ**
5. A <u>radian</u> is the measure of an angle on the unit circle where the rays of the angle intercept an arc with length 1 unit. **true**
6. If the measures of three sides of a triangle are known, then the <u>Law of Sines</u> can be used to solve the triangle. **false; Law of Cosines**
7. An angle measuring $\underline{60°}$ is a quadrantal angle. **False; see margin.**
8. For all values of x, $\cos(x + \underline{180°}) = \cos x$. **false; 360°**
9. In a coordinate plane, the <u>initial</u> side of an angle is the ray that rotates about the center. **false; terminal**

Lesson-by-Lesson Review

13-1 Right Triangle Trigonometry

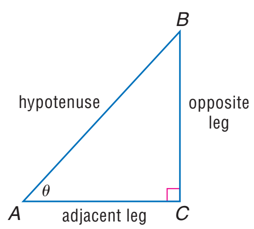
See pages 701–708.

Concept Summary

- If θ is the measure of an acute angle of a right triangle, *opp* is the measure of the leg opposite θ, *adj* is the measure of the leg adjacent to θ, and *hyp* is the measure of the hypotenuse, then the following are true.

$$\sin \theta = \frac{opp}{hyp} \quad \cos \theta = \frac{adj}{hyp} \quad \tan \theta = \frac{opp}{adj}$$

$$\csc \theta = \frac{hyp}{opp} \quad \sec \theta = \frac{hyp}{adj} \quad \cot \theta = \frac{adj}{opp}$$



www.algebra2.com/vocabulary_review

FOLDABLES™ Study Organizer

For more information about Foldables, see *Teaching Mathematics with Foldables.*

Remind students to review the Foldable and make sure that their definitions are accurate and complete. Ask them to check over their notes to make sure they have a diagram for each concept and application that they have worked with in this chapter. Have them make any needed additions.

Encourage students to refer to their Foldables while completing the Study Guide and Review and to use them in preparing for the Chapter Test.

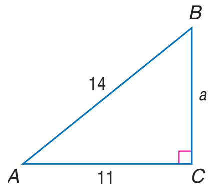
Example **Solve $\triangle ABC$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.**

Find a. $a^2 + b^2 = c^2$ Pythagorean Theorem

$a^2 + 11^2 = 14^2$ $b = 11$ and $c = 14$

$a = \sqrt{14^2 - 11^2}$ Solve for a.

$a \approx 8.7$ Use a calculator.



Find A. $\cos A = \frac{11}{14}$ $\cos A = \frac{\text{adj}}{\text{hyp}}$

Use a calculator to find the angle whose cosine is $\frac{11}{14}$.

KEYSTROKES: [2nd] [COS^{-1}] 11 [÷] 14 [)] [ENTER] 38.2132107

To the nearest degree, $A \approx 38°$.

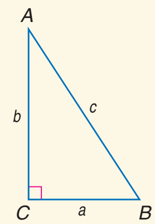
Find B. $38° + B \approx 90°$ Angles A and B are complementary.

$B \approx 52°$ Solve for B.

Therefore, $a \approx 8.7$, $A \approx 38°$, and $B \approx 52°$.

10–15. See margin.

Exercises **Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.** *See Examples 4 and 5 on page 704.*



10. $c = 16, a = 7$ **11.** $A = 25°, c = 6$

12. $B = 45°, c = 12$ **13.** $B = 83°, b = \sqrt{31}$

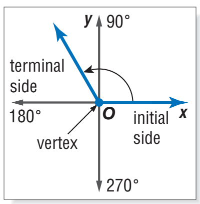
14. $a = 9, B = 49°$ **15.** $\cos A = \frac{1}{4}, a = 4$

13-2 Angles and Angle Measure

See pages 709–715.

Concept Summary

- An angle in standard position has its vertex at the origin and its initial side along the positive x-axis.
- The measure of an angle is determined by the amount of rotation from the initial side to the terminal side. If the rotation is in a counterclockwise direction, the measure of the angle is positive. If the rotation is in a clockwise direction, the measure of the angle is negative.



Examples **Rewrite the degree measure in radians and the radian measure in degrees.**

1 **240°**

$$240° = 240°\left(\frac{\pi \text{ radians}}{180°}\right)$$

$$= \frac{240\pi}{180} \text{ radians or } \frac{4\pi}{3}$$

2 $\frac{\pi}{12}$

$$\frac{\pi}{12} = \left(\frac{\pi}{12} \cancel{\text{radians}}\right)\left(\frac{180°}{\pi \cancel{\text{radians}}}\right)$$

$$= \frac{180°}{12} \text{ or } 15°$$

Answers

7. 0°, 90°, 180°, 270°, 360°, or any multiple of these

10. $A \approx 26°$, $B \approx 64°$, $b \approx 14.4$

11. $B = 65°$, $a \approx 2.5$, $b \approx 5.4$

12. $A = 45°$, $a \approx 8.5$, $b \approx 8.5$

13. $A = 7°$, $a \approx 0.7$, $c \approx 5.6$

14. $A = 41°$, $b \approx 10.4$, $c \approx 13.7$

15. $A \approx 76°$, $B \approx 14°$, $b \approx 1.0$, $c \approx 4.1$

Answers

24. $\sin\theta = \frac{5\sqrt{29}}{29}$, $\cos\theta = \frac{2\sqrt{29}}{29}$, $\tan\theta = \frac{5}{2}$, $\csc\theta = \frac{\sqrt{29}}{5}$, $\sec\theta = \frac{\sqrt{29}}{2}$, $\cot\theta = \frac{2}{5}$

25. $\sin\theta = -\frac{8}{17}$, $\cos\theta = \frac{15}{17}$, $\tan\theta = -\frac{8}{15}$, $\csc\theta = -\frac{17}{8}$, $\sec\theta = \frac{17}{15}$, $\cot\theta = -\frac{15}{8}$

Exercises Rewrite each degree measure in radians and each radian measure in degrees. *See Example 2 on page 711.*

16. 255° $\frac{17\pi}{12}$ 17. −210° $-\frac{7\pi}{6}$ 18. $\frac{7\pi}{4}$ 315° 19. -4π −720°

Find one angle with positive measure and one angle with negative measure coterminal with each angle. *See Example 4 on page 712.*

20. 205° 565°, −155° 21. −40° 320°, −400° 22. $\frac{4\pi}{3}$ $\frac{10\pi}{3}$, $-\frac{2\pi}{3}$ 23. $-\frac{7\pi}{4}$ $\frac{\pi}{4}$, $-\frac{15\pi}{4}$

13-3 Trigonometric Functions of General Angles

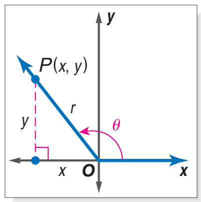
See pages 717–724.

Concept Summary

- You can find the exact values of the six trigonometric functions of θ given the coordinates of a point $P(x, y)$ on the terminal side of the angle.

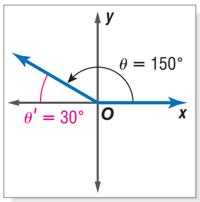
$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}, x \neq 0$

$\csc\theta = \frac{r}{y}, y \neq 0$ $\sec\theta = \frac{r}{x}, x \neq 0$ $\cot\theta = \frac{x}{y}, y \neq 0$



Example **Find the exact value of cos 150°.**

Because the terminal side of 150° lies in Quadrant II, the reference angle θ' is 180° − 150° or 30°. The cosine function is negative in Quadrant II, so cos 150° = −cos 30° or $-\frac{\sqrt{3}}{2}$.



Exercises Find the exact value of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point. *See Example 1 on pages 717 and 718.* 24–25. See margin.

24. $P(2, 5)$ 25. $P(15, -8)$

Find the exact value of each trigonometric function. *See Example 4 on page 720.*

26. cos 3π −1 27. tan 120° $-\sqrt{3}$ 28. sin $\frac{5\pi}{4}$ $-\frac{\sqrt{2}}{2}$ 29. sec (−30°) $-\frac{2\sqrt{3}}{3}$

13-4 Law of Sines

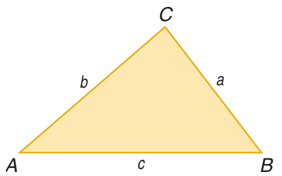
See pages 725–732.

Concept Summary

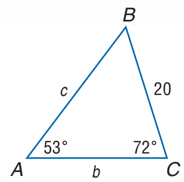
- You can find the area of $\triangle ABC$ if the measures of two sides and their included angle are known.

area $= \frac{1}{2}bc \sin A$ area $= \frac{1}{2}ac \sin B$ area $= \frac{1}{2}ab \sin C$

- Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



Example Solve $\triangle ABC$.



First, find the measure of the third angle.

$53° + 72° + B = 180°$ The sum of the angle measures is 180°.

$B = 55°$ $180 - (53 + 72) = 55$

Now use the Law of Sines to find b and c. Write two equations, each with one variable.

$\frac{\sin A}{a} = \frac{\sin C}{c}$	Law of Sines	$\frac{\sin B}{b} = \frac{\sin A}{a}$
$\frac{\sin 53°}{20} = \frac{\sin 72°}{c}$	Replace A with 53°, B with 55°, C with 72°, and a with 20.	$\frac{\sin 55°}{b} = \frac{\sin 53°}{20}$
$c = \frac{20 \sin 72°}{\sin 53°}$	Solve for the variable.	$b = \frac{20 \sin 55°}{\sin 53°}$
$c \approx 23.8$	Use a calculator.	$b \approx 20.5$

Therefore, $B = 55°$, $b \approx 20.5$, and $c \approx 23.8$.

31, 32, 34, 35. See margin.

Exercises **Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.** *See Examples 3–5 on pages 727 and 728.*

30. $a = 24, b = 36, A = 64°$ **no**

31. $A = 40°, b = 10, a = 8$

32. $b = 10, c = 15, C = 66°$

33. $A = 82°, a = 9, b = 12$ **no**

34. $A = 105°, a = 18, b = 14$

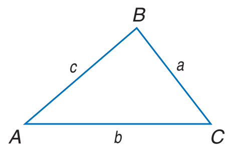
35. $B = 46°, C = 83°, b = 65$

13-5 Law of Cosines

See pages 733–738.

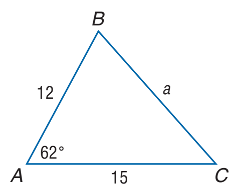
Concept Summary

- Law of Cosines: $a^2 = b^2 + c^2 - 2bc \cos A$
  $b^2 = a^2 + c^2 - 2ac \cos B$
  $c^2 = a^2 + b^2 - 2ab \cos C$



Example **Solve $\triangle ABC$ for $A = 62°$, $b = 15$, and $c = 12$.**

You are given the measure of two sides and the included angle. Begin by drawing a diagram and using the Law of Cosines to determine a.



$a^2 = b^2 + c^2 - 2bc \cos A$ Law of Cosines

$a^2 = 15^2 + 12^2 - 2(15)(12) \cos 62°$ $b = 15$, $c = 12$, and $A = 62°$

$a^2 = 200$ Simplify.

$a \approx 14.1$ Take the square root of each side.

Next, you can use the Law of Sines to find the measure of angle C.

$\frac{\sin 62°}{14.1} \approx \frac{\sin C}{12}$ Law of Sines

$\sin C \approx \frac{12 \sin 62°}{14.1}$ or about 48.7° Use a calculator.

The measure of the angle B is approximately $180 - (62 + 48.7)$ or 69.3°.

Therefore, $a \approx 14.1$, $C \approx 48.7°$, $B \approx 69.3°$.

Answers

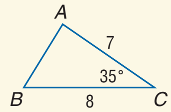
31. two; $B \approx 54°$, $C \approx 86°$, $c \approx 12.4$; $B \approx 126°$, $C \approx 14°$, $c \approx 3.0$
32. one; $B = 38°$, $A = 76°$, $a = 15.9$
34. one; $B = 49°$, $C = 26°$, $c = 8.2$
35. one; $A = 51°$, $a = 70.2$, $c = 89.7$

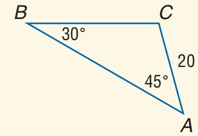
Chapter 13 For More ...
- Extra Practice, see pages 857–859.
- Mixed Problem Solving, see page 874.

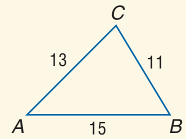
Answers

36. cosines; 4.6, $A \approx 84°$, $B \approx 61°$
37. sines; $C = 105°$, $a \approx 28.3$, $c \approx 38.6$
38. cosines; $A \approx 45°$, $B \approx 58°$, $C \approx 77°$
39. cosines; $A \approx 33°$, $B \approx 82°$, $c \approx 6.4$
40. sines; $B \approx 52°$, $C \approx 92°$, $c \approx 10.2$; $B \approx 128°$, $C \approx 16°$, $c \approx 2.8$
41. cosines; $B \approx 26°$, $C \approx 125°$, $a \approx 8.3$

Exercises Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. *See Examples 1 and 2 on pages 734 and 735.* **36–41. See margin.**

36.


37.


38.


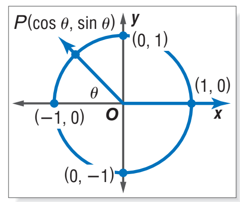
39. $C = 65°$, $a = 4$, $b = 7$
40. $A = 36°$, $a = 6$, $b = 8$
41. $b = 7.6$, $c = 14.1$, $A = 29°$

13-6 Circular Functions

See pages 739–745.

Concept Summary

- If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos\theta = x$ and $\sin\theta = y$. Therefore, the coordinates of P can be written as $P(\cos\theta, \sin\theta)$.



Example Find the exact value of $\cos\left(-\frac{7\pi}{4}\right)$.

$$\cos\left(-\frac{7\pi}{4}\right) = \cos\left(-\frac{7\pi}{4} + 2\pi\right) = \cos\frac{\pi}{4} \text{ or } \frac{\sqrt{2}}{2}$$

Exercises Find the exact value of each function. *See Example 2 on page 741.*

42. $\sin(-150°)$ $-\frac{1}{2}$
43. $\cos 300°$ $\frac{1}{2}$
44. $(\sin 45°)(\sin 225°)$ $-\frac{1}{2}$
45. $\sin\frac{5\pi}{4}$ $-\frac{\sqrt{2}}{2}$
46. $(\sin 30°)^2 + (\cos 30°)^2$ 1
47. $\frac{4\cos 150° + 2\sin 300°}{3}$ $-\sqrt{3}$

13-7 Inverse Trigonometric Functions

See pages 746–751.

Concept Summary

- $y = \text{Sin } x$ if and only if $y = \sin x$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
- $y = \text{Cos } x$ if and only if $y = \cos x$ and $0 \le x \le \pi$.
- $y = \text{Tan } x$ if and only if $y = \tan x$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Example Find the value of $\text{Cos}^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$ in radians. Round to the nearest hundredth.

KEYSTROKES: 2nd [COS^{-1}] TAN (−) 2nd [π] ÷ 6)) ENTER 2.186276035

Therefore, $\text{Cos}^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right] \approx 2.19$ radians.

Exercises Find each value. Write angle measures in radians. Round to the nearest hundredth. *See Example 3 on page 748.* **48. −1.57 50. 0.75**

48. $\text{Sin}^{-1}(-1)$
49. $\text{Tan}^{-1}\sqrt{3}$ 1.05
50. $\tan\left(\text{Arcsin}\,\frac{3}{5}\right)$
51. $\cos(\text{Sin}^{-1} 1)$ 0

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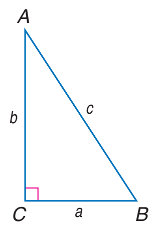
Chapter 13

Practice Test

Vocabulary and Concepts

1. **Draw** a right triangle and label one of the acute angles θ. Then label the hypotenuse *hyp*, the side opposite θ *opp*, and the side adjacent θ *adj*. **See margin.**
2. **State** the Law of Sines for $\triangle ABC$. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
3. **Describe** a situation in which you would solve a triangle by first applying the Law of Cosines. **Sample answer: when the measures of two sides and the included angle are given**

Skills and Applications



Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

4. $a = 7$, $A = 49°$ **$b \approx 6.1$, $c \approx 9.3$, $B = 41°$**
5. $B = 75°$, $b = 6$
6. $A = 22°$, $c = 8$ **$a \approx 3.0$, $b \approx 7.4$, $B = 68°$**
7. $a = 7$, $c = 16$

5. $a \approx 1.6$, $c \approx 6.2$, $A = 15°$ 7. $b \approx 14.4$, $A \approx 26°$, $B \approx 64°$

Rewrite each degree measure in radians and each radian measure in degrees.

8. $275°$ **$\frac{55\pi}{36}$**
9. $-\frac{\pi}{6}$ **$-30°$**
10. $\frac{11\pi}{2}$ **$990°$**
11. $330°$ **$\frac{11\pi}{6}$**
12. $-600°$ **$-\frac{10\pi}{3}$**
13. $-\frac{7\pi}{4}$ **$-315°$**

Find the exact value of each expression. Write angle measures in degrees.

14. $\cos(-120°)$ **$-\frac{1}{2}$**
15. $\sin \frac{7\pi}{4}$ **$-\frac{\sqrt{2}}{2}$**
16. $\cot 300°$ **$-\frac{\sqrt{3}}{3}$**
17. $\sec\left(-\frac{7\pi}{6}\right)$ **$-\frac{2\sqrt{3}}{3}$**
18. $\mathrm{Sin}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **$-60°$**
19. Arctan 1 **$45°$**
20. $\tan 135°$ **-1**
21. $\csc \frac{5\pi}{6}$ **2**

22. Determine the number of possible solutions for a triangle in which $A = 40°$, $b = 10$, and $a = 14$. If a solution exists, solve the triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. **one; $B \approx 27°$, $C \approx 113°$, $c \approx 20.0$**

23. Suppose θ is an angle in standard position whose terminal side lies in Quadrant II. Find the exact values of the remaining five trigonometric functions for θ for $\cos\theta = -\frac{\sqrt{3}}{2}$. **$\sin\theta = \frac{1}{2}$, $\tan\theta = -\frac{\sqrt{3}}{3}$, $\sec\theta = -\frac{2\sqrt{3}}{3}$, $\csc\theta = 2$, $\cot\theta = -\sqrt{3}$**

24. **GEOLOGY** From the top of the cliff, a geologist spots a dry riverbed. The measurement of the angle of depression to the riverbed is 70°. The cliff is 50 meters high. How far is the riverbed from the base of the cliff? **18.2 m**

25. **STANDARDIZED TEST PRACTICE** Triangle ABC has a right angle at C, angle $B = 30°$, and $BC = 6$. Find the area of triangle ABC. **C**

Ⓐ 6 units^2 Ⓑ $\sqrt{3}$ units^2 Ⓒ $6\sqrt{3}$ units^2 Ⓓ 12 units^2

www.algebra2.com/chapter_test

Chapter 13

Practice Test

Assessment Options

Vocabulary Test A vocabulary test/review for Chapter 13 can be found on p. 830 of the *Chapter 13 Resource Masters*.

Chapter Tests There are six Chapter 13 Tests and an Open-Ended Assessment task available in the *Chapter 13 Resource Masters*.

Chapter 13 Tests

Form	Type	Level	Pages
1	MC	basic	817–818
2A	MC	average	819–820
2B	MC	average	821–822
2C	FR	average	823–824
2D	FR	average	825–826
3	FR	advanced	827–828

MC = multiple-choice questions
FR = free-response questions

Open-Ended Assessment
Performance tasks for Chapter 13 can be found on p. 829 of the *Chapter 13 Resource Masters*. A sample scoring rubric for these tasks appears on p. A28.

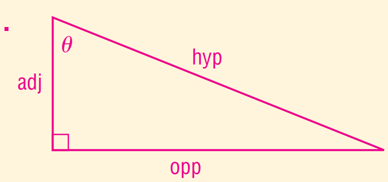
TestCheck and Worksheet Builder

This **networkable software** has three modules for assessment.

- **Worksheet Builder** to make worksheets and tests.
- **Student Module** to take tests on-screen.
- **Management System** to keep student records.

Answer

1.



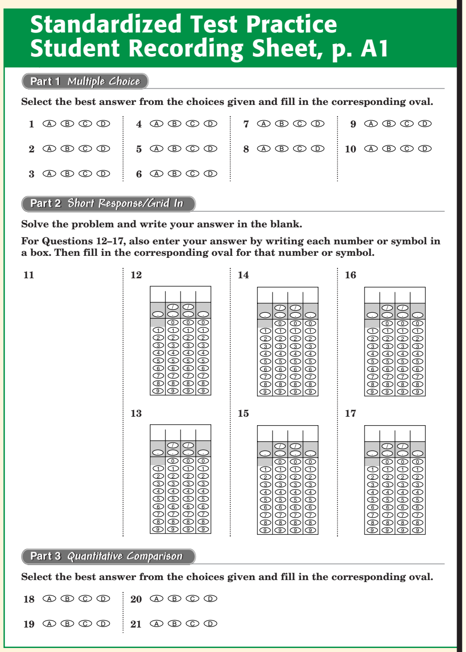
Portfolio Suggestion

Introduction In this chapter, you studied a number of different approaches to naming and measuring angles and their trigonometric functions.

Ask Students Of the seven lessons in this chapter, pick the one that you are still having some trouble understanding. Describe what questions you still have about this lesson. Explain how this lesson might have been explained in a way that would be clearer to you. Place this in your portfolio.

These two pages contain practice questions in the various formats that can be found on the most frequently given standardized tests.

A practice answer sheet for these two pages can be found on p. A1 of the *Chapter 13 Resource Masters*.


Standardized Test Practice Student Recording Sheet, p. A1

Part 1 Multiple Choice

Select the best answer from the choices given and fill in the corresponding oval.

1 Ⓐ Ⓑ Ⓒ Ⓓ　4 Ⓐ Ⓑ Ⓒ Ⓓ　7 Ⓐ Ⓑ Ⓒ Ⓓ　9 Ⓐ Ⓑ Ⓒ Ⓓ

2 Ⓐ Ⓑ Ⓒ Ⓓ　5 Ⓐ Ⓑ Ⓒ Ⓓ　8 Ⓐ Ⓑ Ⓒ Ⓓ　10 Ⓐ Ⓑ Ⓒ Ⓓ

3 Ⓐ Ⓑ Ⓒ Ⓓ　6 Ⓐ Ⓑ Ⓒ Ⓓ

Part 2 Short Response/Grid In

Solve the problem and write your answer in the blank.

For Questions 12–17, also enter your answer by writing each number or symbol in a box. Then fill in the corresponding oval for that number or symbol.

11　12　14　16

13　15　17

Part 3 Quantitative Comparison

Select the best answer from the choices given and fill in the corresponding oval.

18 Ⓐ Ⓑ Ⓒ Ⓓ　20 Ⓐ Ⓑ Ⓒ Ⓓ

19 Ⓐ Ⓑ Ⓒ Ⓓ　21 Ⓐ Ⓑ Ⓒ Ⓓ

Teaching Tip In Questions 7 and 9, students may find it helpful to make sketches to increase their understanding of the problem.

Additional Practice

See pp. 835–836 in the *Chapter 13 Resource Masters* for additional standardized test practice.

Chapter 13 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. If $3n + k = 30$ and n is a positive even integer, then which of the following statements must be true? **C**

I. k is divisible by 3.
II. k is an even integer.
III. k is less than 20.

Ⓐ I only　Ⓑ II only
Ⓒ I and II only　Ⓓ I, II, and III

2. If $4x^2 + 5x = 80$ and $4x^2 - 5y = 30$, then what is the value of $6x + 6y$? **C**

Ⓐ 10　Ⓑ 50　Ⓒ 60　Ⓓ 110

3. If $a = b + cb$, then what does $\frac{b}{a}$ equal in terms of c? **B**

Ⓐ $\frac{1}{c}$　Ⓑ $\frac{1}{1 + c}$
Ⓒ $1 - c$　Ⓓ $1 + c$

4. What is the value of $\sum_{n=1}^{5} 3n^2$? **D**

Ⓐ 55　Ⓑ 58
Ⓒ 75　Ⓓ 165

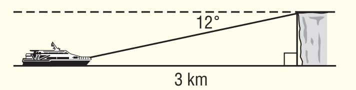
5. There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble? **D**

Ⓐ 4　Ⓑ 6　Ⓒ 8　Ⓓ 12

The Princeton Review Test-Taking Tip

Questions 1–10 The answer choices to multiple-choice questions can provide clues to help you solve a problem. In Question 5, you can add the values in the answer choices to the number of yellow marbles and the total number of marbles to find which is the correct answer.

6. From a lookout point on a cliff above a lake, the angle of depression to a boat on the water is 12°. The boat is 3 kilometers from the shore just below the cliff. What is the height of the cliff from the surface of the water to the lookout point? **D**



Ⓐ $\frac{3}{\sin 12°}$　Ⓑ $\frac{3}{\tan 12°}$
Ⓒ $\frac{3}{\cos 12°}$　Ⓓ $3 \tan 12°$

7. If $x + y = 90°$ and x and y are positive, then $\frac{\cos x}{\sin y} =$ **C**

Ⓐ 0.　Ⓑ $\frac{1}{2}$.
Ⓒ 1.　Ⓓ cannot be determined

8. A child flying a kite holds the string 4 feet above the ground. The taut string is 40 feet long and makes an angle of 35° with the horizontal. How high is the kite off the ground? **A**

Ⓐ $4 + 40 \sin 35°$　Ⓑ $4 + 40 \cos 35°$
Ⓒ $4 + 40 \tan 35°$　Ⓓ $4 + \frac{40}{\sin 35°}$

9. If $\sin \theta = -\frac{1}{2}$ and $180° < \theta < 270°$, then $\theta =$ **B**

Ⓐ 200°.　Ⓑ 210°.
Ⓒ 225°.　Ⓓ 240°.

10. If $\cos \theta = \frac{8}{17}$ and the terminal side of the angle is in quadrant IV, then $\sin \theta =$ **C**

Ⓐ $-\frac{15}{8}$.　Ⓑ $-\frac{17}{15}$.
Ⓒ $-\frac{15}{17}$.　Ⓓ $\frac{15}{17}$.

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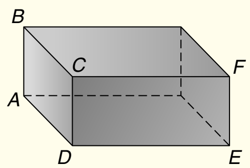
Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. The length, width, and height of the rectangular box illustrated below are each integers greater than 1. If the area of *ABCD* is 18 square units and the area of *CDEF* is 21 square units, what is the volume of the box? **126 units^3**

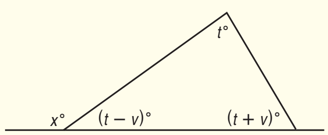


12. When six consecutive integers are multiplied, their product is 0. What is their greatest possible sum? **15**

13. The average (arithmetic mean) score for the 25 players on a team is n. Their scores range from 60 to 100, inclusive. The average score of 20 of the players is 70. What is the difference between the greatest and least possible values of n? **8**

14. The variables a, b, c, d, and e are integers in a sequence, where $a = 2$ and $b = 12$. To find the next term, double the last term and add that result to one less than the next-to-last term. For example, $c = 25$, because $2(12) = 24$, $2 - 1 = 1$, and $24 + 1 = 25$. What is the value of e? **146**

15. In the figure, if $t = 2v$, what is the value of x? **150**



16. If $b = 4$, then what is the value of a in the equations below? **5**
$3a + 4b + 2c = 33$
$2b + 4c = 12$

17. At the head table at a banquet, 3 men and 3 women sit in a row. In how many ways can the row be arranged so that the men and women alternate? **72**

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Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- Ⓐ **the quantity in Column A is greater,**
- Ⓑ **the quantity in Column B is greater,**
- Ⓒ **the two quantities are equal, or**
- Ⓓ **the relationship cannot be determined from the information given.**

Column A	Column B

18. A container holds a certain number of tiles. The tiles are either red or white. One tile is chosen from the container at random.

probability of choosing a red or a white tile	200%

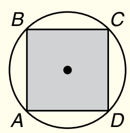
B

19. $\Diamond x = (4x)^4 + \frac{x}{4}$

5	$\Diamond \frac{1}{4}$

A

20.

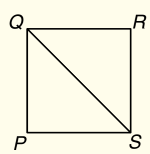


The area of square *ABCD* is 64 units^2.

area of circle *O*	192 units^2

B

21. *PQRS* is a square.



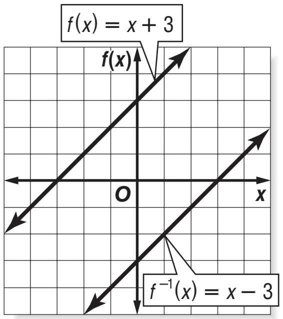
$\frac{QS}{RS}$	2

B

Chapter 13 Standardized Test Practice 759

Page 699, Chapter 13 Getting Started

9.



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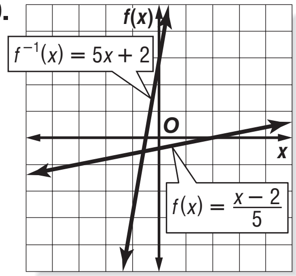

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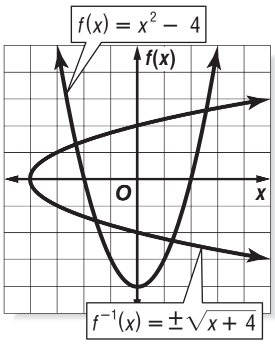

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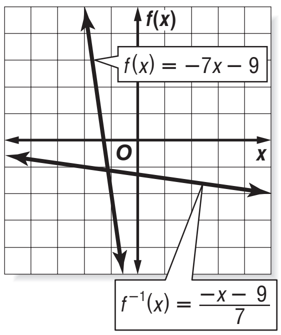


Pages 706–708, Lesson 13-1

15. $\sin\theta = \frac{4}{11}$; $\cos\theta = \frac{\sqrt{105}}{11}$; $\tan\theta = \frac{4\sqrt{105}}{105}$; $\csc\theta = \frac{11}{4}$; $\sec\theta = \frac{11\sqrt{105}}{105}$; $\cot\theta = \frac{\sqrt{105}}{4}$

16. $\sin\theta = \frac{3}{5}$; $\cos\theta = \frac{4}{5}$; $\tan\theta = \frac{3}{4}$; $\csc\theta = \frac{5}{3}$; $\sec\theta = \frac{5}{4}$; $\cot\theta = \frac{4}{3}$

17. $\sin\theta = \frac{\sqrt{7}}{4}$; $\cos\theta = \frac{3}{4}$; $\tan\theta = \frac{\sqrt{7}}{3}$; $\csc\theta = \frac{4\sqrt{7}}{7}$; $\sec\theta = \frac{4}{3}$; $\cot\theta = \frac{3\sqrt{7}}{7}$

18. $\sin\theta = \frac{9\sqrt{106}}{106}$; $\cos\theta = \frac{5\sqrt{106}}{106}$; $\tan\theta = \frac{9}{5}$; $\csc\theta = \frac{\sqrt{106}}{9}$; $\sec\theta = \frac{\sqrt{106}}{5}$; $\cot\theta = \frac{5}{9}$

19. $\sin\theta = \frac{\sqrt{5}}{5}$; $\cos\theta = \frac{2\sqrt{5}}{5}$; $\tan\theta = \frac{1}{2}$; $\csc\theta = \sqrt{5}$; $\sec\theta = \frac{\sqrt{5}}{2}$; $\cot\theta = 2$

20. $\sin\theta = \frac{\sqrt{15}}{8}$; $\cos\theta = \frac{7}{8}$; $\tan\theta = \frac{\sqrt{15}}{7}$; $\csc\theta = \frac{8\sqrt{15}}{15}$; $\sec\theta = \frac{8}{7}$; $\cot\theta = \frac{7\sqrt{15}}{15}$

27a. $\sin 30° = \frac{\text{opp}}{\text{hyp}}$ sine ratio

$\sin 30° = \frac{x}{2x}$ Replace opp with x and hyp with $2x$.

$\sin 30° = \frac{1}{2}$ Simplify.

27b. $\cos 30° = \frac{\text{adj}}{\text{hyp}}$ cosine ratio

$\cos 30° = \frac{\sqrt{3}x}{2x}$ Replace adj with $\sqrt{3}x$ and hyp with $2x$.

$\cos 30° = \frac{\sqrt{3}}{2}$ Simplify.

27c. $\sin 60° = \frac{\text{opp}}{\text{hyp}}$ sine ratio

$\sin 60° = \frac{\sqrt{3}x}{2x}$ Replace opp with $\sqrt{3}x$ and hyp with $2x$.

$\sin 60° = \frac{\sqrt{3}}{2}$ Simplify.

28a. $\sin 45° = \frac{\text{opp}}{\text{hyp}}$ sine ratio

$\sin 45° = \frac{x}{\sqrt{2}x}$ Replace opp with x and hyp with $\sqrt{2}x$.

$\sin 45° = \frac{1}{\sqrt{2}}$ Simplify.

$\sin 45° = \frac{\sqrt{2}}{2}$ Rationalize the denominator.

28b. $\cos 45° = \frac{\text{adj}}{\text{hyp}}$ cosine ratio

$\cos 45° = \frac{x}{\sqrt{2}x}$ Replace adj with x and hyp with $\sqrt{2}x$.

$\cos 45° = \frac{1}{\sqrt{2}}$ Simplify.

$\cos 45° = \frac{\sqrt{2}}{2}$ Rationalize the denominator.

28c. $\tan 45° = \frac{\text{opp}}{\text{adj}}$ tangent ratio

$\tan 45° = \frac{x}{x}$ Replace opp with x and adj with x.

$\tan 45° = 1$ Simplify.

29. $B = 74°$, $a \approx 3.9$, $b \approx 13.5$

30. $A = 63°$, $a \approx 13.7$, $c \approx 15.4$

31. $B = 56°$, $b \approx 14.8$, $c \approx 17.9$

32. $A = 75°$, $a \approx 24.1$, $b \approx 6.5$

33. $A = 60°$, $a \approx 19.1$, $c = 22$

34. $B = 45°$, $a = 7$, $b = 7$

35. $A = 72°$, $b \approx 1.3$, $c \approx 4.1$

36. $B = 80°$, $a \approx 2.6$, $c \approx 15.2$

37. $A \approx 63°$, $B \approx 27°$, $a \approx 11.5$

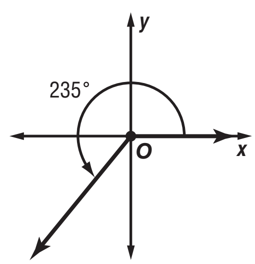
38. $A \approx 26°$, $B \approx 64°$, $b \approx 8.1$

39. $A \approx 49°$, $B \approx 41°$, $a = 8$, $c \approx 10.6$

40. $A \approx 19°$, $B \approx 71°$, $b \approx 14.1$, $c = 15$

Pages 712–714, Lesson 13-2

19.



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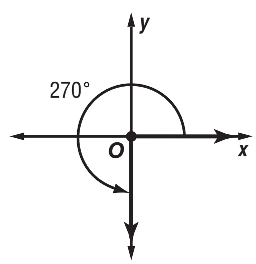

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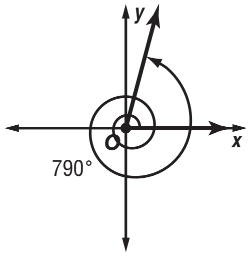

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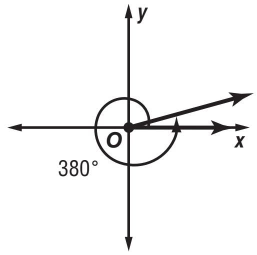

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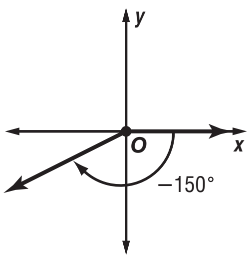

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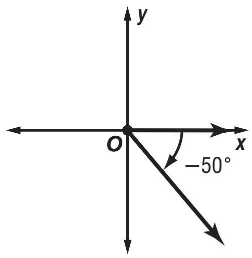

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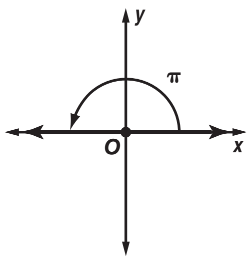

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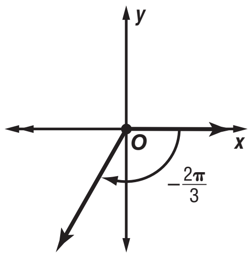
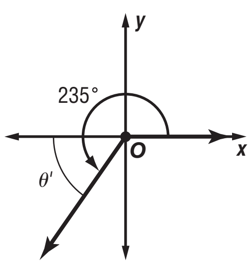


61. Student answers should include the following.
- An angle with a measure of more than 180° gives an indication of motion in a circular path that ended at a point more than halfway around the circle from where it started.
- Negative angles convey the same meaning as positive angles, but in an opposite direction. The standard convention is that negative angles represent rotations in a clockwise direction.
- Rates over 360° per minute indicate that an object is rotating or revolving more than one revolution per minute.

Page 716, Follow-Up of Lesson 13-2
Algebra Activity

7. To find the length of the apothem, you need to write this equation: $\cos\theta = \frac{a}{\text{length of radius}}$. If the radius is 1, then $\cos\theta = a$. If the radius is not 1, then a = length of radius · $\cos\theta$.

Pages 722–723, Lesson 13-3

7.


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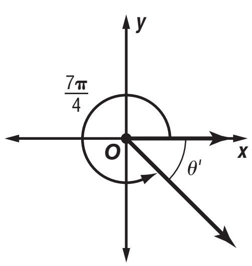

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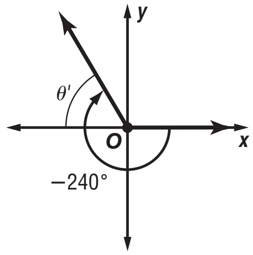

17. $\sin\theta = \frac{24}{25}$, $\cos\theta = \frac{7}{25}$, $\tan\theta = \frac{24}{7}$, $\csc\theta = \frac{25}{24}$, $\sec\theta = \frac{25}{7}$, $\cot\theta = \frac{7}{24}$

18. $\sin\theta = \frac{\sqrt{5}}{5}$, $\cos\theta = \frac{2\sqrt{5}}{5}$, $\tan\theta = \frac{1}{2}$, $\csc\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$, $\cot\theta = 2$

19. $\sin\theta = -\frac{8\sqrt{89}}{89}$, $\cos\theta = \frac{5\sqrt{89}}{89}$, $\tan\theta = -\frac{8}{5}$, $\csc\theta = -\frac{\sqrt{89}}{8}$, $\sec\theta = \frac{\sqrt{89}}{5}$, $\cot\theta = -\frac{5}{8}$

20. $\sin\theta = -\frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = -\frac{3}{4}$, $\csc\theta = -\frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = -\frac{4}{3}$

21. $\sin\theta = -1$, $\cos\theta = 0$, $\tan\theta =$ undefined, $\csc\theta = -1$, $\sec\theta =$ undefined, $\cot\theta = 0$

22. $\sin\theta = 0$, $\cos\theta = -1$, $\tan\theta = 0$, $\csc\theta =$ undefined, $\sec\theta = -1$, $\cot\theta =$ undefined

23. $\sin\theta = -\frac{\sqrt{2}}{2}$, $\cos\theta = \frac{\sqrt{2}}{2}$, $\tan\theta = -1$, $\csc\theta = -\sqrt{2}$, $\sec\theta = \sqrt{2}$, $\cot\theta = -1$

24. $\sin\theta = -\frac{\sqrt{6}}{3}$, $\cos\theta = -\frac{\sqrt{3}}{3}$, $\tan\theta = \sqrt{2}$, $\csc\theta = -\frac{\sqrt{6}}{2}$, $\sec\theta = -\sqrt{3}$, $\cot\theta = \frac{\sqrt{2}}{2}$

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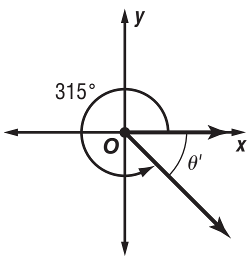

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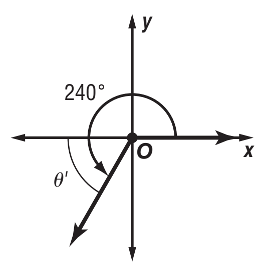

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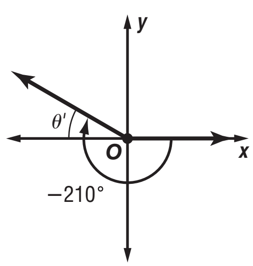

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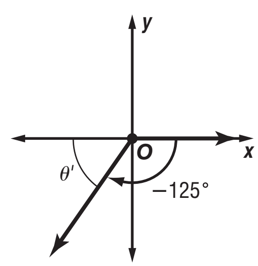

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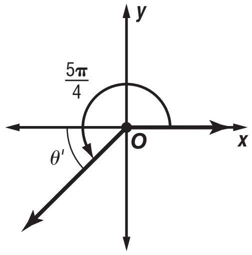

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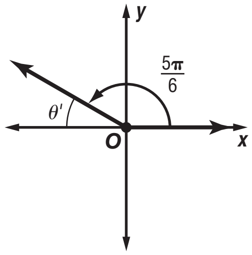

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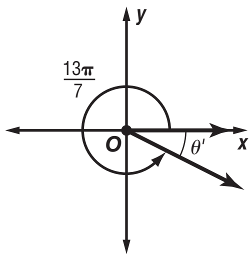

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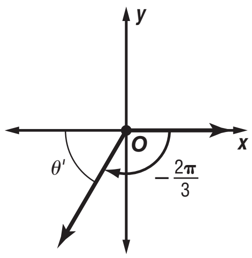


50. $\sin\theta = -\frac{2\sqrt{5}}{5}$, $\cos\theta = -\frac{\sqrt{5}}{5}$, $\tan\theta = 2$, $\csc\theta = -\frac{\sqrt{5}}{2}$, $\sec\theta = -\sqrt{5}$

51. $\sin\theta = -\frac{3\sqrt{10}}{10}$, $\cos\theta = -\frac{\sqrt{10}}{10}$, $\tan\theta = 3$, $\csc\theta = -\frac{\sqrt{10}}{3}$, $\cot\theta = \frac{1}{3}$

52. $\sin\theta = -\frac{1}{5}$, $\cos\theta = \frac{2\sqrt{6}}{5}$, $\tan\theta = -\frac{\sqrt{6}}{12}$, $\sec\theta = \frac{5\sqrt{6}}{12}$, $\cot\theta = -2\sqrt{6}$

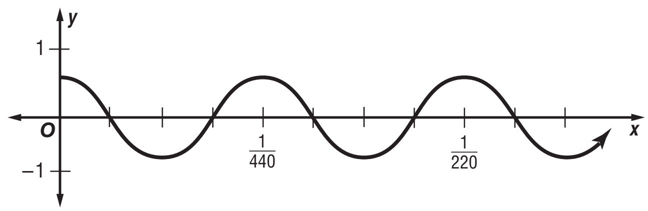
Pages 736–737, Lesson 13-5

35. Answers should include the following.

- The Law of Cosines can be used when you know all three sides of a triangle or when you know two sides and the included angle. It can even be used with two sides and the nonincluded angle. This set of conditions leaves a quadratic equation to be solved. It may have one, two, or no solution just like the SSA case with the Law of Sines.
- Given the latitude of a point on the surface of Earth, you can use the radius of the Earth and the orbiting height of a satellite in geosynchronous orbit to create a triangle. This triangle will have two known sides and the measure of the included angle. Find the third side using the Law of Cosines and then use the Law of Sines to determine the angles of the triangle. Subtract 90 degrees from the angle with its vertex on Earth's surface to find the angle at which to aim the receiver dish.

Page 744, Lesson 13-6

34.



Notes