Chapter 14

Trigonometric Graphs and Identities

Chapter Overview and Pacing

LESSON OBJECTIVES

	PACING (days)			
	Regular		Block	
	Basic/ Average	Advanced	Basic/ Average	Advanced
14-1 **Graphing Trigonometric Functions** *(pp. 762–768)* • Graph trigonometric functions. • Find the amplitude and period of variation of the sine, cosine, and tangent functions.	optional	2	optional	1
14-2 **Translations of Trigonometric Graphs** *(pp. 769–776)* • Graph horizontal translations of trigonometric graphs and find phase shifts. • Graph vertical translations of trigonometric graphs.	optional	2	optional	1
14-3 **Trigonometric Identities** *(pp. 777–781)* • Use identities to find trigonometric values. • Use trigonometric identities to simplify expressions.	optional	2	optional	1
14-4 **Verifying Trigonometric Identities** *(pp. 782–785)* • Verify trigonometric identities by transforming one side of an equation into the form of the other side. • Verify trigonometric identities by transforming each side of the equation into the same form.	optional	2	optional	1
14-5 **Sum and Difference of Angles Formulas** *(pp. 786–790)* • Find values of sine and cosine involving sum and difference formulas. • Verify identities by using sum and difference formulas.	optional	1	optional	0.5
14-6 **Double-Angle and Half-Angle Formulas** *(pp. 791–797)* • Find values of sine and cosine involving double-angle formulas. • Find values of sine and cosine involving half-angle formulas.	optional	1	optional	0.5
14-7 **Solving Trigonometric Equations** *(pp. 798–804)* ***Preview:*** Solving Trigonometric Equations • Solve trigonometric equations. • Use trigonometric equations to solve real-world problems.	optional	2 (with 14-7 Preview)	optional	1 (with 14-7 Preview)
Study Guide and **Practice Test** *(pp. 805–809)* **Standardized Test Practice** *(pp. 810–811)*	optional	1	optional	0.5
Chapter Assessment	optional	1	optional	1
TOTAL	**0**	**14**	**0**	**7.5**

Pacing suggestions for the entire year can be found on pages T20–T21.

Timesaving Tools

TeacherWorks™

All-In-One Planner and Resource Center

See pages T12–T13.

Chapter Resource Manager

CHAPTER 14 RESOURCE MASTERS									
Study Guide and Intervention	Practice (Skills and Average)	Reading to Learn Mathematics	Enrichment	Assessment	Applications*	5-Minute Check Transparencies	Interactive Chalkboard	Alge2PASS: Tutorial Plus (lessons)	Materials
837–838	839–840	841	842			14-1	14-1		graphing calculator, posterboard
843–844	845–846	847	848	893	GCS 53, SC 27	14-2	14-2	27	graphing calculator, grid paper, string, masking tape, rope
849–850	851–852	853	854		GCS 54	14-3	14-3		
855–856	857–858	859	860	893, 895		14-4	14-4	28	
861–862	863–864	865	866			14-5	14-5		
867–868	869–870	871	872	894		14-6	14-6		
873–874	875–876	877	878	894	SC 28, SM 145–148	14-7	14-7		(*Preview:* graphing calculator)
				879–892, 896–898					

**Key to Abbreviations:* GCS = Graphing Calculator and Speadsheet Masters,
SC = School-to-Career Masters,
SM = Science and Mathematics Lab Manual

Chapter 14

Mathematical Connections and Background

Continuity of Instruction

Prior Knowledge

In the previous chapter students investigated the six trigonometric functions and worked with angles measured in degrees or radians, as rays in standard position, and as points on the unit circle. Also, they explored periodicity and inverse trigonometric functions.

This Chapter

Students use the trigonometric functions to explore amplitude and period, and they investigate phase shifts and vertical shifts in the graphs of trigonometric functions. They learn how to verify and use trigonometric identities, including sum and difference formulas, double-angle formulas, and half-angle formulas. Finally they solve trigonometric equations using the ideas of factoring, the zero product property, trigonometric inverses, and periodic behavior.

Future Connections

Students will continue to study amplitude, period, and frequency for trigonometric and other periodic functions. In later math courses they will use trigonometric formulas and identities, and frequently when they study graphs they will analyze vertical and horizontal translations of graphs and how those changes are related to changes in the algebraic description of the graph.

14-1 Graphing Trigonometric Functions

This chapter continues the extensive investigation of trigonometric functions from the previous chapter. This lesson focuses on graphs of the trigonometric functions. Each of the sine, cosine, and tangent functions repeats a pattern of values, or is periodic. For the sine and cosine functions, the period is 2π radians or 360°, while for the tangent function the period is π radians or 180°. For periodic functions the distance between a horizontal center line and the maximum or minimum value is called the amplitude of the graph. For $y = \sin x$ and $y = \cos x$, the horizontal center line is the x-axis and the maximum and minimum values are ± 1, so the amplitude is 1.

The lesson also describes these properties algebraically. For the sine function $y = a \sin b\theta$ and the cosine function $y = a \cos b\theta$, the period is $2\pi \div |b|$ and the amplitude is $|a|$. For $y = a \tan b\theta$, the tangent function, the period is $\pi \div |b|$. The tangent function has no finite maximum or minimum, so amplitude is not defined for the tangent function.

14-2 Translations of Trigonometric Graphs

In this lesson students explore the graphs of $y = \sin \theta$ and $y = \cos \theta$. More specifically, they use the functions $y = a \sin b(\theta - h) + k$ and $y = a \cos b(\theta - h) + k$ and see how changing each of the values a, b, h, and k affects the graph. Also, they explore how to sketch a graph for a given set of values of the four variables.

One change in the graph of a periodic function is to move the horizontal center line of the graph. When $k = 0$ the horizontal center line is the x-axis and the vertical shift is zero. A positive value of k represents a vertical shift upward while a negative value of k represents a downward vertical shift. The amplitude is determined by the value of $|a|$, so the maximum values of the function are $|a|$ units above k and the minimum values of the function are $|a|$ units below k. A third change is the period of the function. The expression for the length of one period has the variable b in the denominator, so as the value of $|b|$ increases the period of the function decreases. The fourth variable, h, is associated with the phase shift of the function. If h is positive, the entire graph is shifted to the right; if h is negative, the entire graph is shifted to the left. Students also use the equation $y = a \tan b(\theta - h) + k$ and explore phase shifts, periods, and vertical shifts for the tangent function.

14-3 Trigonometric Identities

This lesson and the next three deal with trigonometric identities. Students learn the definition of an identity, and they work with arguments that are half of a given angle, twice a given angle, or the sum or difference of two given angles. In this lesson students work with the definitions of the six trigonometric functions in terms of x, y, and r. By dividing each side of $x^2 + y^2 = r^2$ by r^2, y^2, or x^2, the results are three identities called the Pythagorean Identities. For other identities, called the Reciprocal Identities, students note that the definitions for sine and cosecant, for cosine and secant, and for tangent and cotangent are reciprocals. Also, they see that the ratios $\sin \theta \div \cos \theta$ and $\cos \theta \div \sin \theta$ can be simplified to $\tan \theta$ and $\cot \theta$, respectively, resulting in two identities called the Quotient Identities. Students explore how to use identities to simplify trigonometric expressions, and they use identities to evaluate a complicated trigonometric expression for a given argument.

14-4 Verifying Trigonometric Identities

In this lesson students continue exploring how to identify and use trigonometric identities. For each equation, the goal is to transform each side, replacing expressions with equivalent expressions, until the two sides are identical. There are several approaches for writing equivalent expressions. First, students can make substitutions using the Pythagorean Identities. Second, they can use the Distributive Property to factor an expression or to collect like terms. Third, they can transform a term by multiplying the term by an expression equivalent to 1. And fourth, they can rewrite all the trigonometric functions in terms of $\sin \theta$ or $\cos \theta$ by using the Quotient and Reciprocal Identities. Students also relate trigonometric identities to graphs, using a graphing calculator to show that the expressions on each side of a trigonometric identity have the same graph.

14-5 Sum and Difference of Angles Formulas

Students derive and then use formulas for rewriting the two-variable functions $\sin (\alpha \pm \beta)$ and $\cos (\alpha \pm \beta)$ in terms of the one-variable functions $\sin \alpha$, $\sin \beta$, $\cos \alpha$, and $\cos \beta$. The derivation of the difference formula for the cosine function begins with the two ordered pairs on the unit circle that correspond to two angles α and β. The distance d between the two points can be found using the distance formula, or it can be found as the distance between the point (1, 0) and the coordinates of the point on the unit circle associated with angle $(\alpha - \beta)$. After equating the two expressions for d, algebraic manipulation gives an expression for the two-variable function $\cos (\alpha - \beta)$ in terms of one-variable functions. Students use the formulas to find exact values for particular trigonometric expressions. They also use the formulas in problems such as verifying that the equation $\sin (180° + \theta) = -\sin \theta$ is an identity.

14-6 Double-Angle and Half-Angle Formulas

Students begin with the formulas for $\sin (\alpha + \beta)$ and $\cos (\alpha + \beta)$ and replace both α and β with θ. The results, called the Double-Angle Formulas, are equations in which each of $\sin 2\theta$ and $\cos 2\theta$ is expressed in terms of $\sin \theta$ and $\cos \theta$. Then students use an algebraic technique and let α represent 2θ (so $\frac{\alpha}{2}$ represents θ), and derive formulas for $\sin \frac{\alpha}{2}$ and $\cos \frac{\alpha}{2}$ in terms of $\sin \alpha$ and $\cos \alpha$. The two formulas are called the Half-Angle Formulas. Students use the Half-Angle and Double-Angle Formulas, along with other formulas, to find exact values for particular trigonometric expressions. They also substitute the formulas in equations to verify trigonometric identities.

14-7 Solving Trigonometric Equations

In this last lesson of the two-chapter investigation of trigonometric functions, students solve trigonometric equations and review some of the important general ideas of algebra. The first step in solving a trigonometric equation is to use factoring, the zero product property, and identities to rewrite a complicated equation as a string of simpler trigonometric equations. The second step is to use trigonometric inverses to isolate the variable; that is, to solve an equation such as $\cos \theta = 0.5$ for θ. The third step is to use ideas of periodicity to include all the occurrences of that value. Students solve trigonometric equations for arguments measured in degrees or in radians, and they use trigonometric equations and their solutions to solve statements of real-world problems.

Chapter 14

DAILY INTERVENTION and Assessment

	Type	Student Edition	Teacher Resources	Technology/Internet
INTERVENTION	Ongoing	Prerequisite Skills, pp. 761, 768, 776, 781, 785, 790, 797 Practice Quiz 1, p. 781 Practice Quiz 2, p. 797	5-Minute Check Transparencies Quizzes, *CRM* pp. 893–894 Mid-Chapter Test, *CRM* p. 895 Study Guide and Intervention, *CRM* pp. 837–838, 843–844, 849–850, 855–856, 861–862, 867–868, 873–874	Alge2PASS: Tutorial Plus www.algebra2.com/self_check_quiz www.algebra2.com/extra_examples
	Mixed Review	pp. 768, 776, 781, 785, 790, 797, 804	Cumulative Review, *CRM* p. 896	
	Error Analysis	Find the Error, p. 766 Common Misconceptions, p. 782	Find the Error, *TWE* p. 766 Tips for New Teachers, *TWE* p. 793	
	Standardized Test Practice	pp. 768, 776, 781, 783, 784, 785, 790, 796, 804, 809, 810–811	*TWE* p. 783 Standardized Test Practice, *CRM* pp. 897–898	Standardized Test Practice CD-ROM www.algebra2.com/standardized_test
ASSESSMENT	Open-Ended Assessment	Writing in Math, pp. 768, 776, 781, 785, 790, 796, 804 Open Ended, pp. 766, 774, 779, 784, 788, 794, 802	Modeling: *TWE* pp. 768, 790 Speaking: *TWE* pp. 781, 784, 803 Writing: *TWE* pp. 776, 797 Open-Ended Assessment, *CRM* p. 891	
	Chapter Assessment	Study Guide, pp. 805–808 Practice Test, p. 809	Multiple-Choice Tests (Forms 1, 2A, 2B), *CRM* pp. 879–884 Free-Response Tests (Forms 2C, 2D, 3), *CRM* pp. 885–890 Vocabulary Test/Review, *CRM* p. 892	TestCheck and Worksheet Builder (see below) MindJogger Videoquizzes www.algebra2.com/vocabulary_review www.algebra2.com/chapter_test

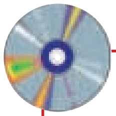
Key to Abbreviations: TWE = Teacher Wraparound Edition; CRM = Chapter Resource Masters

Additional Intervention Resources

The Princeton Review's *Cracking the SAT & PSAT*

The Princeton Review's *Cracking the ACT*

ALEKS



TestCheck and Worksheet Builder

This **networkable** software has three modules for intervention and assessment flexibility:

- **Worksheet Builder** to make worksheet and tests
- **Student Module** to take tests on screen (optional)
- **Management System** to keep student records (optional)

Special banks are included for SAT, ACT, TIMSS, NAEP, and End-of-Course tests.

Intervention Technology

Alge2PASS: Tutorial Plus CD-ROM offers a complete, self-paced algebra curriculum.

Algebra 2 Lesson	Alge2PASS Lesson
14-2	**27** *Graphing Trigonometric Functions*
14-4	**28** *Trigonometric Identities*

ALEKS is an online mathematics learning system that adapts assessment and tutoring to the student's needs. Subscribe at www.k12aleks.com.

Intervention at Home

Log on for student study help.

- For each lesson in the Student Edition, there are Extra Examples and Self-Check Quizzes.
  www.algebra2.com/extra_examples
  www.algebra2.com/self_check_quiz
- For chapter review, there is vocabulary review, test practice, and standardized test practice.
  www.algebra2.com/vocabulary_review
  www.algebra2.com/chapter_test
  www.algebra2.com/standardized_test

For more information on Intervention and Assessment, see pp. T8–T11.

Reading and Writing in Mathematics

Glencoe Algebra 2 provides numerous opportunities to incorporate reading and writing into the mathematics classroom.

Student Edition

- Foldables Study Organizer, p. 761
- Concept Check questions require students to verbalize and write about what they have learned in the lesson. (pp. 766, 774, 779, 784, 788, 794, 802, 805)
- Writing in Math questions in every lesson, pp. 768, 776, 781, 785, 790, 796, 804
- Reading Study Tip, pp. 786, 788
- WebQuest, pp. 775, 804

Teacher Wraparound Edition

- Foldables Study Organizer, pp. 761, 805
- Study Notebook suggestions, pp. 766, 774, 779, 783, 788, 794, 802
- Modeling activities, pp. 768, 790
- Speaking activities, pp. 781, 784, 803
- Writing activities, pp. 776, 797
- ELL Resources, pp. 760, 767, 775, 780, 785, 789, 796, 803, 805

Additional Resources

- Vocabulary Builder worksheets require students to define and give examples for key vocabulary terms as they progress through the chapter. (*Chapter 14 Resource Masters*, pp. vii-viii)
- Reading to Learn Mathematics master for each lesson (*Chapter 14 Resource Masters*, pp. 841, 847, 853, 859, 865, 871, 877)
- *Vocabulary PuzzleMaker* software creates crossword, jumble, and word search puzzles using vocabulary lists that you can customize.
- *Teaching Mathematics with Foldables* provides suggestions for promoting cognition and language.
- *Reading and Writing in the Mathematics Classroom*
- *WebQuest and Project Resources*

For more information on Reading and Writing in Mathematics, see pp. T6–T7.

Chapter 14 Notes

What You'll Learn

Have students read over the list of objectives and make a list of any words with which they are not familiar.

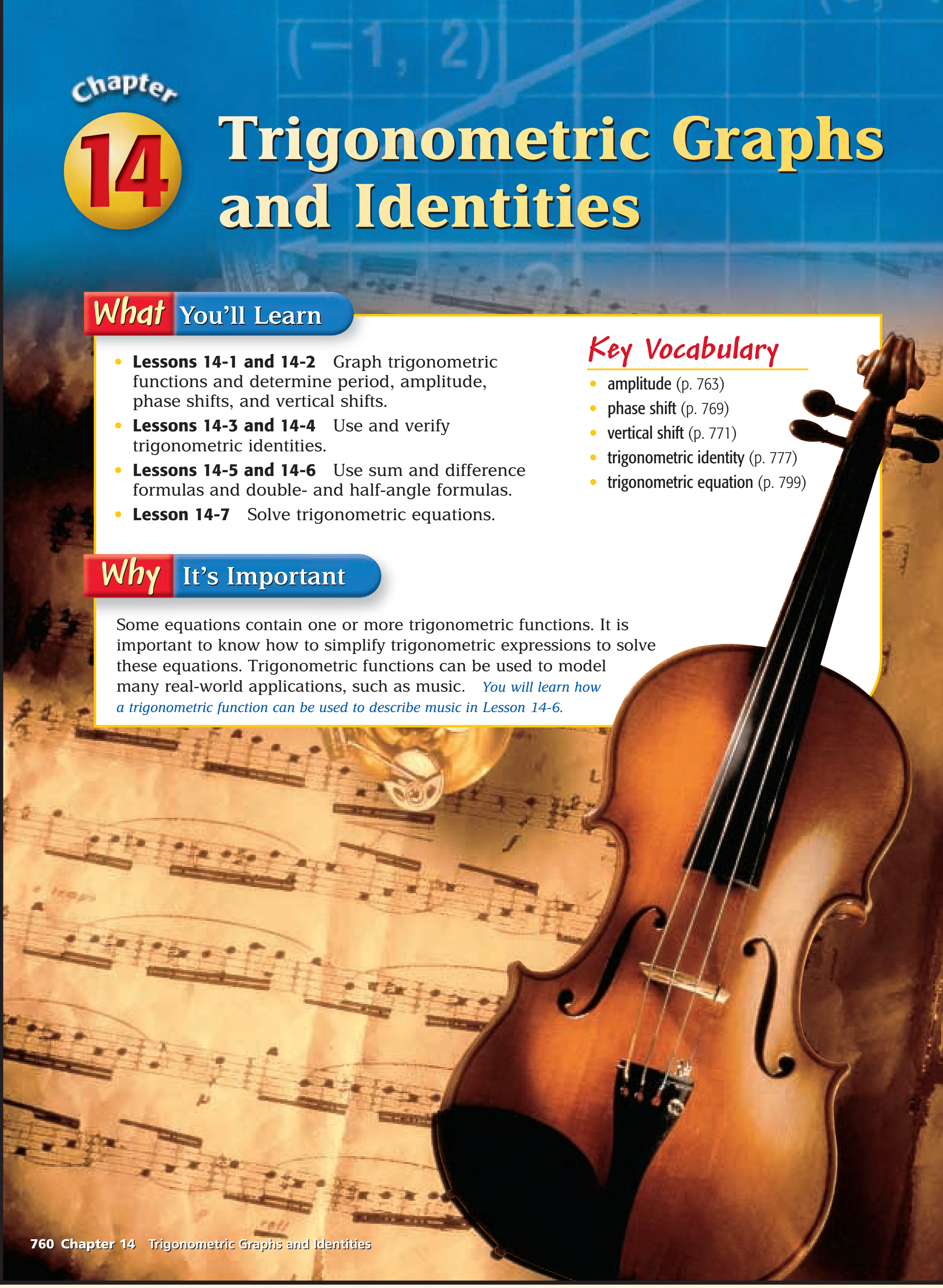
Why It's Important

Point out to students that this is only one of many reasons why each objective is important. Others are provided in the introduction to each lesson.

Lesson	NCTM Standards	Local Objectives
14-1	1, 2, 6, 7, 8, 9, 10	
14-2	1, 2, 3, 4, 6, 8, 9, 10	
14-3	1, 2, 6, 8, 9, 10	
14-4	2, 8	
14-5	1, 2, 6, 7, 8, 9, 10	
14-6	1, 2, 7, 8, 9, 10	
14-7 Preview	2, 10	
14-7	1, 2, 6, 7, 8, 9, 10	

Key to NCTM Standards:

1=Number & Operations, 2=Algebra, 3=Geometry, 4=Measurement, 5=Data Analysis & Probability, 6=Problem Solving, 7=Reasoning & Proof, 8=Communication, 9=Connections, 10=Representation



Chapter 14 Trigonometric Graphs and Identities

What You'll Learn

- **Lessons 14-1 and 14-2** Graph trigonometric functions and determine period, amplitude, phase shifts, and vertical shifts.
- **Lessons 14-3 and 14-4** Use and verify trigonometric identities.
- **Lessons 14-5 and 14-6** Use sum and difference formulas and double- and half-angle formulas.
- **Lesson 14-7** Solve trigonometric equations.

Key Vocabulary

- **amplitude** (p. 763)
- **phase shift** (p. 769)
- **vertical shift** (p. 771)
- **trigonometric identity** (p. 777)
- **trigonometric equation** (p. 799)

Why It's Important

Some equations contain one or more trigonometric functions. It is important to know how to simplify trigonometric expressions to solve these equations. Trigonometric functions can be used to model many real-world applications, such as music. *You will learn how a trigonometric function can be used to describe music in Lesson 14-6.*

Vocabulary Builder

ELL

The Key Vocabulary list introduces students to some of the main vocabulary terms included in this chapter. For a more thorough vocabulary list with pronunciations of new words, give students the Vocabulary Builder worksheets found on pages vii and viii of the *Chapter 14 Resource Masters*. Encourage them to complete the definition of each term as they progress through the chapter. You may suggest that they add these sheets to their study notebooks for future reference when studying for the Chapter 14 test.

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 14.

For Lessons 14-1 and 14-2 — Trigonometric Values

Find the exact value of each trigonometric function. *(For review, see Lesson 13-3.)*

1. $\sin 135°$ $\frac{\sqrt{2}}{2}$ **2.** $\tan 315°$ -1 **3.** $\cos 90°$ 0 **4.** $\tan 45°$ 1

5. $\sin \frac{5\pi}{4}$ $-\frac{\sqrt{2}}{2}$ **6.** $\cos \frac{7\pi}{6}$ $-\frac{\sqrt{3}}{2}$ **7.** $\sin \frac{11\pi}{6}$ $-\frac{1}{2}$ **8.** $\tan \frac{3\pi}{2}$ **not defined**

For Lessons 14-3, 14-5, and 14-6 — Circular Functions

Find the exact value of each trigonometric function. *(For review, see Lesson 13-6.)*

9. $\cos(-150°)$ $-\frac{\sqrt{3}}{2}$ **10.** $\sin 510°$ $\frac{1}{2}$ **11.** $\cot \frac{9\pi}{4}$ 1 **12.** $\sec \frac{13\pi}{6}$ $\frac{2\sqrt{3}}{3}$

13. $\tan\left(-\frac{3\pi}{2}\right)$ **not defined** **14.** $\csc(-720°)$ **not defined** **15.** $\cos \frac{7\pi}{3}$ $-\frac{1}{2}$ **16.** $\tan \frac{8\pi}{3}$ $-\sqrt{3}$

For Lesson 14-4 — Factor Polynomials

Factor completely. If the polynomial is not factorable, write *prime*. *(For review, see Lesson 5-4.)*

17. $-15x^2 - 5x$ $-5x(3x + 1)$ **18.** $2x^4 - 4x^2$ $2x^2(x^2 - 2)$ **19.** $x^3 + 4$ **prime**

20. $x^2 - 6x + 8$ $(x - 4)(x - 2)$ **21.** $2x^2 - 3x - 2$ $(2x + 1)(x - 2)$ **22.** $3x^3 - 2x^2 - x$ $x(3x + 1)(x - 1)$

For Lesson 14-7 — Solve Quadratic Equations

Solve each equation by factoring. *(For review, see Lesson 6-3.)*

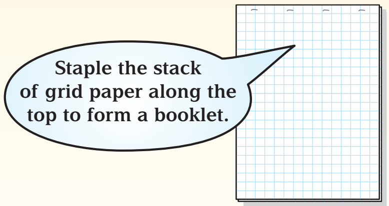
23. $x^2 - 5x - 24 = 0$ $8, -3$ **24.** $x^2 - 2x - 48 = 0$ $8, -6$ **25.** $x^2 + 3x - 40 = 0$ $-8, 5$

26. $x^2 - 12x = 0$ $0, 12$ **27.** $-2x^2 - 11x - 12 = 0$ $-4, -\frac{3}{2}$ **28.** $x^2 - 16 = 0$ $-4, 4$

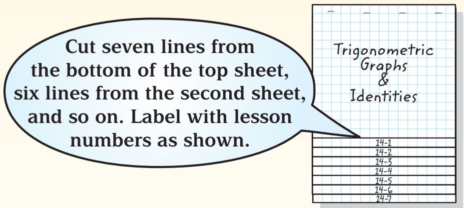
FOLDABLES™ Study Organizer

Make this Foldable to help you organize information about trigonometric graphs and identities. Begin with eight sheets of grid paper.

Step 1 Staple



Step 2 Cut and Label



Reading and Writing As you read and study the chapter, use each page to write notes and to graph examples for each lesson.

Getting Started

This section provides a review of the basic concepts needed before beginning Chapter 14. Page references are included for additional student help.

Prerequisite Skills in the Getting Ready for the Next Lesson section at the end of each exercise set review a skill needed in the next lesson.

For Lesson	Prerequisite Skill
14-2	Graphs of Quadratic Functions (p. 768)
14-3	Reference Angles (p. 776)
14-4	Properties of Equality (p. 781)
14-5	Simplifying Radical Expressions (p. 785)
14-6	Solving Equations Using the Square Root Property (p. 790)
14-7	Solving Equations Using the Zero Product Property (p. 797)

FOLDABLES™ Study Organizer

For more information about Foldables, see *Teaching Mathematics with Foldables.*

Writing Instructions and Sequencing Data After students make their Foldable, have them label each tab to correspond to a lesson in this chapter. Students use their Foldable to take notes, define terms, record concepts, and write examples. After each lesson, ask students to write a set of instructions on how to do something presented in the lesson. For example, a student might write instructions for graphing trigonometric functions. Have students follow their own instructions to check them for accuracy. Use their notes and textbook to make needed revisions.

14-1 Lesson Notes

1 Focus

5-Minute Check Transparency 14-1 Use as a quiz or review of Chapter 13.

Mathematical Background notes are available for this lesson on p. 760C.

Building on Prior Knowledge

In Chapter 13, students learned the sine, cosine, and tangent of 30°, 45°, and 60° angles. In this lesson, students will learn the sine, cosine, and tangent of angles with other measures and use all of these values to create graphs of the six trigonometric functions.

Why can you predict the behavior of tides?

Ask students:

- Why would you need to know the times for the high and low tides? **Sample answer: roads might be flooded at high tide**
- How is the period of a tide defined? **Sample answer: the length of time between two high tides (or two low tides)**
- What is a tidal range? **Sample answer: the difference in water level between low tide and high tide**

14-1 Graphing Trigonometric Functions

What You'll Learn

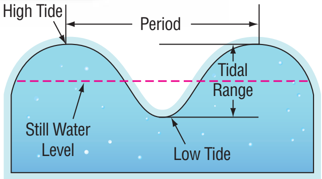
- Graph trigonometric functions.
- Find the amplitude and period of variation of the sine, cosine, and tangent functions.

Vocabulary

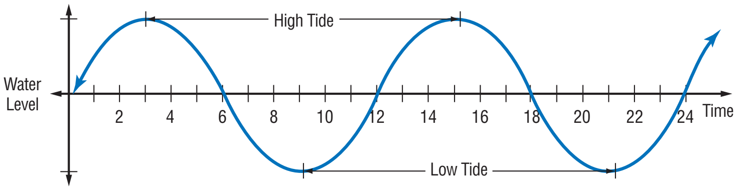
- amplitude

Why can you predict the behavior of tides?

The rise and fall of tides can have great impact on the communities and ecosystems that depend upon them. One type of tide is a semidiurnal tide. This means that bodies of water, like the Atlantic Ocean, have two high tides and two low tides a day. Because tides are periodic, they behave the same way each day.



GRAPH TRIGONOMETRIC FUNCTIONS The diagram below illustrates the water level as a function of time for a body of water with semidiurnal tides.



Study Tip

Look Back
To review **period** and **periodic functions**, see Lesson 13-6.

In each cycle of high and low tides, the pattern repeats itself. Recall that a function whose graph repeats a basic pattern is called a *periodic function*.

To find the period, start from any point on the graph and proceed to the right until the pattern begins to repeat. The simplest approach is to begin at the origin. Notice that after about 12 hours the graph begins to repeat. Thus, the period of the function is about 12 hours.

To graph the periodic functions $y = \sin \theta$, $y = \cos \theta$, or $y = \tan \theta$, use values of θ expressed either in degrees or radians. Ordered pairs for points on these graphs are of the form $(\theta, \sin \theta)$, $(\theta, \cos \theta)$, and $(\theta, \tan \theta)$, respectively.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
nearest tenth	0	0.5	0.7	0.9	1	0.9	0.7	0.5	0	−0.5	−0.7	−0.9	−1	−0.9	−0.7	−0.5	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
nearest tenth	1	0.9	0.7	0.5	0	−0.5	−0.7	−0.9	−1	−0.9	−0.7	−0.5	0	0.5	0.7	0.9	1
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nd	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nd	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0
nearest tenth	0	0.6	1	1.7	nd	−1.7	−1	−0.6	0	0.6	1	1.7	nd	−1.7	−1	−0.6	0
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

nd = not defined

Resource Manager

Workbook and Reproducible Masters

Chapter 14 Resource Masters

- Study Guide and Intervention, pp. 837–838
- Skills Practice, p. 839
- Practice, p. 840
- Reading to Learn Mathematics, p. 841
- Enrichment, p. 842

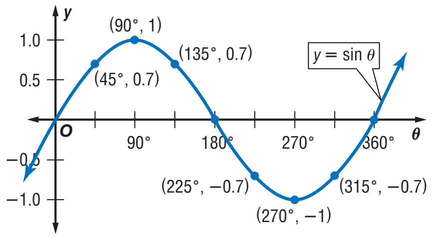
Transparencies

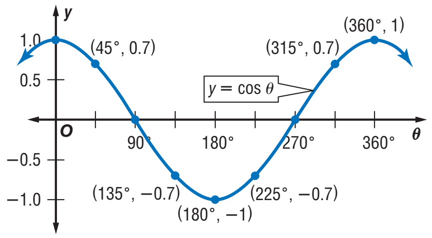
5-Minute Check Transparency 14-1
Real-World Transparency 14
Answer Key Transparencies

Technology

Interactive Chalkboard

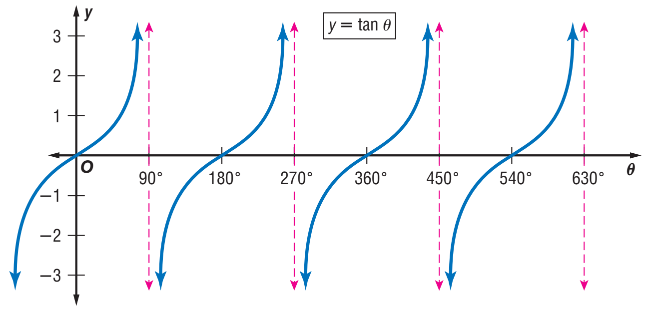
After plotting several points, complete the graphs of $y = \sin \theta$ and $y = \cos \theta$ by connecting the points with a smooth, continuous curve. Recall from Chapter 13 that each of these functions has a period of 360° or 2π radians. That is, the graph of each function repeats itself every 360° or 2π radians.





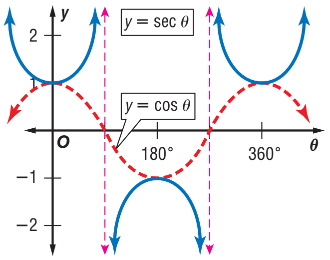
Notice that both the sine and cosine have a maximum value of 1 and a minimum value of −1. The **amplitude** of the graph of a periodic function is the absolute value of half the difference between its maximum value and its minimum value. So, for both the sine and cosine functions, the amplitude of their graphs is $\left|\frac{1-(-1)}{2}\right|$ or 1.

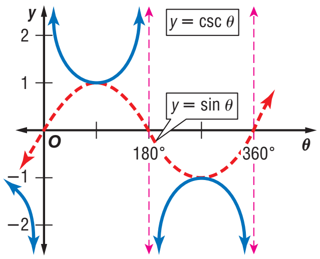
The graph of the tangent function can also be drawn by plotting points. By examining the values for $\tan \theta$ in the table, you can see that the tangent function is not defined for 90°, 270°, …, $90° + k \cdot 180°$, where k is an integer. The graph is separated by vertical asymptotes whose x-intercepts are the values for which $y = \tan \theta$ is not defined.

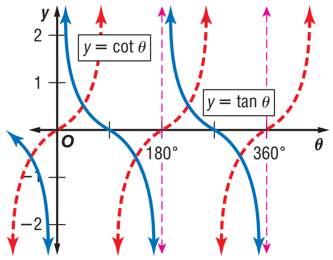


The period of the tangent function is 180° or π radians. Since the tangent function has no maximum or minimum value, it has no amplitude.

The graphs of the secant, cosecant, and cotangent functions are shown below. Compare them to the graphs of the cosine, sine, and tangent functions, which are shown in red.







Notice that the period of the secant and cosecant functions is 360° or 2π radians. The period of the cotangent is 180° or π radians. Since none of these functions have a maximum or minimum value, they have no amplitude.

2 Teach

GRAPH TRIGONOMETRIC FUNCTIONS

Teaching Tip Have students brainstorm to make a list of real-world phenomena that fluctuate in a regular periodic pattern. (Examples: temperatures, the number of people in a mall over the course of a week)

Teaching Tip You might wish to have students draw their own graphs of the sine, cosine, and tangent functions using the data from the table on p. 762. Their graphs can be compared to those shown on p. 763. This immediate feedback can be beneficial in helping students acquire the skills for graphing trigonometric functions, which are more complicated than most of the graphing students have done to this point.

Teaching Tip Review how the sine and cosine functions are related in a right triangle.

DAILY INTERVENTION

Differentiated Instruction

Visual/Spatial Have groups of students make posters showing sketches of the graphs of the six trigonometric functions. Encourage students to color-code the key features of all the graphs, such as period, amplitude, asymptotes, and so on.

VARIATIONS OF TRIGONOMETRIC FUNCTIONS

Concept Check

After discussing the Key Concept box about amplitudes and periods, ask: What is an equation involving the sine function with a period of 90° and an amplitude of $\frac{1}{2}$? **One of the following: $y = \frac{1}{2}\sin 4\theta$, $y = \frac{1}{2}\sin(-4\theta)$, $y = -\frac{1}{2}\sin 4\theta$, or $y = -\frac{1}{2}\sin(-4\theta)$**

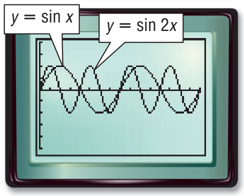
VARIATIONS OF TRIGONOMETRIC FUNCTIONS Just as with other functions, a trigonometric function can be used to form a family of graphs by changing the period and amplitude.

Graphing Calculator Investigation

Period and Amplitude

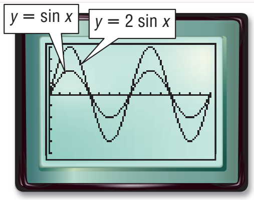
On a TI-83 Plus graphing calculator, set the MODE to degrees.

Think and Discuss



[0, 720] scl: 45 by [−2.5, 2.5] scl: 0.5

1. Graph $y = \sin x$ and $y = \sin 2x$. What is the maximum value of each function? **1**
2. How many times does each function reach a maximum value? **2, 4**
3. Graph $y = \sin\left(\frac{x}{2}\right)$. What is the maximum value of this function? How many times does this function reach its maximum value? **1; 1**
4. Use the equations $y = \sin bx$ and $y = \cos bx$. Repeat Exercises 1–3 for maximum values and the other values of b. What conjecture can you make about the effect of b on the maximum values and the periods of these functions? **The greater the value of b, the smaller the period. b has no effect on the maximum value.**



[0, 720] scl: 45 by [−2.5, 2.5] scl: 0.5

5. Graph $y = \sin x$ and $y = 2\sin x$. What is the maximum value of each function? What is the period of each function? **1, 2; 360°**
6. Graph $y = \frac{1}{2}\sin x$. What is the maximum value of this function? What is the period of this function? **$\frac{1}{2}$; 360°**
7. Use the equations $y = a\sin x$ and $y = a\cos x$. Repeat Exercises 5 and 6 for other values of a. What conjecture can you make about the effect of a on the amplitudes and periods of $y = a\sin x$ and $y = a\cos x$?

Study Tip

Amplitude and Period

Note that the amplitude affects the graph along the vertical axis and the period affects it along the horizontal axis.

7. When a is positive, the amplitude is a. When a is negative, then amplitude is $|a|$. a has no effect on the period.

The results of the investigation suggest the following generalization.

Key Concept — *Amplitudes and Periods*

- **Words** For functions of the form $y = a\sin b\theta$ and $y = a\cos b\theta$, the amplitude is $|a|$, and the period is $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$.
  For functions of the form $y = a\tan b\theta$, the amplitude is not defined, and the period is $\frac{180°}{|b|}$ or $\frac{\pi}{|b|}$.
- **Examples**

$y = 3\sin 4\theta$	amplitude 3 and period $\frac{360°}{4}$ or 90°
$y = -6\cos 5\theta$	amplitude $|-6|$ or 6 and period $\frac{2\pi}{5}$
$y = 2\tan\frac{1}{3}\theta$	no amplitude and period 3π

Graphing Calculator Investigation

Graphing Trigonometric Functions To set the calculator for degrees, press MODE and move the cursor to highlight **DEGREE** and press ENTER. Also, be sure to have students clear the **Y=** lists before beginning Exercise 1.

You can use the amplitude and period of a trigonometric function to help you graph the function.

Example 1 Graph Trigonometric Functions

Find the amplitude and period of each function. Then graph the function.

a. $y = \cos 3\theta$

First, find the amplitude.

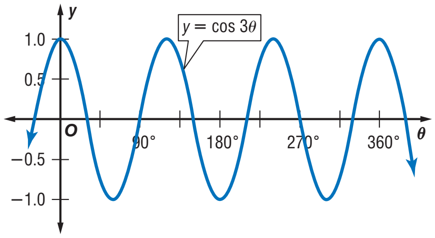
$|a| = |1|$ The coefficient of cos 3θ is 1.

Next, find the period.

$\frac{360°}{|b|} = \frac{360°}{|3|}$ $b = 3$

$= 120°$

Use the amplitude and period to graph the function.



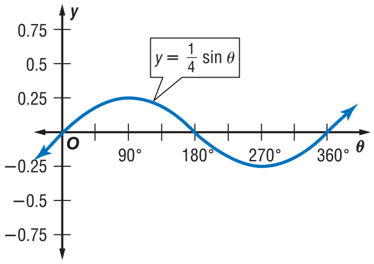
b. $y = \frac{1}{4}\sin\theta$

Amplitude: $|a| = \left|\frac{1}{4}\right|$

$= \frac{1}{4}$

Period: $\frac{360°}{|b|} = \frac{360°}{|1|}$

$= 360°$



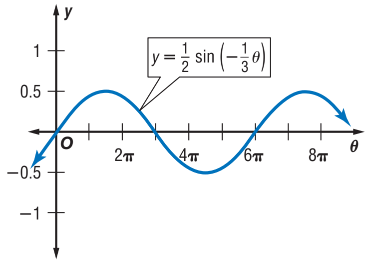
c. $y = \frac{1}{2}\sin\left(-\frac{1}{3}\theta\right)$

Amplitude: $|a| = \left|\frac{1}{2}\right|$

$= \frac{1}{2}$

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{\left|-\frac{1}{3}\right|}$

$= 6\pi$



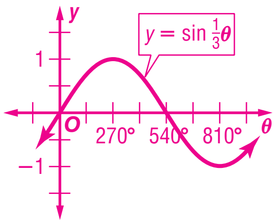
In-Class Example

Power Point®

1 Find the amplitude and period of each function. Then graph the function.

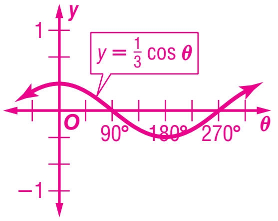
a. $y = \sin\frac{1}{3}\theta$

ampl: 1; period: 1080°



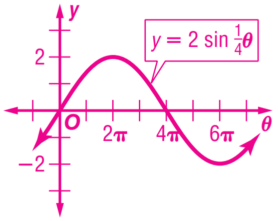
b. $y = \frac{1}{3}\cos\theta$

ampl: $\frac{1}{3}$; period: 360°

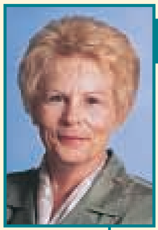


c. $y = 2\sin\frac{1}{4}\theta$

ampl: 2; period: 1440°



Teacher to Teacher



Berchie Holliday Author, Cincinnati, OH

"I have my students construct a unit circle on a coordinate plane with a toothpick length radius. They mark every 15°. Students form right triangles inside the circle and break toothpicks to match the lengths of each vertical leg. (See Figure 1 at the right.) They transfer each leg to its appropriate degree mark on a second x-axis and place a dot at the top of each toothpick. (See Figure 2.) Finally, students connect the dots with a smooth curve."

Figure 1

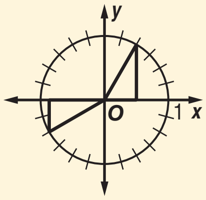
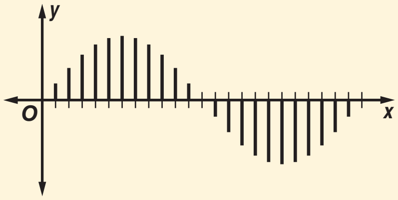


Figure 2



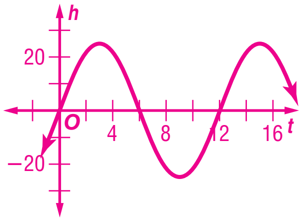
In-Class Example

Power Point®

2 **OCEANOGRAPHY** Refer to the application at the beginning of the lesson. The tidal range in the Bay of Fundy in Canada measures 50 feet.

a. Write a function to represent the height h of the tide. Assume that the tide is at equilibrium at $t = 0$ and that the high tide is beginning. $y = 25 \sin \frac{\pi}{6}t$

b. Graph the tide function.



3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 14.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

DAILY INTERVENTION **FIND THE ERROR** Students should quickly notice that Dante must be incorrect because his graph does not have an amplitude of 3. Ask students if they can identify the mistake that Dante made.

You can use trigonometric functions to describe real-world situations.

Example 2 Use Trigonometric Functions

OCEANOGRAPHY **Refer to the application at the beginning of the lesson. Suppose the tidal range of a city on the Atlantic coast is 18 feet. A tide is at *equilibrium* when it is at its normal level, halfway between its highest and lowest points.**

a. Write a function to represent the height h of the tide. Assume that the tide is at equilibrium at $t = 0$ and that the high tide is beginning.

Since the height of the tide is 0 at $t = 0$, use the sine function $h = a \sin bt$, where a is the amplitude of the tide and t is the time in hours.

Find the amplitude. The difference between high tide and low tide is the tidal range or 18 feet.

$$a = \frac{18}{2} \text{ or } 9$$

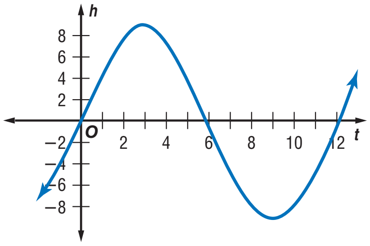
Find the value of b. Each tide cycle lasts about 12 hours.

$$\frac{2\pi}{|b|} = 12 \qquad \text{period} = \frac{2\pi}{|b|}$$

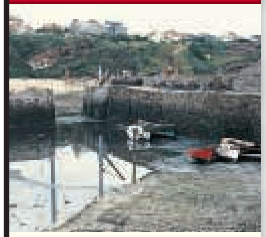
$$b = \frac{2\pi}{12} \text{ or } \frac{\pi}{6} \qquad \text{Solve for } b.$$

Thus, an equation to represent the height of the tide is $h = 9 \sin \frac{\pi}{6}t$.

b. Graph the tide function.



More About. . .



Oceanography

Lake Superior has one of the smallest tidal ranges. It can be measured in inches, while the tidal range in the Bay of Funday in Canada measures up to 50 feet.

Source: Office of Naval Research

Check for Understanding

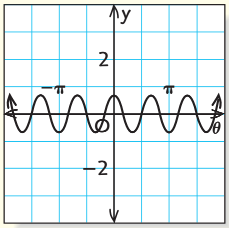
Concept Check

1. **OPEN ENDED** Explain why $y = \tan \theta$ has no amplitude. **See margin.**
2. **Explain** what it means to say that the period of a function is 180°.
3. **FIND THE ERROR** Dante and Jamile graphed $y = 3 \cos \frac{2}{3}\theta$.

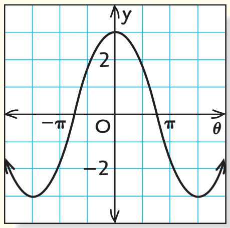
2. Sample answer: The graph repeats itself every 180°.

3. Jamile; the amplitude is 3, and the period is 3π.

Dante



Jamile



Who is correct? Explain your reasoning.

Answer

1. Sample answer: Amplitude is half the difference between the maximum and minimum values of a graph; $y = \tan \theta$ has no maximum or minimum value.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4–12	1
13, 14	2

Find the amplitude, if it exists, and period of each function. Then graph each function. 4–12. See pp. 811A–811N.

4. $y = \frac{1}{2}\sin\theta$
5. $y = 2\sin\theta$
6. $y = \frac{2}{3}\cos\theta$
7. $y = \frac{1}{4}\tan\theta$
8. $y = \csc 2\theta$
9. $y = 4\sin 2\theta$
10. $y = 4\cos\frac{3}{4}\theta$
11. $y = \frac{1}{2}\sec 3\theta$
12. $y = \frac{3}{4}\cos\frac{1}{2}\theta$

Application

BIOLOGY **For Exercises 13 and 14, use the following information.**
In a certain wildlife refuge, the population of field mice can be modeled by $y = 3000 + 1250\sin\frac{\pi}{6}t$, where y represents the number of mice and t represents the number of months past March 1 of a given year.

13. Determine the period of the function. What does this period represent?

14. What is the maximum number of mice and when does this occur? 4250; June 1

13. 12 months; Sample answer: The pattern in the population will repeat itself every 12 months.

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
15–35	1
36–41	2

Extra Practice

See page 859.

Find the amplitude, if it exists, and period of each function. Then graph each function. 15–32. See pp. 811A–811N.

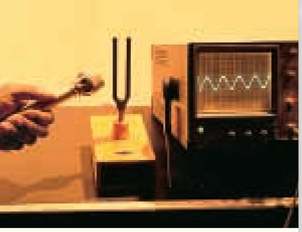
15. $y = 3\sin\theta$
16. $y = 5\cos\theta$
17. $y = 2\csc\theta$
18. $y = 2\tan\theta$
19. $y = \frac{1}{5}\sin\theta$
20. $y = \frac{1}{3}\sec\theta$
21. $y = \sin 4\theta$
22. $y = \sin 2\theta$
23. $y = \sec 3\theta$
24. $y = \cot 5\theta$
25. $y = 4\tan\frac{1}{3}\theta$
26. $y = 2\cot\frac{1}{2}\theta$
27. $y = 6\sin\frac{2}{3}\theta$
28. $y = 3\cos\frac{1}{2}\theta$
29. $y = 3\csc\frac{1}{2}\theta$
30. $y = \frac{1}{2}\cot 2\theta$
31. $2y = \tan\theta$
32. $\frac{3}{4}y = \frac{2}{3}\sin\frac{3}{5}\theta$

★ 33. Draw a graph of a sine function with an amplitude $\frac{3}{5}$ and a period of 90°. Then write an equation for the function. See pp. 811A–811N for graph; $y = \frac{3}{5}\sin 4\theta$.

★ 34. Draw a graph of a cosine function with an amplitude of $\frac{7}{8}$ and a period of $\frac{2\pi}{5}$. Then write an equation for the function. See pp. 811A–811N for graph; $y = \frac{7}{8}\cos 5\theta$.

35. **COMMUNICATIONS** The carrier wave for a certain FM radio station can be modeled by the equation $y = A\sin(10^7 \cdot 2\pi t)$, where A is the amplitude of the wave and t is the time in seconds. Determine the period of the carrier wave. $\frac{1}{10^7}$

More About. . .



Medicine

The tuning fork was invented in 1711 by English trumpeter John Shore.

Source: www.encarta.msn.com

MEDICINE **For Exercises 36 and 37, use the following information.**
Doctors may use a tuning fork that resonates at a given frequency as an aid to diagnose hearing problems. The sound wave produced by a tuning fork can be modeled using a sine function.

36. If the amplitude of the sine function is 0.25, write the equations for tuning forks that resonate with a frequency of 64, 256, and 512 Hertz.

37. How do the periods of the tuning forks compare? See margin.

36. $y = 0.25\sin 128\pi t$, $y = 0.25\sin 512\pi t$, $y = 0.25\sin 1024\pi t$

38. **CRITICAL THINKING** A function is called *even* if the graphs of $y = f(x)$ and $y = f(-x)$ are exactly the same. Which of the six trigonometric functions are even? Justify your answer with a graph of each function.

38. $f(x) = \cos x$ and $f(x) = \sec x$; See pp. 811A–811N for graphs.

www.algebra2.com/self_check_quiz

Study Guide and Intervention, p. 837 (shown) and p. 838

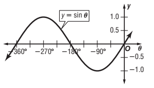
Graph Trigonometric Functions To graph a trigonometric function, make a table of values for known degree measures (0°, 30°, 45°, 60°, 90°, and so on). Round function values to the nearest tenth, and plot the points. Then connect the points with a smooth, continuous curve. The *period* of the sine, cosine, secant, and cosecant functions is 360° or 2π radians.

Amplitude of a Function	The **amplitude** of the graph of a periodic function is the absolute value of half the difference between its maximum and minimum values.

Example **Graph $y = \sin\theta$ for $-360° \le \theta \le 0°$.**

First make a table of values.

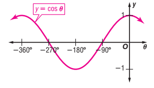
θ	−360°	−330°	−315°	−300°	−270°	−240°	−225°	−210°	−180°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
θ	−150°	−135°	−120°	−90°	−60°	−45°	−30°	0°	
$\sin\theta$	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	



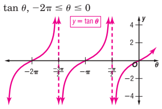
Exercises

Graph the following functions for the given domain.

1. $\cos\theta$, $-360° \le \theta \le 0°$

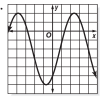


2. $\tan\theta$, $-2\pi \le \theta \le 0$

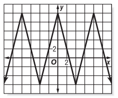


What is the amplitude of each function?

3. 4



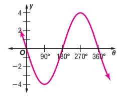
4. 8



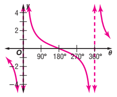
Skills Practice, p. 839 and Practice, p. 840 (shown)

Find the amplitude, if it exists, and period of each function. Then graph each function.

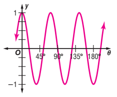
1. $y = -4\sin\theta$ 4; 360°



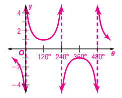
2. $y = \cot\frac{1}{2}\theta$ no amplitude; 360°



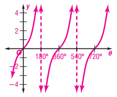
3. $y = \cos 5\theta$ 1; 72°



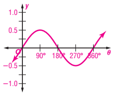
4. $y = \csc\frac{3}{4}\theta$ no amplitude; 480°



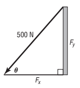
5. $y = 2\tan\frac{1}{2}\theta$ no amplitude; 360°



6. $2y = \sin\theta$ $\frac{1}{2}$; 360°



FORCE **For Exercises 7 and 8, use the following information.**
An anchoring cable exerts a force of 500 Newtons on a pole. The force has the horizontal and vertical components F_x and F_y. (A force of one Newton (N), is the force that gives an acceleration of 1 m/sec² to a mass of 1 kg.)



7. The function $F_x = 500\cos\theta$ describes the relationship between the angle θ and the horizontal force. What are the amplitude and period of this function? 500; 360°

8. The function $F_y = 500\sin\theta$ describes the relationship between the angle θ and the vertical force. What are the amplitude and period of this function? 500; 360°

WEATHER **For Exercises 9 and 10, use the following information.**
The function $y = 60 + 25\sin\frac{\pi}{6}t$, where t is in months and $t = 0$ corresponds to April 15, models the average high temperature in degrees Fahrenheit in Centerville.

9. Determine the period of this function. What does this period represent? 12; a calendar year

10. What is the maximum high temperature and when does this occur? 85°F; July 15

Reading to Learn Mathematics, p. 841

ELL

Pre-Activity **Why can you predict the behavior of tides?**

Read the introduction to Lesson 14-1 at the top of page 762 in your textbook.

Consider the tides of the Atlantic Ocean as a function of time. Approximately what is the period of this function? 12 hours

Reading the Lesson

1. Determine whether each statement is *true* or *false*.
   - a. The period of a function is the distance between the maximum and minimum points. false
   - b. The amplitude of a function is the difference between its maximum and minimum values. false
   - c. The amplitude of the function $y = \sin\theta$ is 2π. false
   - d. The function $y = \cot\theta$ has no amplitude. true
   - e. The period of the function $y = \sec\theta$ is π. false
   - f. The amplitude of the function $y = 2\cos\theta$ is 4. false
   - g. The function $y = \sin 2\theta$ has a period of π. true
   - h. The period of the function $y = \cot 3\theta$ is $\frac{\pi}{3}$. true
   - i. The amplitude of the function $y = -5\sin\theta$ is −5. false
   - j. The period of the function $y = \csc\frac{1}{4}\theta$ is 4π. false
   - k. The graph of the function $y = \sin\theta$ has no asymptotes. true
   - l. The graph of the function $y = \tan\theta$ has an asymptote at $\theta = 180°$. false
   - m. When $\theta = 360°$, the values of $\cos\theta$ and $\sec\theta$ are equal. true
   - n. When $\theta = 270°$, $\cot\theta$ is undefined. false
   - o. When $\theta = 180°$, $\csc\theta$ is undefined. true

Helping You Remember

2. What is an easy way to remember the periods of $y = a\sin b\theta$ and $y = a\cos b\theta$? Sample answer: The period of the functions $y = \sin\theta$ and $y = \cos\theta$ is 360° or 2π. Divide 360° or 2π by the absolute value of the coefficient of θ, depending on whether you want to find the period in degrees or in radians.

Answer

37. Sample answer: The amplitudes are the same. As the frequency increases, the period decreases.

Enrichment, p. 842

Blueprints

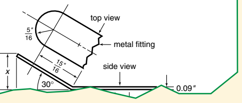
Interpreting blueprints requires the ability to select and use trigonometric functions and geometric properties. The figure below represents a plan for an improvement to a roof. The metal fitting shown makes a 30° angle with the horizontal. The vertices of the geometric shapes are *not* labeled in these plans. Relevant information must be selected and the appropriate function used to find the unknown measures.

Example **Find the unknown measures in the figure at the right.**

The measures x and y are the legs of a right triangle.

The measure of the hypotenuse is $\frac{15}{16}$ in. + $\frac{5}{16}$ in. or $\frac{20}{16}$ in.

Roofing Improvement



About the Exercises...

Organization by Objective

- **Graph Trigonometric Functions:** 15–34
- **Variations of Trigonometric Functions:** 15–34

Odd/Even Assignments

Exercises 15–34 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 15–31 odd, 35–56

Average: 15–35 odd, 36–56

Advanced: 16–34 even, 36–52 (optional: 53–56)

4 Assess

Open-Ended Assessment

Modeling Provide students with a coordinate grid and a length of string. Give students one of the six basic trigonometric functions and have them model the graph using the string. Have students check their model with a graphing calculator.

Getting Ready for Lesson 14-2

PREREQUISITE SKILL Students will graph horizontal and vertical translations of trigonometric functions in Lesson 14-2. Students will apply what they learned about families of quadratic functions. Use Exercises 53–56 to determine your students' familiarity with the graphs of families of functions.

BOATING **For Exercises 39–41, use the following information.**
A marker buoy off the coast of Gulfport, Mississippi, bobs up and down with the waves. The distance between the highest and lowest point is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds.

39. Write an equation for the motion of the buoy. Assume that it is at equilibrium at $t = 0$ and that it is on the way up from the normal water level. $y = 2 \sin \frac{\pi}{5} t$

40. Draw a graph showing the height of the buoy as a function of time. **See margin.**

41. What is the height of the buoy after 12 seconds? **about 1.9 ft**

42. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson. **See margin.**

Why can you predict the behavior of tides?

Include the following in your answer:

- an explanation of why certain tidal characteristics follow the patterns seen in the graph of the sine function, and
- a description of how to determine the amplitude of a function using the maximum and minimum values.

Standardized Test Practice

43. What is the period of $f(x) = \frac{1}{2} \cos 3x$? **A**

Ⓐ 120°   Ⓑ 180°   Ⓒ 360°   Ⓓ 720°

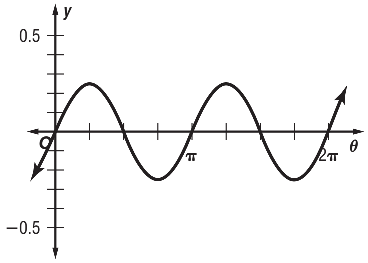
44. Identify the equation of the graphed function. **C**

Ⓐ $y = \frac{1}{2} \sin 4\theta$

Ⓑ $y = 2 \sin \frac{1}{4}\theta$

Ⓒ $y = \frac{1}{4} \sin 2\theta$

Ⓓ $y = 4 \sin \frac{1}{2}\theta$



Maintain Your Skills

Mixed Review

Solve each equation. *(Lesson 13-7)*

45. $x = \text{Sin}^{-1} 1$ **90°**

46. $\text{Arcsin}(-1) = y$ **−90°**

47. $\text{Arccos} \frac{\sqrt{2}}{2} = x$ **45°**

Find the exact value of each function. *(Lesson 13-6)*

48. sin 390° $\frac{1}{2}$

49. sin (−315°) $\frac{\sqrt{2}}{2}$

50. cos 405° $\frac{\sqrt{2}}{2}$

51. **PROBABILITY** There are 8 girls and 8 boys on the Faculty Advisory Board. Three are juniors. Find the probability of selecting a boy or a girl from the committee who is not a junior. *(Lesson 12-5)* $\frac{13}{16}$

52. Find the first five terms of the sequence in which $a_1 = 3$, $a_{n+1} = 2a_n + 5$. *(Lesson 11-5)* **3, 11, 27, 59, 123**

Getting Ready for the Next Lesson

53–56. See pp. 811A–811N.

PREREQUISITE SKILL **Graph each pair of functions on the same set of axes.**
*(To review **graphs of quadratic functions**, see Lesson 6-6.)*

53. $y = x^2, y = 3x^2$

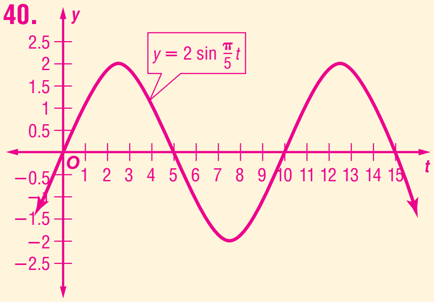
54. $y = 3x^2, y = 3x^2 - 4$

55. $y = 2x^2, y = 2(x+1)^2$

56. $y = x^2 + 2, y = (x - 3)^2 + 2$

Answers

40.



42. Sample answer: Tides display periodic behavior. This means that their pattern repeats at regular intervals. Answers should include the following information.

- **Tides rise and fall in a periodic manner, similar to the sine function.**
- **In $f(x) = a \sin bx$, the amplitude is the absolute value of a.**

14-2 Translations of Trigonometric Graphs

What You'll Learn

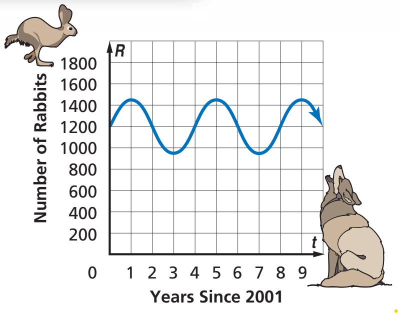
- Graph horizontal translations of trigonometric graphs and find phase shifts.
- Graph vertical translations of trigonometric graphs.

Vocabulary

- phase shift
- vertical shift
- midline

How can translations of trigonometric graphs be used to show animal populations?

In predator-prey ecosystems, the number of predators and the number of prey tend to vary in a periodic manner. In a certain region with coyotes as predators and rabbits as prey, the rabbit population R can be modeled by the equation $R = 1200 + 250 \sin \frac{1}{2}\pi t$, where t is the time in years since January 1, 2001.



HORIZONTAL TRANSLATIONS Recall that a translation is a type of transformation in which the image is identical to the preimage in all aspects except its location on the coordinate plane. A horizontal translation shifts to the left or right, and not upward or downward.

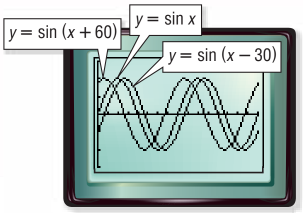
Graphing Calculator Investigation

Horizontal Translations

On a TI-83 Plus, set the MODE to degrees.

Think and Discuss 1–3. See margin.

1. Graph $y = \sin x$ and $y = \sin (x - 30)$. How do the two graphs compare?
2. Graph $y = \sin (x + 60)$. How does this graph compare to the other two?
3. What conjecture can you make about the effect of h in the function $y = \sin (x - h)$?
4. Test your conjecture on the following pairs of graphs.
   - $y = \cos x$ and $y = \cos (x + 30)$ See pp. 811A–811N for graphs;
   - $y = \tan x$ and $y = \tan (x - 45)$ the conjecture holds.
   - $y = \sec x$ and $y = \sec (x + 75)$



[0, 720] scl: 45 by [−1.5, 1.5] scl: 0.5

TEACHING TIP

For Exercise 4, point out that since the calculator has no preprogrammed button for the secant function, they will need to graph $y = \frac{1}{\cos x}$.

Notice that when a constant is added to an angle measure in a trigonometric function, the graph is shifted to the left or to the right. If (x, y) are coordinates of $y = \sin x$, then $(x \pm h, y)$ are coordinates of $y = \sin (x \mp h)$. A horizontal translation of a trigonometric function is called a **phase shift**.

14-2 Lesson Notes

1 Focus

5-Minute Check Transparency 14-2 Use as a quiz or review of Lesson 14-1.

Mathematical Background notes are available for this lesson on page 760C.

How can translations of trigonometric graphs be used to show animal populations?

Ask students:

- Why might the two animal populations vary? Sample answer: As the number of predators increases, more prey are eaten and there are fewer prey left. Lower numbers of prey means increased competition for food by the predator species, so the number of predators decreases as their food supply diminishes.
- What are the minimum and maximum rabbit populations shown by the graph? 950 rabbits, 1450 rabbits
- How could you find the range of the population without calculating the maximum and minimum values or graphing the function? The range, 500, is twice the amplitude, 250, of the graph.

Answers

See p. 770 for Graphing Calculator answers.

Resource Manager

Workbook and Reproducible Masters

Chapter 14 Resource Masters

- Study Guide and Intervention, pp. 843–844
- Skills Practice, p. 845
- Practice, p. 846
- Reading to Learn Mathematics, p. 847
- Enrichment, p. 848
- Assessment, p. 893

Graphing Calculator and Spreadsheet Masters, p. 53

School-to-Career Masters, p. 27

Transparencies

5-Minute Check Transparency 14-2
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lesson 27
Interactive Chalkboard

2 Teach

Building on Prior Knowledge

In Lesson 6-6, students learned about the translations of graphs of quadratic functions. In this lesson, students will use similar techniques to study the translations of graphs of the trigonometric functions.

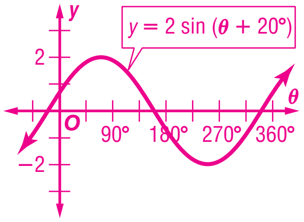
HORIZONTAL TRANSLATIONS

In-Class Example

Power Point®

1 State the amplitude, period, and phase shift for each function. Then graph the function.

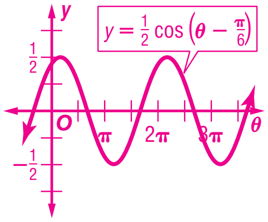
a. $y = 2 \sin (\theta + 20°)$ **amplitude: 2; period: 360°; phase shift: 20° left**



b. $y = \frac{1}{2} \cos \left(\theta - \frac{\pi}{6}\right)$

amplitude: $\frac{1}{2}$; period: 2π;

phase shift: $\frac{\pi}{6}$ right



Answers (p. 769)

Graphing Calculator Investigation

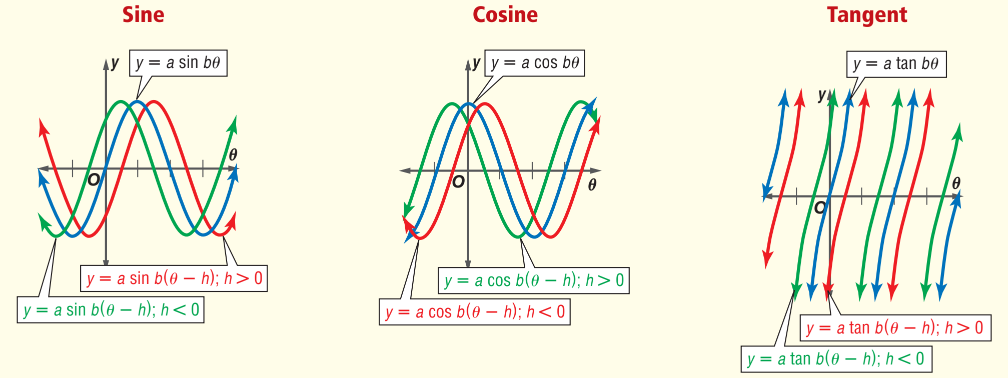
1. **The graph of $y = \sin (x - 30)$ is shifted 30° to the right of the graph of $y = \sin x$.**
2. **The graph of $y = \sin (x + 60)$ is shifted 60° to the left of the graph of $y = \sin x$.**
3. **Sample answer: When h is positive the graph shifts right h units. When h is negative the graph shifts left h units.**

Key Concept — Phase Shift

- **Words:** The phase shift of the functions $y = a \sin b(\theta - h)$, $y = a \cos b(\theta - h)$, and $y = a \tan b(\theta - h)$ is h, where $b > 0$.

  If $h > 0$, the shift is to the right. If $h < 0$, the shift is to the left.

- **Models:**



The secant, cosecant, and cotangent can be graphed using the same rules.

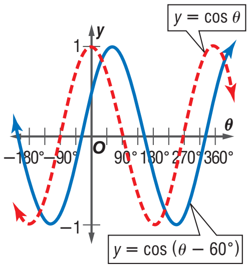
Example 1 Graph Horizontal Translations

State the amplitude, period, and phase shift for each function. Then graph the function.

a. $y = \cos (\theta - 60°)$

Since $a = 1$ and $b = 1$, the amplitude and period of the function are the same as $y = \cos \theta$. However, $h = 60°$, so the phase shift is 60°. Because $h > 0$, the parent graph is shifted to the right.

To graph $y = \cos (\theta - 60°)$, consider the graph of $y = \cos \theta$. Graph this function and then shift the graph 60° to the right. The graph $y = \cos (\theta - 60°)$ is the graph of $y = \cos \theta$ shifted to the right.



Study Tip

Verifying a Graph
After drawing the graph of a trigonometric function, select value of θ and evaluate them in the equation to verify your graph.

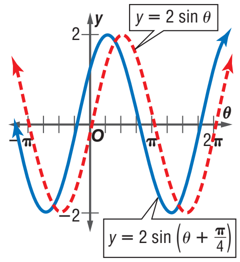
b. $y = 2 \sin \left(\theta + \frac{\pi}{4}\right)$

Amplitude: $a = |2|$ or 2

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{|1|}$ or 2π

Phase Shift: $h = -\frac{\pi}{4}$ $\left(\theta + \frac{\pi}{4}\right) = \theta - \left(-\frac{\pi}{4}\right)$

The phase shift is to the left since $-\frac{\pi}{4} < 0$.



Graphing Calculator Investigation

Graphing the Secant Function The graph of the secant function should look like a pattern of U shapes, alternately opening upward and downward. Students might want to extend the activity by investigating the graphs of cosecant (csc) and cotangent (cot). The graphing calculator does not have a button for graphing either of these functions so students should enter **Y= 1/sin *x*** to graph the cosecant function, and they should enter **Y= 1/tan *x*** to graph the cotangent function.

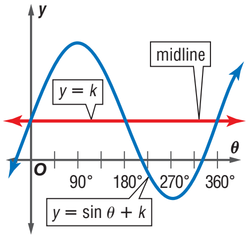
Study Tip

Look Back
Pay close attention to trigonometric functions for the placement of parentheses. Note that $\sin(\theta + x) \neq \sin\theta + x$. The expression on the left represents a phase shift while the expression on the right represents a vertical shift.

VERTICAL TRANSLATIONS In Chapter 6, you learned that the graph of $y = x^2 + 4$ is a vertical translation of the parent graph of $y = x^2$. Similarly, graphs of trigonometric functions can be translated vertically through a **vertical shift**.

When a constant is added to a trigonometric function, the graph is shifted upward or downward. If (x, y) are coordinates of $y = \sin x$, then $(x, y \pm k)$ are coordinates of $y = \sin x \pm k$.

A new horizontal axis called the **midline** becomes the reference line about which the graph oscillates. For the graph of $y = \sin\theta + k$, the midline is the graph of $y = k$.



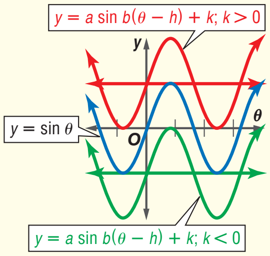
Key Concept Vertical Shift

- **Words** The vertical shift of the functions $y = a \sin b(\theta - h) + k$, $y = a \cos b(\theta - h) + k$, and $y = a \tan b(\theta - h) + k$ is k.

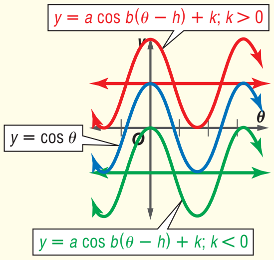
  If $k > 0$, the shift is up. If $k < 0$, the shift is down. The midline is $y = k$.

- **Models:**

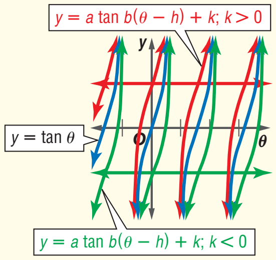
Sine



Cosine



Tangent



The secant, cosecant, and cotangent can be graphed using the same rules.

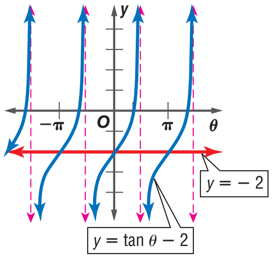
Example 2 Graph Vertical Translations

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

a. $y = \tan\theta - 2$

Since $\tan\theta - 2 = \tan\theta + (-2)$, $k = -2$, and the vertical shift is -2. Draw the midline, $y = -2$. The tangent function has no amplitude and the period is the same as that of $\tan\theta$.

Draw the graph of the function relative to the midline.



www.algebra2.com/extra_examples

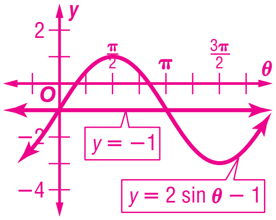
VERTICAL TRANSLATIONS

Teaching Tip Make sure students are clear about the difference between the terms *vertical* and *horizontal*. Students can use the words *vertigo* (a sense of dizziness some people experience when looking down from a great height) and *horizon* to help link vertical and horizontal to their relative direction.

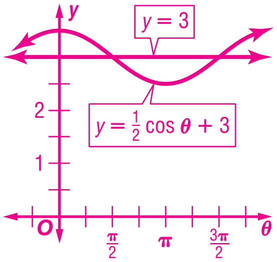
In-Class Example Power Point®

2 State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

a. $y = 2\sin\theta - 1$
vertical shift: -1; midline: $y = -1$; amplitude: 2; period: 2π



b. $y = \frac{1}{2}\cos\theta + 3$
vertical shift: $+3$; midline: $y = 3$; amplitude: $\frac{1}{2}$; period: 2π



Interactive Chalkboard
PowerPoint® Presentations

This CD-ROM is a customizable Microsoft® PowerPoint® presentation that includes:

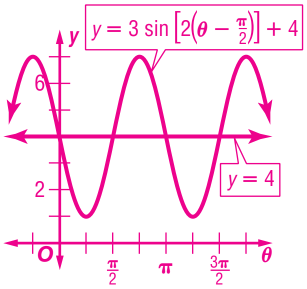
- Step-by-step, dynamic solutions of each In-Class Example from the Teacher Wraparound Edition
- Additional, Your Turn exercises for each example
- The 5-Minute Check Transparencies
- Hot links to Glencoe Online Study Tools

Teaching Tip Suggest that students write the general form of the sine, cosine, and tangent functions given in the Key Concept box on p. 771 at the top of separate index cards. Using the steps listed in the Concept Summary on p. 772, have students identify the parts of the equation used in each step to find the information listed in each step. They can then use their index cards to work Example 3.

In-Class Example

Power Point®

3 State the vertical shift, amplitude, period, and phase shift of $y = 3 \sin\left[2\left(\theta - \frac{\pi}{2}\right)\right] + 4$. Then graph the function. **The vertical shift is +4. The amplitude is 3. The period is π. The phase shift is $\frac{\pi}{2}$ to the right.**



Study Tip

Look Back
It may be helpful to first graph the parent graph $y = \sin \theta$ in one color. Then apply the vertical shift and graph the function in another color. Then apply the change in amplitude and graph the function in the final color.

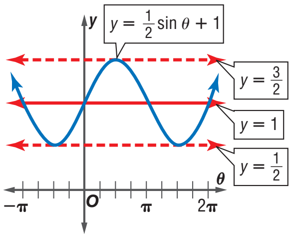
b. $y = \frac{1}{2} \sin \theta + 1$

Vertical shift: $k = 1$, so the midline is the graph of $y = 1$.

Amplitude: $|a| = \left|\frac{1}{2}\right|$ or $\frac{1}{2}$

Period: $\frac{2\pi}{|b|} = 2\pi$

Since the amplitude of the function is $\frac{1}{2}$, draw dashed lines parallel to the midline that are $\frac{1}{2}$ unit above and below the midline. Then draw the sine curve.



In general, use the following steps to graph any trigonometric function.

Concept Summary — Graphing Trigonometric Functions

Step 1 Determine the vertical shift, and graph the midline.

Step 2 Determine the amplitude, if it exists. Use dashed lines to indicate the maximum and minimum values of the function.

Step 3 Determine the period of the function and graph the appropriate function.

Step 4 Determine the phase shift and translate the graph accordingly.

Example 3 Graph Transformations

State the vertical shift, amplitude, period, and phase shift of $y = 4 \cos\left[\frac{1}{2}\left(\theta - \frac{\pi}{3}\right)\right] - 6$. Then graph the function.

The function is written in the form $y = a \cos [b(\theta - h)] + k$. Identify the values of k, a, b, and h.

$k = -6$, so the vertical shift is -6.

$a = 4$, so the amplitude is $|4|$ or 4.

$b = \frac{1}{2}$, so the period is $\frac{2\pi}{\left|\frac{1}{2}\right|}$ or 4π.

$h = \frac{\pi}{3}$, so the phase shift is $\frac{\pi}{3}$ to the right.

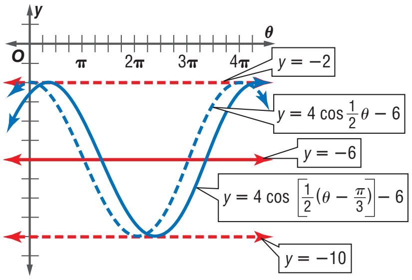
Then graph the function.

Step 1 The vertical shift is -6. Graph the midline $y = -6$.

Step 2 The amplitude is 4. Draw dashed lines 4 units above and below the midline at $y = -2$ and $y = -10$.

Step 3 The period is 4π, so the graph will be stretched. Graph $y = 4 \cos \frac{1}{2}\theta - 6$ using the midline as a reference.

Step 4 Shift the graph $\frac{\pi}{3}$ to the right.



You can use information about amplitude, period, and translations of trigonometric functions to model real-world applications.

Example 4 Use Translations to Solve a Problem

HEALTH **Suppose a person's resting blood pressure is 120 over 80. This means that the blood pressure oscillates between a maximum of 120 and a minimum of 80. If this person's resting heart rate is 60 beats per minute, write a sine function that represents the blood pressure for *t* seconds. Then graph the function.**

Explore You know that the function is periodic and can be modeled using sine.

Plan Let P represent blood pressure and let t represent time in seconds. Use the equation $P = a \sin [b(t - h)] + k$.

Solve

- Write the equation for the midline. Since the maximum is 120 and the minimum is 80, the midline lies halfway between these values.

  $P = \frac{120 + 80}{2}$ or 100

- Determine the amplitude by finding the difference between the midline value and the maximum and minimum values.

  $a = |120 - 100|$ $\qquad$ $a = |80 - 100|$

  $= |20|$ or 20 $\qquad$ $= |-20|$ or 20

  Thus, $a = 20$.

- Determine the period of the function and solve for b. Recall that the period of a function can be found using the expression $\frac{2\pi}{|b|}$. Since the heart rate is 60 beats per minute, there is one heartbeat, or cycle, per second. So, the period is 1 second.

  $1 = \frac{2\pi}{|b|}$ Write an equation.

  $|b| = 2\pi$ Multiply each side by $|b|$.

  $b = \pm 2\pi$ Solve.

  For this example, let $b = 2\pi$. The use of the positive or negative value depends upon whether you begin a cycle with a maximum value (positive) or a minimum value (negative).

- There is no phase shift, so $h = 0$. So, the equation is $P = 20 \sin 2\pi t + 100$.

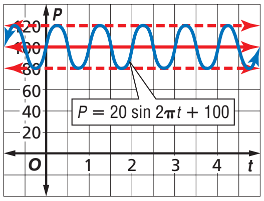
- Graph the function.

  Step 1 Draw the midline $P = 100$.

  Step 2 Draw maximum and minimum reference lines.

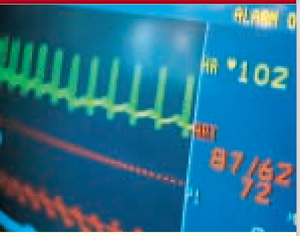
  Step 3 Use the period to draw the graph of the function.

  Step 4 There is no phase shift.



Examine Notice that each cycle begins at the midline, rises to 120, drops to 80, and then returns to the midline. This represents the blood pressure of 120 over 80 for one heartbeat. Since each cycle lasts 1 second, there will be 60 cycles, or heartbeats, in 1 minute. Therefore, the graph accurately represents the information.

More About. . .



Health

Blood pressure can change from minute to minute and can be affected by the slightest of movements, such as tapping your fingers or crossing your arms.

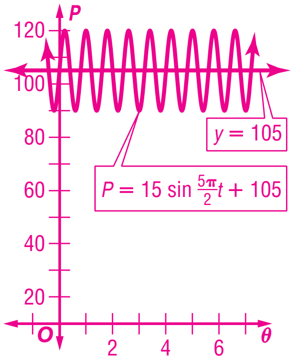
Source: American Heart Association

In-Class Example

Power Point®

4 **HEALTH** Refer to Example 4 in the Student Edition. Write a sine function that represents the blood pressure for *t* seconds of a person with a resting blood pressure of 120 over 90 and a heart rate of 75 beats per minute. Then graph the function.

$P = 15 \sin \frac{5\pi}{2} t + 105$



DAILY INTERVENTION

Differentiated Instruction

Kinesthetic Make coordinate axes with masking tape on the classroom floor. Give students at least 15 feet of rope and have them stand along the x-axis, positioning the rope to model the graph of $y = \sin x$. As you call out equations of functions whose graphs are horizontal phase shifts of the graph of $y = \sin x$, students can step left or right to model the translated graph. Similarly, call out functions whose graphs are vertical shifts of the graph of $y = \sin x$.

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 14.
- record the Key Concepts box on p. 771 and the Concept Summary on p. 772.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective

- **Horizontal Translations:** 19–24, 33–42
- **Vertical Translations:** 25–42

Odd/Even Assignments

Exercises 19–42 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 19–29 odd, 33–43 odd, 44–73

Average: 19–43 odd, 44–73

Advanced: 20–42 even, 44–46, 48–65 (optional: 66–73)

Check for Understanding

Concept Check

1–3. See margin.

1. **Identify** the vertical shift, amplitude, period, and phase shift of the graph of $y = 3 \cos (2x - 90°) + 15$.
2. **Define** the midline of a trigonometric graph.
3. **OPEN ENDED** Write the equation of a trigonometric function with a phase shift of $-45°$.

Guided Practice

GUIDED PRACTICE KEY

Exercises	Examples
4–7	1
8–11	2
12–15	3
16–18	4

State the amplitude, period, and phase shift for each function. Then graph the function. **4–15. See pp. 811A–811N for graphs.**

4. $y = \sin \left(\theta - \frac{\pi}{2}\right)$ **1; 2π; $\frac{\pi}{2}$**
5. $y = \tan (\theta + 60°)$ **no amplitude; 180°; −60°**
6. $y = \cos (\theta - 45°)$ **1; 360°; 45°**
7. $y = \sec \left(\theta + \frac{\pi}{3}\right)$ **no amplitude; 2π; $-\frac{\pi}{3}$**

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

8. $y = \cos \theta + \frac{1}{4}$ **$\frac{1}{4}$; $y = \frac{1}{4}$; 1; 360°**
9. $y = \sec \theta - 5$ **−5; $y = -5$; no amplitude; 360°**
10. $y = \tan \theta + 4$ **4; $y = 4$; no amplitude; 180°**
11. $y = \sin \theta + 0.25$ **0.25; $y = 0.25$; 1; 360°**

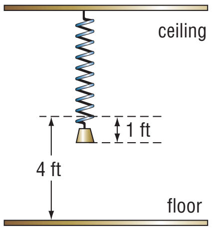
State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function. **12. 10; 3; 180°; 30°** **13. −6; no amplitude; 60°; −45°**

12. $y = 3 \sin [2(\theta - 30°)] + 10$
13. $y = 2 \cot (3\theta + 135°) - 6$
14. $y = \frac{1}{2} \sec \left[4\left(\theta - \frac{\pi}{4}\right)\right] + 1$ **1; no amplitude; $\frac{\pi}{2}$; $\frac{\pi}{4}$**
15. $y = \frac{2}{3} \cos \left[\frac{1}{2}\left(\theta + \frac{\pi}{6}\right)\right] - 2$ **−2; $\frac{2}{3}$; 4π; $-\frac{\pi}{6}$**

Application

PHYSICS **For Exercises 16–18, use the following information.**

A weight is attached to a spring and suspended from the ceiling. At equilibrium, the weight is located 4 feet above the floor. The weight is pulled down 1 foot and released.



16. Determine the vertical shift, amplitude, and period of a function that represents the height of the weight above the floor if the weight returns to its lowest position every 4 seconds. **4; 1; 4 s**
17. Write the equation for the height h of the weight above the floor as a function of time t seconds. **$h = 4 - \cos \frac{\pi}{2}t$ or $h = 4 - \cos 90°t$**
18. Draw a graph of the function you wrote in Exercise 17. **See pp. 811A–811N.**

★ indicates increased difficulty

Practice and Apply

State the amplitude, period, and phase shift for each function. Then graph the function. **20. no amplitude; 180°; 30°**

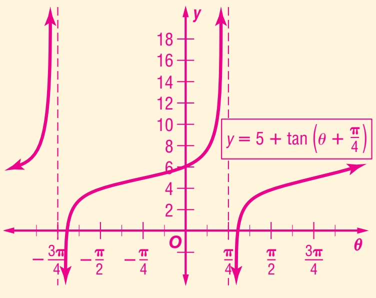
19–24. See pp. 811A–811N for graphs.

19. $y = \cos (\theta + 90°)$ **1; 360°; −90°**
20. $y = \cot (\theta - 30°)$
21. $y = \sin \left(\theta - \frac{\pi}{4}\right)$ **1; 2π; $\frac{\pi}{4}$**
22. $y = \cos \left(\theta + \frac{\pi}{3}\right)$ **1; 2π; $-\frac{\pi}{3}$**
23. $y = \frac{1}{4} \tan (\theta + 22.5°)$ **no amplitude; 180°; −22.5°**
24. $y = 3 \sin (\theta - 75°)$ **3; 360°; 75°**

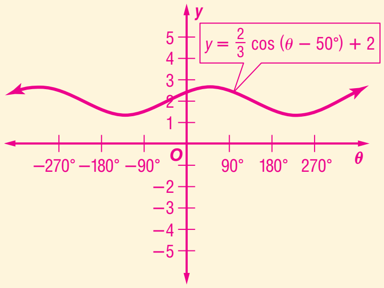
Answers

1. **vertical shift: 15; amplitude: 3; period: 180°; phase shift: 45°**
2. **The midline of a trigonometric function is the line about which the graph of the function oscillates after a vertical shift.**
3. **Sample answer: $y = \sin (\theta + 45°)$**

31.



32.



Homework Help

For Exercises	See Examples
19–24	1
25–30	2
31, 32	1, 2
33–42	3
37–40	4

Extra Practice

See page 859.

28. $-\frac{3}{4}$; $y = -\frac{3}{4}$; no amplitude; 360°

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function. 25–30. See pp. 811A–811N for graphs.

25. $y = \sin \theta - 1$ -1; $y = -1$; 1; 360°

26. $y = \sec \theta + 2$ 2; $y = 2$; no amplitude; 360°

27. $y = \cos \theta - 5$ -5; $y = -5$; 1; 360°

28. $y = \csc \theta - \frac{3}{4}$

29. $y = \frac{1}{2}\sin \theta + \frac{1}{2}$ $\frac{1}{2}$; $y = \frac{1}{2}$; $\frac{1}{2}$; 360°

30. $y = 6 \cos \theta + 1.5$ 1.5; $y = 1.5$; 6; 360°

★ 31. Graph $y = 5 + \tan\left(\theta + \frac{\pi}{4}\right)$. Describe the transformation to the parent graph $y = \tan \theta$. See margin for graph; translation $\frac{\pi}{4}$ units left and 5 units up.

★ 32. Draw a graph of the function $y = \frac{2}{3}\cos(\theta - 50°) + 2$. How does this graph compare to the graph of $y = \cos \theta$? See margin for graph; translation 50° right and 2 units up with an amplitude of $\frac{2}{3}$ unit.

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function. 33–42. See pp. 811A–811N.

33. $y = 2 \sin [3(\theta - 45°)] + 1$

34. $y = 4 \cos [2(\theta + 30°)] - 5$

35. $y = 3 \csc \left[\frac{1}{2}(\theta + 60°)\right] - 3.5$

36. $y = 6 \cot \left[\frac{2}{3}(\theta - 90°)\right] + 0.75$

37. $y = \frac{1}{4}\cos(2\theta - 150°) + 1$

38. $y = \frac{2}{5}\tan(6\theta + 135°) - 4$

39. $y = 3 + 2 \sin \left[2\left(\theta + \frac{\pi}{4}\right)\right]$

40. $y = 4 + 5 \sec \left[\frac{1}{3}\left(\theta + \frac{2\pi}{3}\right)\right]$

41. Graph $y = 3 - \frac{1}{2}\cos \theta$ and $y = 3 + \frac{1}{2}\cos(\theta + \pi)$. How do the graphs compare?

42. Compare the graphs of $y = -\sin\left[\frac{1}{4}\left(\theta - \frac{\pi}{2}\right)\right]$ and $y = \cos\left[\frac{1}{4}\left(\theta + \frac{3\pi}{2}\right)\right]$.

WebQuest

Translations of trigonometric graphs can be used to describe temperature trends. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

43. **MUSIC** When represented on an oscilloscope, the note A above middle C has period of $\frac{1}{440}$. Which of the following can be an equation for an oscilloscope graph of this note? The amplitude of the graph is K. c

a. $y = K \sin 220\pi t$ b. $y = K \sin 440\pi t$ c. $y = K \sin 880\pi t$

ZOOLOGY For Exercises 44–46, use the following information.

The population of predators and prey in a closed ecological system tends to vary periodically over time. In a certain system, the population of owls O can be represented by $O = 150 + 30 \sin\left(\frac{\pi}{10}t\right)$ where t is the time in years since January 1, 2001. In that same system, the population of mice M can be represented by $M = 600 + 300 \sin\left(\frac{\pi}{10}t + \frac{\pi}{20}\right)$. 44. 180; 5 yr

44. Find the maximum number of owls. After how many years does this occur?

45. What is the minimum number of mice? How long does it take for the population of mice to reach this level? 300; 14.5 yr

46. Why would the maximum owl population follow behind the population of mice? See margin.

47. $h = 9 + 6 \sin\left[\frac{\pi}{9}(t - 1.5)\right]$

47. **TIDES** The height of the water in a harbor rose to a maximum height of 15 feet at 6:00 P.M. and then dropped to a minimum level of 3 feet by 3:00 A.M. Assume that the water level can be modeled by the sine function. Write an equation that represents the height h of the water t hours after noon on the first day.

Online Research **Data Update** Use the Internet or another resource to find tide data for a location of your choice. Write a sine function to represent your data. Then graph the function. Visit www.algebra2.com/data_update to learn more.

www.algebra2.com/self_check_quiz

Answer

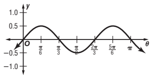
46. Sample answer: When the prey (mouse) population is at its greatest the predator will consume more and the predator population will grow while the prey population falls.

Study Guide and Intervention, p. 843 (shown) and p. 844

Horizontal Translations When a constant is subtracted from the angle measure in a trigonometric function, a **phase shift** of the graph results.

Phase Shift	The horizontal phase shift of the graphs of the functions $y = a \sin b(\theta - h)$, $y = a \cos b(\theta - h)$, and $y = a \tan b(\theta - h)$ is h, where $b > 0$. If $h > 0$, the shift is to the right. If $h < 0$, the shift is to the left.

Example **State the amplitude, period, and phase shift for $y = \frac{1}{2}\cos 3\left(\theta - \frac{\pi}{2}\right)$. Then graph the function.**



Amplitude: $a = \left|\frac{1}{2}\right|$ or $\frac{1}{2}$

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{|3|}$ or $\frac{2\pi}{3}$

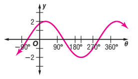
Phase Shift: $h = \frac{\pi}{2}$

The phase shift is to the right since $\frac{\pi}{2} > 0$.

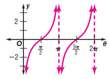
Exercises

State the amplitude, period, and phase shift for each function. Then graph the function.

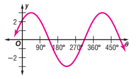
1. $y = 2 \sin(\theta + 60°)$ 2; 360°; 60° to the left



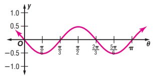
2. $y = \tan\left(\theta - \frac{\pi}{2}\right)$ no amplitude; π; $\frac{\pi}{2}$ to the right



3. $y = 3 \cos(\theta - 45°)$ 3; 360°; 45° to the right



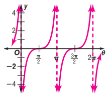
4. $y = \frac{1}{2}\sin 3\left(\theta - \frac{\pi}{3}\right)$ $\frac{1}{2}$; $\frac{2\pi}{3}$; $\frac{\pi}{3}$ to the right



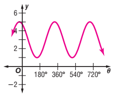
Skills Practice, p. 845 and Practice, p. 846 (shown)

State the vertical shift, amplitude, period, and phase shift for each function. Then graph the function.

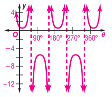
1. $y = \frac{1}{2}\tan\left(\theta - \frac{\pi}{2}\right)$ no vertical shift; no amplitude; π; $\frac{\pi}{2}$



2. $y = 2 \cos(\theta + 30°) + 3$ 3; 2; 360; −30°



3. $y = 3 \csc(2\theta + 60°) - 2.5$ −2.5; no amplitude; 180°; −60°



ECOLOGY For Exercises 4–6, use the following information.

The population of an insect species in a stand of trees follows the growth cycle of a particular tree species. The insect population can be modeled by the function $y = 40 + 30 \sin 6t$, where t is the number of years since the stand was first cut in November, 1920.

4. How often does the insect population reach its maximum level? every 60 yr

5. When did the population last reach its maximum? 1995

6. What condition in the stand do you think corresponds with a minimum insect population? Sample answer: The species on which the insect feeds has been cut.

BLOOD PRESSURE For Exercises 7–9, use the following information.

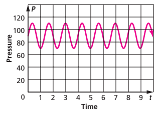
Jason's blood pressure is 110 over 70, meaning that the pressure oscillates between a maximum of 110 and a minimum of 70. Jason's heart rate is 45 beats per minute. The function that represents Jason's blood pressure P can be modeled using a sine function with no phase shift.

7. Find the amplitude, midline, and period in seconds of the function. 20; $P = 90$; $1\frac{1}{3}$ s

8. Write a function that represents Jason's blood pressure P after t seconds. $P = 20 \sin 270t + 90$

9. Graph the function.

Jason's Blood Pressure



Reading to Learn Mathematics, p. 847

ELL

Pre-Activity **How can translations of trigonometric graphs be used to show animal populations?**

Read the introduction to Lesson 14-2 at the top of page 769 in your textbook.

According to the model given in your textbook, what would be the estimated rabbit population for January 1, 2005? 1200

Reading the Lesson

1. Determine whether the graph of each function represents a shift of the parent function *to the left*, *to the right*, *upward*, or *downward*. (Do not actually graph the functions.)
   - a. $y = \sin(\theta + 90°)$ to the left
   - b. $y = \sin \theta + 3$ upward
   - c. $y = \cos\left(\theta - \frac{\pi}{3}\right)$ to the right
   - d. $y = \tan \theta - 4$ downward

2. Determine whether the graph of each function has an *amplitude change*, *period change*, *phase shift*, or *vertical shift* compared to the graph of the parent function. (More than one of these may apply to each function. Do not actually graph the functions.)
   - a. $y = 3 \sin\left(\theta + \frac{5\pi}{6}\right)$ amplitude change and phase shift
   - b. $y = \cos(2\theta + 70°)$ period change and phase shift
   - c. $y = -4 \cos 3\theta$ amplitude change and period change
   - d. $y = \sec \frac{1}{2}\theta + 3$ period change and vertical shift
   - e. $y = \tan\left(\theta - \frac{\pi}{4}\right) - 1$ phase shift and vertical shift
   - f. $y = 2 \sin\left(\frac{1}{3}\theta + \frac{\pi}{6}\right) - 4$ amplitude change, period change, phase shift, and vertical shift

Helping You Remember

3. Many students have trouble remembering which of the functions $y = \sin(\theta + \alpha)$ and $y = \sin(\theta - \alpha)$ represents a shift to the left and which represents a shift to the right. Using $\alpha = 45°$, explain a good way to remember which is which.

Sample answer: Although sine curves are infinitely repeating periodic graphs, think of $y = \sin x$ starting a period or cycle at (0, 0). Then $y = \sin(\theta + 45°)$ "starts early" at (−45°), a shift of 45° to the left, while $y = \sin(\theta - 45°)$ "starts late" at 45°, a shift of 45° to the right.

Enrichment, p. 848

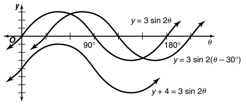
Translating Graphs of Trigonometric Functions

Three graphs are shown at the right:

$y = 3 \sin 2\theta$

$y = 3 \sin 2(\theta - 30°)$

$y + 4 = 3 \sin 2\theta$



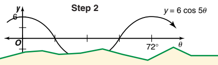
Replacing θ with $(\theta - 30°)$ translates the graph to the right. Replacing y with $y + 4$ translates the graph 4 units down.

Example **Graph one cycle of $y = 6 \cos(5\theta + 80°) + 2$.**

Step 1 Transform the equation into the form $y - k = a \cos b(\theta - h)$.

$y - 2 = 6 \cos 5(\theta + 16°)$

Step 2



4 Assess

Open-Ended Assessment

Writing Have students write a summary explaining how to use the equation for a trigonometric function to identify how its graph is shifted vertically and/or horizontally from its parent graph.

Getting Ready for Lesson 14-3

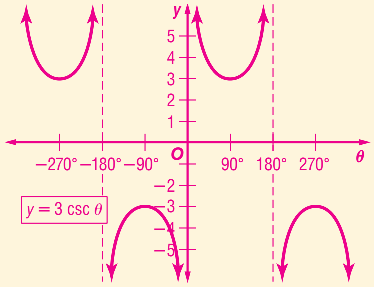
PREREQUISITE SKILL Students will find the value of a trigonometric function in Lesson 14-3. Students should be familiar with values of the sine, cosine, and tangent functions for various reference angles in order to determine the sign of the value. Use Exercises 66–73 to determine your students' familiarity with reference angles.

Assessment Options

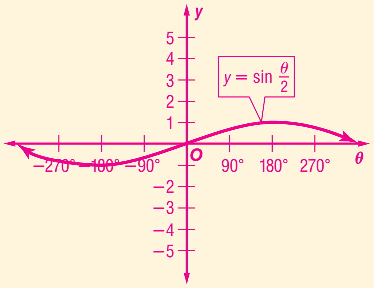
Quiz (Lessons 14-1 and 14-2) is available on p. 893 of the *Chapter 14 Resource Masters*.

Answers

52. amplitude: does not exist; period: 360° or 2π



53. amplitude: 1; period: 720° or 4π



48. CRITICAL THINKING The graph of $y = \cot \theta$ is a transformation of the graph of $y = \tan \theta$. Determine a, b, and h so that $\cot \theta = a \tan [b(\theta - h)]$ for all values of θ for which each function is defined. $a = -1$, $b = 1$, $h = \frac{\pi}{2}$

49. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson. **See pp. 811A–811N.**

How can translations of trigonometric graphs be used to show animal populations?

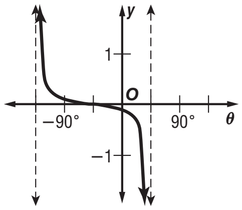
Include the following in your answer:

- a description of what each number in the equation $R = 1200 + 250 \sin \frac{1}{2}\pi t$ represents, and
- a comparison of the graphs of $y = a \cos bx$, $y = a \cos bx + k$, and $y = a \cos [b(x - h)]$.

Standardized Test Practice

50. Which equation is represented by the graph? **B**

Ⓐ $y = \cot (\theta + 45°)$
Ⓑ $y = \cot (\theta - 45°)$
Ⓒ $y = \tan (\theta + 45°)$
Ⓓ $y = \tan (\theta - 45°)$



51. Identify the equation for a sine function of period 90°, after a phase shift 20° to the left. **D**

Ⓐ $y = \sin [0.25(\theta - 20°)]$
Ⓑ $y = \sin [4(\theta - 20°)]$
Ⓒ $y = \sin [0.25(\theta + 20°)]$
Ⓓ $y = \sin [4(\theta + 20°)]$

Maintain Your Skills

Mixed Review

Find the amplitude, if it exists, and period of each function. Then graph each function. *(Lesson 14-1)* **52–54. See margin.**

52. $y = 3 \csc \theta$
53. $y = \sin \frac{\theta}{2}$
54. $y = 3 \tan \frac{2}{3}\theta$

Find each value. *(Lesson 13-7)*

55. $\sin \left(\text{Cos}^{-1} \frac{2}{3}\right)$ **0.75**
56. $\cos \left(\text{Cos}^{-1} \frac{4}{7}\right)$ **0.57**
57. $\text{Sin}^{-1} \left(\sin \frac{5}{6}\right)$ **0.83**
58. $\cos \left(\text{Tan}^{-1} \frac{3}{4}\right)$ **0.8**

59. GEOMETRY Find the total number of diagonals that can be drawn in a decagon. *(Lesson 12-2)* **35**

Solve each equation. Round to the nearest hundredth. *(Lesson 10-4)*

60. $4^x = 24$ **2.29**
61. $4.3^{3x + 1} = 78.5$ **0.66**
62. $7^{x - 2} = 53^{-x}$ **0.66**

Simplify each expression. *(Lesson 9-4)*

63. $\frac{3}{a - 2} + \frac{2}{a - 3}$
64. $\frac{w + 12}{4w - 16} - \frac{w + 4}{2w - 8}$ $-\frac{1}{4}$
65. $\frac{3y + 1}{2y - 10} + \frac{1}{y^2 - 2y - 15}$

63. $\frac{5a - 13}{(a - 2)(a - 3)}$

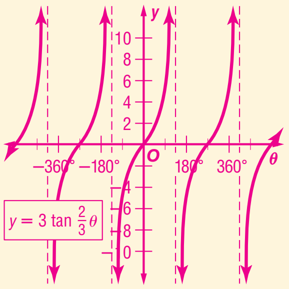
65. $\frac{3y^2 + 10y + 5}{2(y - 5)(y + 3)}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find the value of each function.**
*(To review **reference angles**, see Lesson 13-3.)*

66. $\cos 150°$ $-\frac{\sqrt{3}}{2}$
67. $\tan 135°$ -1
68. $\sin \frac{3\pi}{2}$ -1
69. $\cos \left(-\frac{\pi}{3}\right)$ $\frac{1}{2}$
70. $\sin (-\pi)$ 0
71. $\tan \left(-\frac{5\pi}{6}\right)$ $\frac{\sqrt{3}}{3}$
72. $\cos 225°$ $-\frac{\sqrt{2}}{2}$
73. $\tan 405°$ 1

54. amplitude: does not exist; period: 270° or $\frac{3\pi}{2}$



14-3 Trigonometric Identities

What You'll Learn

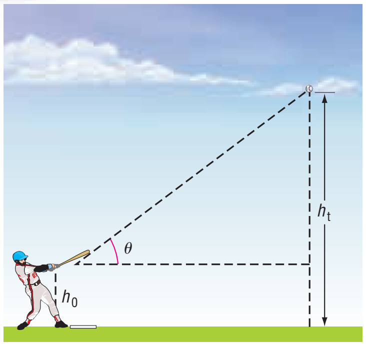
- Use identities to find trigonometric values.
- Use trigonometric identities to simplify expressions.

Vocabulary

- trigonometric identity

How can trigonometry be used to model the path of a baseball?

A model for the height of a baseball after it is hit as a function of time can be determined using trigonometry. If the ball is hit with an initial velocity of v feet per second at an angle of θ from the horizontal, then the height h of the ball after t seconds can be represented by $h = \left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0$, where h_0 is the height of the ball in feet the moment it is hit.



FIND TRIGONOMETRIC VALUES In the equation above, the second term $\left(\frac{\sin\theta}{\cos\theta}\right)t$ can also be written as $(\tan\theta)t$. $\left(\frac{\sin\theta}{\cos\theta}\right)t = (\tan\theta)t$ is an example of a trigonometric identity. A **trigonometric identity** is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

The identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is true except for angle measures such as 90°, 270°, 450°, …, $90° + 180° \cdot k$. The cosine of each of these angle measures is 0, so none of the expressions tan 90°, tan 270°, tan 450°, and so on, are defined. An identity similar to this is $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

These identities are sometimes called *quotient identities*. These and other basic trigonometric identities are listed below.

Key Concept *Basic Trigonometric Identities*

Quotient Identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	
Reciprocal Identities	$\csc\theta = \frac{1}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{1}{\tan\theta}$
Pythagorean Identities	$\cos^2\theta + \sin^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $\cot^2\theta + 1 = \csc^2\theta$		

You can use trigonometric identities to find values of trigonometric functions.

14-3 Lesson Notes

1 Focus

5-Minute Check Transparency 14-3 Use as a quiz or review of Lesson 14-2.

Mathematical Background notes are available for this lesson on page 760D.

How can trigonometry be used to model the path of a baseball?

Ask students:

- What assumptions are made when using this function as a model for the height of the ball above the ground? **Sample answers: There is no wind; there is no friction as the ball passes through the air; the ball does not hit an obstruction.**
- To hit the ball the farthest horizontal distance, is it better for the angle θ to be a large angle or a small angle? **Answers will vary. Ignoring friction, an angle measure of 45° will result in the farthest horizontal distance.**
- What might be a reasonable value for h_0? **Sample answer: 3 ft**

Resource Manager

Workbook and Reproducible Masters

Chapter 14 Resource Masters

- Study Guide and Intervention, pp. 849–850
- Skills Practice, p. 851
- Practice, p. 852
- Reading to Learn Mathematics, p. 853
- Enrichment, p. 854

Graphing Calculator and Spreadsheet Masters, p. 54

Transparencies

5-Minute Check Transparency 14-3
Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

FIND TRIGONOMETRIC VALUES

Teaching Tip You can use the familiar definitions of sine, cosine, and tangent as the ratios of the opposite side, adjacent side, and hypotenuse of a right triangle to show why $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

In-Class Example PowerPoint®

1

a. Find $\tan\theta$ if $\sec\theta = -2$ and $180° < \theta < 270°$. **$\tan\theta = \sqrt{3}$**

b. Find $\sin\theta$ if $\cos\theta = -\frac{1}{2}$ and $90° < \theta < 180°$. **$\sin\theta = \frac{\sqrt{3}}{2}$**

SIMPLIFY EXPRESSIONS

In-Class Example PowerPoint®

2 Simplify $\sin\theta(\csc\theta - \sin\theta)$. **$\cos^2\theta$**

Example 1 Find a Value of a Trigonometric Function

a. Find $\cos\theta$ if $\sin\theta = -\frac{3}{5}$ and $90° < \theta < 180°$.

$\cos^2\theta + \sin^2\theta = 1$	Trigonometric identity
$\cos^2\theta = 1 - \sin^2\theta$	Subtract $\sin^2\theta$ from each side.
$\cos^2\theta = 1 - \left(\frac{3}{5}\right)^2$	Substitute $\frac{3}{5}$ for $\sin\theta$.
$\cos^2\theta = 1 - \frac{9}{25}$	Square $\frac{3}{5}$.
$\cos^2\theta = \frac{16}{25}$	Subtract.
$\cos\theta = \pm\frac{4}{5}$	Take the square root of each side.

Since θ is in the second quadrant, $\cos\theta$ is negative. Thus, $\cos\theta = -\frac{4}{5}$.

b. Find $\csc\theta$ if $\cot\theta = -\frac{1}{4}$ and $270° < \theta < 360°$.

$\cot^2\theta + 1 = \csc^2\theta$	Trigonometric identity
$\left(-\frac{1}{4}\right)^2 + 1 = \csc^2\theta$	Substitute $-\frac{1}{4}$ for $\cot\theta$.
$\frac{1}{16} + 1 = \csc^2\theta$	Square $-\frac{1}{4}$.
$\frac{17}{16} = \csc^2\theta$	Add.
$\pm\frac{\sqrt{17}}{4} = \csc\theta$	Take the square root of each side.

Since θ is in the fourth quadrant, $\csc\theta$ is negative. Thus, $\csc\theta = -\frac{\sqrt{17}}{4}$.

SIMPLIFY EXPRESSIONS Trigonometric identities can also be used to simplify expressions containing trigonometric functions. Simplifying an expression that contains trigonometric functions means that the expression is written as a numerical value or in terms of a single trigonometric function, if possible.

Example 2 Simplify an Expression

Simplify $\frac{\csc^2\theta - \cot^2\theta}{\cos\theta}$.

$\frac{\csc^2\theta - \cot^2\theta}{\cos\theta} = \frac{\frac{1}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}}{\cos\theta}$	$\csc^2\theta = \frac{1}{\sin^2\theta}$, $\cot^2\theta = \frac{\cos^2\theta}{\sin^2\theta}$
$= \frac{\frac{1 - \cos^2\theta}{\sin^2\theta}}{\cos\theta}$	Add.
$= \frac{\frac{\sin^2\theta}{\sin^2\theta}}{\cos\theta}$	$1 - \cos^2\theta = \sin^2\theta$
$= \frac{1}{\cos\theta}$	$\frac{\sin^2\theta}{\sin^2\theta} = 1$
$= \sec\theta$	$\frac{1}{\cos\theta} = \sec\theta$

DAILY INTERVENTION

Differentiated Instruction

Logical Have students work in groups of three. Ask each group to choose one of the identities in the Key Concept box on p. 777 and work together to demonstrate that it is true. Students should verify their results using the definitions of sine, cosine, and tangent in terms of the sides of a right triangle.

Example 3 Simplify and Use an Expression

BASEBALL **Refer to the application at the beginning of the lesson. Rewrite the equation in terms of tan θ.**

$$h = \left(\frac{-16}{v^2 \cos^2 \theta}\right)t^2 + \left(\frac{\sin \theta}{\cos \theta}\right)t + h_0 \qquad \text{Original equation}$$

$$= \frac{-16}{v^2}\left(\frac{1}{\cos^2 \theta}\right)t^2 + \left(\frac{\sin \theta}{\cos \theta}\right)t + h_0 \qquad \text{Factor.}$$

$$= \frac{-16}{v^2}\left(\frac{1}{\cos^2 \theta}\right)t^2 + (\tan \theta)t + h_0 \qquad \frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$= \frac{-16}{v^2}(\sec^2 \theta)t^2 + (\tan \theta)t + h_0 \qquad \text{Since } \frac{1}{\cos \theta} = \sec \theta, \frac{1}{\cos^2 \theta} = \sec^2 \theta.$$

$$= \frac{-16}{v^2}(1 + \tan^2 \theta)t^2 + (\tan \theta)t + h_0 \qquad \sec^2 \theta = 1 + \tan^2 \theta$$

Thus, $\left(\frac{-16}{v^2 \cos^2 \theta}\right)t^2 + \left(\frac{\sin \theta}{\cos \theta}\right)t + h_0 = \frac{-16}{v^2}(1 + \tan^2 \theta)t^2 + (\tan \theta)t + h_0$.

Check for Understanding

Concept Check

1–3. See margin.

1. **Describe** how you can determine the quadrant in which the terminal side of angle α lies if $\sin \alpha = -\frac{1}{4}$.
2. **Explain** why the Pythagorean identities are so named.
3. **OPEN ENDED** Explain what it means to simplify a trigonometric expression.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4–7	1
9–11	2
12	3

Find the value of each expression.

4. $\tan \theta$, if $\sin \theta = \frac{1}{2}$; $90° \le \theta < 180°$ $-\frac{\sqrt{3}}{3}$
5. $\csc \theta$, if $\cos \theta = -\frac{3}{5}$; $180° \le \theta < 270°$ $-\frac{5}{4}$
6. $\cos \theta$, if $\sin \theta = \frac{4}{5}$; $0° \le \theta < 90°$ $\frac{3}{5}$
7. $\sec \theta$, if $\tan \theta = -1$; $270° < \theta < 360°$ $\sqrt{2}$

Simplify each expression.

8. $\csc \theta \cos \theta \tan \theta$ **1**
9. $\sec^2 \theta - 1$ $\tan^2 \theta$
10. $\frac{\tan \theta}{\sin \theta}$ $\sec \theta$
11. $\sin \theta\,(1 + \cot^2 \theta)$ $\csc \theta$

Application

12. **PHYSICAL SCIENCE** When a person moves along a circular path, the body leans away from a vertical position. The nonnegative acute angle that the body makes with the vertical is called the *angle of inclination* and is represented by the equation $\tan \theta = \frac{v^2}{gR}$, where R is the radius of the circular path, v is the speed of the person in meters per second, and g is the acceleration due to gravity, 9.8 meters per second squared. Write an equivalent expression using $\sin \theta$ and $\cos \theta$. $\sin \theta = \cos \theta \frac{v^2}{gR}$

Practice and Apply

Practice and Apply

15. $-\sqrt{5}$
16. $2\sqrt{2}$

Find the value of each expression.

13. $\tan \theta$, if $\cot \theta = 2$; $0° \le \theta < 90°$ $\frac{1}{2}$
14. $\sin \theta$, if $\cos \theta = \frac{2}{3}$; $0° \le \theta < 90°$ $\frac{\sqrt{5}}{3}$
15. $\sec \theta$, if $\tan \theta = -2$; $90° < \theta < 180°$
16. $\tan \theta$, if $\sec \theta = -3$; $180° < \theta < 270°$

www.algebra2.com/extra_examples

In-Class Example

Power Point®

3 **BASEBALL** Refer to the application at the beginning of the lesson. Rewrite the equation in terms of sec θ.

$h = \frac{-16}{v^2}(\sec^2 \theta)t^2 + \left(\sqrt{\sec^2 \theta - 1}\right)t + h_0$

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 14.
- record the basic trigonometric identities from the Key Concept box on p. 777.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises…

Organization by Objective
- **Find Trigonometric Values:** 13–24
- **Simplify Expressions:** 25–36

Odd/Even Assignments

Exercises 13–36 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 13–35 odd, 37–41, 43–58

Average: 13–35 odd, 37–41, 43–58

Advanced: 14–36 even, 37–54 (optional: 55–58)

All: Practice Quiz 1 (1–5)

Answers

1. Sample answer: The sine function is negative in the third and fourth quadrants. Therefore, the terminal side of the angle must lie in one of those two quadrants.
2. Sample answer: Pythagorean identities are derived by applying the Pythagorean Theorem to trigonometric concepts.
3. Sample answer: Simplifying a trigonometric expression means writing the expression as a numerical value or in terms of a single trigonometric function, if possible.

Study Guide and Intervention, p. 849 (shown) and p. 850

Find Trigonometric Values A **trigonometric identity** is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

Basic Trigonometric Identities	**Quotient Identities**	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	
	Reciprocal Identities	$\csc\theta = \frac{1}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{1}{\tan\theta}$
	Pythagorean Identities	$\cos^2\theta + \sin^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$

Example **Find the value of cot θ if csc θ = $-\frac{11}{5}$; 180° < θ < 270°.**

$\cot^2\theta + 1 = \csc^2\theta$ Trigonometric identity

$\cot^2\theta + 1 = \left(-\frac{11}{5}\right)^2$ Substitute $-\frac{11}{5}$ for csc θ.

$\cot^2\theta + 1 = \frac{121}{25}$ Square $-\frac{11}{5}$.

$\cot^2\theta = \frac{96}{25}$ Subtract 1 from each side.

$\cot\theta = \pm\frac{4\sqrt{6}}{5}$ Take the square root of each side.

Since θ is in the third quadrant, cot θ is positive, Thus $\cot\theta = \frac{4\sqrt{6}}{5}$.

Exercises

Find the value of each expression.

1. tan θ, if cot θ = 4; 180° < θ < 270° $\frac{1}{4}$

2. csc θ, if cos θ = $\frac{\sqrt{3}}{2}$; 0° ≤ θ < 90° 2

3. cos θ, if sin θ = $\frac{3}{5}$; 0° ≤ θ < 90° $\frac{4}{5}$

4. sec θ, if sin θ = $\frac{1}{3}$; 0° ≤ θ < 90° $\frac{3\sqrt{2}}{4}$

5. cos θ, if tan θ = $-\frac{4}{3}$; 90° < θ < 180° $-\frac{3}{5}$

6. tan θ, if sin θ = $\frac{3}{7}$; 0° ≤ θ < 90° $\frac{3\sqrt{10}}{20}$

7. sec θ, if cos θ = $-\frac{7}{8}$; 90° < θ < 180° $-\frac{8}{7}$

8. sin θ, if cos θ = $\frac{6}{7}$; 270° ≤ θ < 360° $-\frac{\sqrt{13}}{7}$

9. cot θ, if csc θ = $\frac{12}{5}$; 90° < θ < 180° $-\frac{\sqrt{119}}{5}$

10. sin θ, if csc θ = $-\frac{9}{4}$; 270° < θ < 360° $-\frac{4}{9}$

Skills Practice, p. 851 and Practice, p. 852 (shown)

Find the value of each expression.

1. sin θ, if cos θ = $\frac{5}{13}$ and 0° ≤ θ < 90° $\frac{12}{13}$

2. sec θ, if sin θ = $-\frac{15}{17}$ and 180° < θ < 270° $-\frac{17}{8}$

3. cot θ, if cos θ = $\frac{3}{10}$ and 270° < θ < 360° $-\frac{3\sqrt{91}}{91}$

4. sin θ, if cot θ = $\frac{1}{2}$ and 0° ≤ θ < 90° $\frac{2\sqrt{5}}{5}$

5. cot θ, if csc θ = $-\frac{3}{2}$ and 180° < θ < 270° $\frac{\sqrt{5}}{2}$

6. sec θ, if csc θ = −8 and 270° < θ < 360° $\frac{8\sqrt{7}}{21}$

7. sec θ, if tan θ = 4 and 180° < θ < 270° $-\sqrt{17}$

8. sin θ, if tan θ = $-\frac{1}{2}$ and 270° < θ < 360° $-\frac{\sqrt{5}}{5}$

9. cot θ, if tan θ = $\frac{2}{5}$ and 0° ≤ θ < 90° $\frac{5}{2}$

10. cot θ, if cos θ = $\frac{1}{3}$ and 270° < θ < 360° $-\frac{\sqrt{2}}{4}$

Simplify each expression.

11. csc θ tan θ sec *x*

12. $\frac{\sin^2\theta}{\tan^2\theta}$ $\cos^2\theta$

13. $\sin^2\theta\cot^2\theta$ $\cos^2\theta$

14. $\cot^2\theta + 1$ $\csc^2\theta$

15. $\frac{\csc^2\theta - \cot^2\theta}{1 - \cos^2\theta}$ $\csc^2\theta$

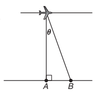
16. $\frac{\csc\theta - \sin\theta}{\cos\theta}$ cot θ

17. sin θ + cos θ cot θ csc θ

18. $\frac{\cos\theta}{1 - \sin\theta} - \frac{\cos\theta}{1 + \sin\theta}$ 2 tan θ

19. $\sec^2\theta\cos^2\theta - \tan^2\theta$ $\sec^2\theta$

20. AERIAL PHOTOGRAPHY The illustration shows a plane taking an aerial photograph of point *A*. Because the point is directly below the plane, there is no distortion in the image. For any point *B* not directly below the plane, however, the increase in distance creates distortion in the photograph. This is because as the distance from the camera to the point being photographed increases, the exposure of the film reduces by (sin θ)(csc θ − sin θ). Express (sin θ)(csc θ − sin θ) in terms of cos θ only. $\cos^2\theta$



21. TSUNAMIS The equation *y* = *a* sin θ*t* represents the height of the waves passing a buoy at a time *t* in seconds. Express *a* in terms of csc θ*t*. *a* = *y* csc θ*t*

Reading to Learn Mathematics, p. 853

ELL

Pre-Activity **How can trigonometry be used to model the path of a baseball?**

Read the introduction to Lesson 14-3 at the top of page 777 in your textbook.

Suppose that a baseball is hit from home plate with an initial velocity of 58 feet per second at an angle of 36° with the horizontal from an initial height of 5 feet. Show the equation that you would use to find the height of the ball 10 seconds after the ball is hit. (Show the formula with the appropriate numbers substituted, but do not do any calculations.)

$h = \left(\frac{-16}{58^2\cos^2 36°}\right)10^2 + \left(\frac{\sin 36°}{\cos 36°}\right)10 + 5$

Reading the Lesson

1. Match each expression from the list on the left with an expression from the list on the right that is equal to it for all values for which each expression is defined. (Some of the expressions from the list on the right may be used more than once or not at all.)

a. $\sec^2\theta - \tan^2\theta$ iii	**i.** $\frac{1}{\sin\theta}$
b. $\cot^2\theta + 1$ v	**ii.** tan θ
c. $\frac{\sin\theta}{\cos\theta}$ ii	**iii.** 1
d. $\sin^2\theta + \cos^2\theta$ iii	**iv.** sec θ
e. csc θ i	**v.** $\csc^2\theta$
f. $\frac{1}{\cos\theta}$ iv	**vi.** cot θ
g. $\frac{\cos\theta}{\sin\theta}$ vi	

2. Write an identity that you could use to find each of the indicated trigonometric values and tell whether that value is positive or negative. (Do not actually find the values.)

a. tan θ, if sin θ = $-\frac{4}{5}$ and 180° < θ < 270° $\tan\theta = \frac{\sin\theta}{\cos\theta}$; positive

b. sec θ, if tan θ = −3 and 90° < θ < 180° $\tan^2\theta + 1 = \sec^2\theta$; negative

Helping You Remember

3. A good way to remember something new is to relate it to something you already know. How can you use the unit circle definitions of the sine and cosine that you learned in Chapter 13 to help you remember the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$?
Sample answer: On a unit circle, $x = \cos\theta$ and $y = \sin\theta$. The equation of the unit circle is $x^2 + y^2 = 1$, so this is equivalent to the equation $\cos^2\theta + \sin^2\theta = 1$.

Homework Help

For Exercises	See Examples
13–24	1
25–36	2
37–43	3

Extra Practice

See page 860.

20. $-\frac{3\sqrt{5}}{5}$

22. $\frac{4}{5}$

Find the value of each expression.

17. csc θ, if cos θ = $-\frac{3}{5}$; 90° < θ < 180° $\frac{5}{4}$

18. cos θ, if sec θ = $\frac{5}{3}$; 270° < θ < 360° $\frac{3}{5}$

19. cos θ, if sin θ = $\frac{1}{2}$; 0° ≤ θ < 90° $\frac{\sqrt{3}}{2}$

20. csc θ, if cos θ = $-\frac{2}{3}$; 180° < θ < 270°

21. tan θ, if cos θ = $\frac{4}{5}$; 0° ≤ θ < 90° $\frac{3}{4}$

22. cos θ, if csc θ = $-\frac{5}{3}$; 270° < θ < 360°

23. sec θ, if sin θ = $\frac{3}{4}$; 90° < θ < 180° $-\frac{4\sqrt{7}}{7}$

24. sin θ, if tan θ = 4; 180° < θ < 270° $-\frac{4\sqrt{17}}{17}$

Simplify each expression.

25. cos θ csc θ cot θ

26. tan θ cot θ 1

27. sin θ cot θ cos θ

28. cos θ tan θ sin θ

29. $2(\csc^2\theta - \cot^2\theta)$ 2

30. $3(\tan^2\theta - \sec^2\theta)$ −3

31. $\frac{\cos\theta\csc\theta}{\tan\theta}$ $\cot^2\theta$

32. $\frac{\sin\theta\csc\theta}{\cot\theta}$ tan θ

33. $\frac{1 - \cos^2\theta}{\sin^2\theta}$ 1

34. $\frac{1 - \sin^2\theta}{\sin^2\theta}$ $\cot^2\theta$

35. $\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}$ $\csc^2\theta$

36. $\frac{\tan^2\theta - \sin^2\theta}{\tan^2\theta\sin^2\theta}$ 1

AMUSEMENT PARKS **For Exercises 37–39, use the following information.**
Suppose a child is riding on a merry-go-round and is seated on an outside horse. The diameter of the merry-go-round is 16 meters.

37. If the sine of the angle of inclination of the child is $\frac{1}{5}$, what is the angle of inclination made by the child? Refer to Exercise 12 for information on the angle of inclination. about 11.5°

38. What is the velocity of the merry-go-round? about 4 m/s

39. If the speed of the merry-go-round is 3.6 meters per second, what is the value of the angle of inclination of a rider? about 9.4°

More About. . .

Amusement Parks

The oldest operational carousel in the United States is the Flying Horse Carousel at Martha's Vineyard, Massachusetts.

Source: Martha's Vineyard Preservation Trust

LIGHTING **For Exercises 40 and 41, use the following information.**
The amount of light that a source provides to a surface is called the illuminance. The illuminance *E* in foot candles on a surface is related to the distance *R* in feet from the light source. The formula $\sec\theta = \frac{I}{ER^2}$, where *I* is the intensity of the light source measured in candles and θ is the angle between the light beam and a line perpendicular to the surface, can be used in situations in which lighting is important.

40. Solve the formula in terms of *E*. $E = \frac{I\cos\theta}{R^2}$

41. Is the equation in Exercise 40 equivalent to $R^2 = \frac{I\tan\theta\cos\theta}{E}$? Explain.
No; $R^2 = \frac{I\tan\theta\cos\theta}{E}$ simplifies to $E = \frac{I\sin\theta}{R^2}$.

ELECTRONICS **For Exercises 42 and 43, use the following information.**
When an alternating current of frequency *f* and a peak current *I* pass through a resistance *R*, then the power delivered to the resistance at time *t* seconds is $P = I^2R - I^2R\cos^2(2ft\pi)$.

42. Write an expression for the power in terms of $\sin^2 2ft\pi$. $P = I^2R\sin^2 2\pi ft$

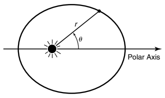
43. Write an expression for the power in terms of $\tan^2 2ft\pi$. $P = I^2R - \frac{I^2R}{1 + \tan^2 2\pi ft}$

44. CRITICAL THINKING If $\tan\beta = \frac{3}{4}$, find $\frac{\sin\beta\sec\beta}{\cot\beta}$. $\frac{9}{16}$

Enrichment, p. 854

Planetary Orbits

The orbit of a planet around the sun is an ellipse with the sun at one focus. Let the pole of a polar coordinate system be that focus and the polar axis be toward the other focus. The polar equation of an ellipse is



$r = \frac{2ep}{1 - e\cos\theta}$. Since $2p = \frac{b^2}{c}$ and $b^2 = a^2 - c^2$,

$2p = \frac{a^2 - c^2}{c} = \frac{a^2}{c}\left(1 - \frac{c^2}{a^2}\right)$. Because $e = \frac{c}{a}$,

$2p = a\left(\frac{a}{c}\right)\left(1 - \left(\frac{c}{a}\right)^2\right) = a\left(\frac{1}{e}\right)(1 - e^2)$.

Therefore $2ep = a(1 - e^2)$. Substituting into the polar equation of an ellipse yields an equation that is useful for finding distances from the planet to the sun.

$r = \frac{a(1 - e^2)}{1 - e\cos\theta}$

45. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

How can trigonometry be used to model the path of a baseball?

Include the following in your answer:

- an explanation of why the equation at the beginning of the lesson is the same as $y = \frac{-16 \sec^2 \theta}{v^2}x^2 + (\tan \theta)x + h_0$, and
- examples of how you might use this equation for other situations.

Standardized Test Practice

46. If $\sin x = m$ and $0 < x < 90°$, then $\tan x =$ **B**

Ⓐ $\frac{1}{m^2}$. Ⓑ $\frac{m}{\sqrt{1 - m^2}}$. Ⓒ $\frac{1 - m^2}{m}$. Ⓓ $\frac{m}{1 - m^2}$.

47. $\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} =$ **A**

Ⓐ $2 \sec^2 x$ Ⓑ $-\sec^2 x$ Ⓒ $2 \csc^2 x$ Ⓓ $-\csc^2 x$

Maintain Your Skills

Mixed Review

48–49. See margin for graphs.

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function. *(Lesson 14-2)*

48. $y = \sin \theta - 1$ **−1; $y = -1$; 1; 360°**

49. $y = \tan \theta + 12$ **12; $y = 12$; no amplitude; 180°**

Find the amplitude, if it exists, and period of each function. Then graph each function. *(Lesson 14-1)* **50–52. See pp. 811A–811N.**

50. $y = \csc 2\theta$

51. $y = \cos 3\theta$

52. $y = \frac{1}{3} \cot 5\theta$

53. Find the sum of a geometric series for which $a_1 = 48$, $a_n = 3$, and $r = \frac{1}{2}$. *(Lesson 11-4)* **93**

54. Write an equation of a parabola with focus at (11, −1) and whose directrix is $y = 2$. *(Lesson 8-2)* $y = -\frac{1}{6}(x - 11)^2 + \frac{1}{2}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Name the property illustrated by each statement.** *(To review **properties of equality**, see Lesson 1-3.)*

55. If $4 + 8 = 12$, then $12 = 4 + 8$. **Symmetric (=)**

56. If $7 + s = 21$, then $s = 14$. **Subt. (=)**

57. If $4x = 16$, then $12x = 48$. **Multiplication (=)**

58. If $q + (8 + 5) = 32$, then $q + 13 = 32$. **Substitution (=)**

Practice Quiz 1

Lessons 14-1 through 14-3

1. Find the amplitude and period of $y = \frac{3}{4} \sin \frac{1}{2}\theta$. Then graph the function. *(Lesson 14-1)* **$\frac{3}{4}$, 720° or 4π** **1–2. See pp. 811A–811N for graphs.**
2. State the vertical shift, amplitude, period, and phase shift for $y = 2 \cos \left[\frac{1}{4}\left(\theta - \frac{\pi}{4}\right)\right] - 5$. Then graph the function. *(Lesson 14-2)* **−5, 2, 8π, $\frac{\pi}{4}$**

Find the value of each expression. *(Lesson 14-3)*

3. $\cos \theta$, if $\sin \theta = \frac{4}{5}$; $90° < \theta < 180°$ $-\frac{3}{5}$
4. $\csc \theta$, if $\cot \theta = -\frac{2}{3}$; $270° < \theta < 360°$ $-\frac{\sqrt{13}}{3}$
5. $\sec \theta$, if $\tan \theta = \frac{1}{2}$; $0° < \theta < 90°$ $\frac{\sqrt{5}}{2}$

www.algebra2.com/self_check_quiz

4 Assess

Open-Ended Assessment

Speaking Have students explain how to rewrite and simplify expressions using trigonometric identities, using an example they worked during the lesson.

Getting Ready for Lesson 14-4

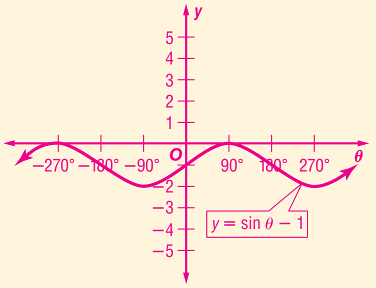
PREREQUISITE SKILL Students will verify trigonometric identities in Lesson 14-4 by rewriting them using the properties of equality. Use Exercises 55–58 to determine your students' familiarity with the properties of equality.

Assessment Options

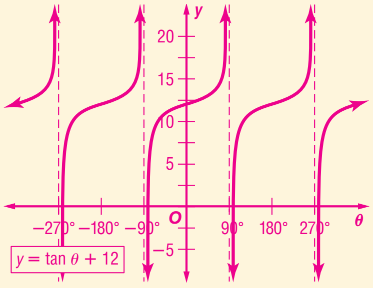
Practice Quiz 1 The quiz provides students with a brief review of the concepts and skills in Lessons 14-1 through 14-3. Lesson numbers are given to the right of the exercises or instruction lines so students can review concepts not yet mastered.

Answers

48.



49.



Answer

45. Sample answer: You can use equations to find the height and the horizontal distance of a baseball after it has been hit. The equations involve using the initial angle the ball makes with the ground with the sine function. Answers should include the following information.

- **Both equations are quadratic in nature with a leading negative coefficient. Thus, both are inverted parabolas which model the path of a baseball.**
- **model rockets, hitting a golf ball, kicking a rock**

14-4 Lesson Notes

1 Focus

5-Minute Check Transparency 14-4 Use as a quiz or review of Lesson 14-3.

Mathematical Background notes are available for this lesson on p. 760D.

How can you verify trigonometric identities?

Ask students:

- Can you give an example of two functions in x whose values are equal for some values of x but not all? **Sample answer: $y = x$ and $y = |x|$ are the same for $x \geq 0$ but not for $x < 0$.**
- Why is it not sufficient to show that two functions have equal values for specific values of x when trying to justify that two functions are equivalent? **The functions might have equal values for some values of x but not others.**

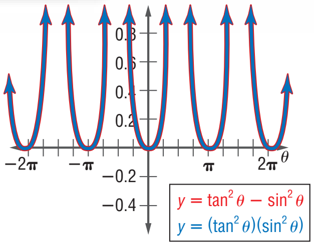
14-4 Verifying Trigonometric Identities

What You'll Learn

- Verify trigonometric identities by transforming one side of an equation into the form of the other side.
- Verify trigonometric identities by transforming each side of the equation into the same form.

How can you verify trigonometric identities?

Examine the graphs of $y = \tan^2\theta - \sin^2\theta$ and $y = \tan^2\theta\sin^2\theta$. Recall that when the graphs of two functions coincide, the functions are equivalent. However, the graphs only show a limited range of solutions. It is not sufficient to show some values of θ and conclude that the statement is true for all values of θ. In order to show that the equation $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$ for all values of θ, you must consider the general case.



TRANSFORM ONE SIDE OF AN EQUATION

You can use the basic trigonometric identities along with the definitions of the trigonometric functions to verify identities. For example, if you wish to show that $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$ is an identity, you need to show that it is true for all values of θ.

Verifying an identity is like checking the solution of an equation. You must simplify one or both sides of an equation *separately* until they are the same. In many cases, it is easier to work with only one side of an equation. You may choose either side, but it is often easier to begin with the more complicated side of the equation. Transform that expression into the form of the simpler side.

Study Tip

Common Misconception

You cannot perform operations to the quantities from each side of an unverified identity as you do with equations. Until an identity is verified it is not considered an equation, so the properties of equality do not apply.

Example 1 Transform One Side of an Equation

Verify that $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$ is an identity.

Transform the left side.

$$\tan^2\theta - \sin^2\theta \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \text{Original equation}$$

$$\frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$$

$$\frac{\sin^2\theta}{\cos^2\theta} - \frac{\sin^2\theta\cos^2\theta}{\cos^2\theta} \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \text{Rewrite using the LCD, } \cos^2\theta.$$

$$\frac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta} \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \text{Subtract.}$$

$$\frac{\sin^2\theta\,(1 - \cos^2\theta)}{\cos^2\theta} \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \text{Factor.}$$

$$\frac{\sin^2\theta\sin^2\theta}{\cos^2\theta} \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad 1 - \cos^2\theta = \sin^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\sin^2\theta}{1} \stackrel{?}{=} \tan^2\theta\sin^2\theta \quad \frac{ab}{c} = \frac{a}{c}\cdot\frac{b}{1}$$

$$\tan^2\theta\sin^2\theta = \tan^2\theta\sin^2\theta \quad \frac{\sin^2\theta}{\cos^2\theta} = \tan\theta$$

Resource Manager

Workbook and Reproducible Masters

Chapter 14 Resource Masters

- Study Guide and Intervention, pp. 855–856
- Skills Practice, p. 857
- Practice, p. 858
- Reading to Learn Mathematics, p. 859
- Enrichment, p. 860
- Assessment, pp. 893, 895

Transparencies

5-Minute Check Transparency 14-4
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lesson 28
Interactive Chalkboard

Standardized Test Practice

Example 2 Find an Equivalent Expression

Multiple-Choice Test Item

$\sin\theta\left(\dfrac{1}{\sin\theta}-\dfrac{\cos\theta}{\cot\theta}\right)=$

Ⓐ $\cos\theta$ Ⓑ $\sin\theta$ Ⓒ $\cos^2\theta$ Ⓓ $\sin^2\theta$

Read the Test Item

Find an expression that is equal to the given expression.

Solve the Test Item

Write a trigonometric identity by using the basic trigonometric identities and the definitions of trigonometric functions to transform the given expression to match one of the choices.

$$\sin\theta\left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\cot\theta}\right)=\sin\theta\left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\frac{\cos\theta}{\sin\theta}}\right) \qquad \cot\theta=\frac{\cos\theta}{\sin\theta}$$

$$=\sin\theta\left(\frac{1}{\sin\theta}-\frac{\cos\theta\sin\theta}{\cos\theta}\right) \qquad \text{Simplify.}$$

$$=\sin\theta\left(\frac{1}{\sin\theta}-\sin\theta\right) \qquad \text{Simplify.}$$

$$=1-\sin^2\theta \qquad \text{Distributive property.}$$

$$=\cos^2\theta \qquad 1-\sin^2\theta=\cos^2\theta$$

Since $\sin\theta\left(\dfrac{1}{\sin\theta}-\dfrac{\cos\theta}{\cot\theta}\right)=\cos^2\theta$, the answer is C.

The Princeton Review

Test-Taking Tip

Verify your answer by choosing values for θ. Then evaluate the original expression and compare to your answer choice.

TRANSFORM BOTH SIDES OF AN EQUATION Sometimes it is easier to transform both sides of an equation separately into a common form. The following suggestions may be helpful as you verify trigonometric identities.

- Substitute one or more basic trigonometric identities to simplify an expression.
- Factor or multiply to simplify an expression.
- Multiply both the numerator and denominator by the same trigonometric expression.
- Write both sides of the identity in terms of sine and cosine only. Then simplify each side as much as possible.

Example 3 Verify by Transforming Both Sides

Verify that $\sec^2\theta-\tan^2\theta=\tan\theta\cot\theta$ is an identity.

$$\sec^2\theta-\tan^2\theta\stackrel{?}{=}\tan\theta\cot\theta \qquad \text{Original equation}$$

$$\frac{1}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}\stackrel{?}{=}\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} \qquad \text{Express all terms using sine and cosine.}$$

$$\frac{1-\sin^2\theta}{\cos^2\theta}\stackrel{?}{=}1 \qquad \text{Subtract on the left. Multiply on the right.}$$

$$\frac{\cos^2\theta}{\cos^2\theta}\stackrel{?}{=}1 \qquad 1-\sin^2\theta=\cos^2\theta$$

$$1=1 \qquad \text{Simplify the left side.}$$

2 Teach

TRANSFORM ONE SIDE OF AN EQUATION

In-Class Examples Power Point®

1 Verify that $\csc\theta\cos\theta\tan\theta=1$ is an identity.

$$\csc\theta\cos\theta\tan\theta\stackrel{?}{=}1$$

$$\frac{1}{\sin\theta}\cdot\cos\theta\cdot\frac{\sin\theta}{\cos\theta}\stackrel{?}{=}1$$

$$1=1$$

2 $\dfrac{\csc\theta}{\cos\theta}-\tan\theta=$ **A**

A $\cot\theta$ **B** $\dfrac{1-\sin\theta}{\cos^2\theta}$

C 0 **D** $\cos^2\theta$

TRANSFORM BOTH SIDES OF AN EQUATION

In-Class Example Power Point®

3 Verify that $\csc\theta+\sec\theta=\dfrac{1+\cot\theta}{\cos\theta}$ is an identity.

$$\csc\theta+\sec\theta\stackrel{?}{=}\frac{1+\cot\theta}{\cos\theta}$$

$$\frac{1}{\sin\theta}+\frac{1}{\cos\theta}\stackrel{?}{=}\frac{1+\frac{\cos\theta}{\sin\theta}}{\cos\theta}$$

$$\frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}\stackrel{?}{=}\frac{\sin\theta\left(1+\frac{\cos\theta}{\sin\theta}\right)}{\sin\theta\cos\theta}$$

$$\frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}=\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}$$

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 14.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

Standardized Test Practice

Example 2 Urge students to read slowly and carefully so they do not mistake cot for cos or csc for cos. In addition, students may find the correct answer, $\cos^2\theta$, but choose choice A because they did not notice the difference between $\cos^2\theta$ and $\cos\theta$.

About the Exercises…

Organization by Objective

- **Transform One Side of an Equation:** 11–24, 26–30
- **Transform Both Sides of an Equation:** 25

Odd/Even Assignments

Exercises 11–30 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Alert! Exercises 37–42 require a graphing calculator.

Assignment Guide

Basic: 11–31 odd, 33–36, 43–54

Average: 11–31 odd, 33–36, 43–54 (optional: 37–42)

Advanced: 12–30 even, 31–50 (optional: 51–54)

4 Assess

Open-Ended Assessment

Speaking Have students explain some of the techniques they have seen or used to verify trigonometric identities in this lesson.

Getting Ready for Lesson 14-5

PREREQUISITE SKILL Students will simplify radical expressions in the process of using the Sum and Difference of Angles Formulas in Lesson 14-5. Use Exercises 51–54 to determine your students' familiarity with simplifying radical expressions.

Assessment Options

Quiz (Lessons 14-3 and 14-4) is available on p. 893 of the *Chapter 14 Resource Masters*.

Mid-Chapter Test (Lessons 14-1 through 14-4) is available on p. 895 of the *Chapter 14 Resource Masters*.

Check for Understanding

Concept Check

1–3. See pp. 811A–811N.

1. **Explain** the steps used to verify the identity $\sin\theta\tan\theta = \sec\theta - \cos\theta$.
2. **Describe** the various methods you can use to show that two trigonometric expressions form an identity.
3. **OPEN ENDED** Write a trigonometric equation that is not an identity. Explain how you know it is not an identity.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4, 6–10	1, 2
5	3

Verify that each of the following is an identity. 4–9. See pp. 811A–811N.

4. $\tan\theta(\cot\theta + \tan\theta) = \sec^2\theta$
5. $\tan^2\theta\cos^2\theta = 1 - \cos^2\theta$
6. $\dfrac{\cos^2\theta}{1 - \sin\theta} = 1 + \sin\theta$
7. $\dfrac{1 + \tan^2\theta}{\csc^2\theta} = \tan^2\theta$
8. $\dfrac{\sin\theta}{\sec\theta} = \dfrac{1}{\tan\theta + \cot\theta}$
9. $\dfrac{\sec\theta + 1}{\tan\theta} = \dfrac{\tan\theta}{\sec\theta - 1}$

Standardized Test Practice

10. Which expression is equivalent to $\dfrac{\sec\theta + \csc\theta}{1 + \tan\theta}$? D

Ⓐ $\sin\theta$ Ⓑ $\cos\theta$ Ⓒ $\tan\theta$ Ⓓ $\csc\theta$

Practice and Apply

Homework Help

For Exercises	See Examples
11–24, 26–32	1, 2
25	3

Extra Practice

See page 860.

Verify that each of the following is an identity. 11–28. See pp. 811A–811N.

11. $\cos^2\theta + \tan^2\theta\cos^2\theta = 1$
12. $\cot\theta(\cot\theta + \tan\theta) = \csc^2\theta$
13. $1 + \sec^2\theta\sin^2\theta = \sec^2\theta$
14. $\sin\theta\sec\theta\cot\theta = 1$
15. $\dfrac{1 - \cos\theta}{1 + \cos\theta} = (\csc\theta - \cot\theta)^2$
16. $\dfrac{1 - 2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta - \cot\theta$
17. $\cot\theta\csc\theta = \dfrac{\cot\theta + \csc\theta}{\sin\theta + \tan\theta}$
18. $\sin\theta + \cos\theta = \dfrac{1 + \tan\theta}{\sec\theta}$
19. $\dfrac{\sec\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta} = \cot\theta$
20. $\dfrac{\sin\theta}{1 - \cos\theta} + \dfrac{1 - \cos\theta}{\sin\theta} = 2\csc\theta$
21. $\dfrac{1 + \sin\theta}{\sin\theta} = \dfrac{\cot^2\theta}{\csc\theta - 1}$
22. $\dfrac{1 + \tan\theta}{1 + \cot\theta} = \dfrac{\sin\theta}{\cos\theta}$
23. $\dfrac{1}{\sec^2\theta} + \dfrac{1}{\csc^2\theta} = 1$
24. $1 + \dfrac{1}{\cos\theta} = \dfrac{\tan^2\theta}{\sec\theta - 1}$
25. $1 - \tan^4\theta = 2\sec^2\theta - \sec^4\theta$
26. $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$
27. $\dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$
28. $\dfrac{\cos\theta}{1 + \sin\theta} + \dfrac{\cos\theta}{1 - \sin\theta} = 2\sec\theta$
29. Verify that $\tan\theta\sin\theta\cos\theta\csc^2\theta = 1$ is an identity. See pp. 811A–811N.
30. Show that $1 + \cos\theta$ and $\dfrac{\sin^2\theta}{1 - \cos\theta}$ form an identity. See pp. 811A–811N.

PHYSICS **For Exercises 31 and 32, use the following information.**
If an object is propelled from ground level, the maximum height that it reaches is given by $h = \dfrac{v^2\sin^2\theta}{2g}$, where θ is the angle between the ground and the initial path of the object, v is the object's initial velocity, and g is the acceleration due to gravity, 9.8 meters per second squared.

31. Verify the identity $\dfrac{v^2\sin^2\theta}{2g} = \dfrac{v^2\tan^2\theta}{2g\sec^2\theta}$. See pp. 811A–811N.
32. A model rocket is launched with an initial velocity of 110 meters per second at an angle of 80° with the ground. Find the maximum height of the rocket. 598.7 m

DAILY INTERVENTION

Differentiated Instruction

Interpersonal Have groups or pairs of students work together to verify some of the identities in Exercises 4–9. Have students record the techniques they found helpful. Ask students to compare their list of techniques to the list of suggestions given above Example 3 on p. 783.

33. **CRITICAL THINKING** Present a logical argument for why the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ is true when $0 \le x \le 1$. **See margin.**

34. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See pp. 811A–811N.**

How can you verify trigonometric identities?

Include the following in your answer:

- an explanation of why you cannot perform operations to each side of an unverified identity,
- an explanation of how you can tell if two expressions are equivalent, and
- an explanation of why you cannot use the graphs of two equations to verify an identity.

Standardized Test Practice

35. Which of the following is not equivalent to $\cos \theta$? **D**

Ⓐ $\frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta}$ Ⓑ $\frac{1 - \sin^2 \theta}{\cos \theta}$ Ⓒ $\cot \theta \sin \theta$ Ⓓ $\tan \theta \csc \theta$

36. Which of the following is equivalent to $\sin \theta + \cot \theta \cos \theta$? **B**

Ⓐ $2 \sin \theta$ Ⓑ $\frac{1}{\sin \theta}$ Ⓒ $\cos^2 \theta$ Ⓓ $\frac{\sin \theta + \cos \theta}{\sin^2 \theta}$

Graphing Calculator

37–42. See pp. 811A–811N for graphs.

VERIFYING TRIGONOMETRIC IDENTITIES You can determine whether or not an equation may be a trigonometric identity by graphing the expressions on either side of the equals sign as two separate functions. If the graphs do not match, then the equation is not an identity. If the two graphs do coincide, the equation *might* be an identity. The equation has to be verified algebraically to ensure that it is an identity.

Determine whether each of the following *may be* or *is not* an identity.

37. $\cot x + \tan x = \csc x \cot x$ **is not**

38. $\sec^2 x - 1 = \sin^2 x \sec^2 x$ **may be**

39. $(1 + \sin x)(1 - \sin x) = \cos^2 x$ **may be**

40. $\frac{1}{\sec x \tan x} = \csc x - \sin x$ **may be**

41. $\frac{\sec^2 x}{\tan x} = \sec x \csc x$ **may be**

42. $\frac{1}{\sec x} + \frac{1}{\csc x} = 1$ **is not**

Maintain Your Skills

Mixed Review **Find the value of each expression.** *(Lesson 14-3)*

43. $\sec \theta$, if $\tan \theta = \frac{1}{2}$; $0° < \theta < 90°$ $\frac{\sqrt{5}}{2}$

44. $\cos \theta$, if $\sin \theta = -\frac{2}{3}$; $180° < \theta < 270°$

45. $\csc \theta$, if $\cot \theta = -\frac{7}{12}$; $90° < \theta < 180°$

46. $\sin \theta$, if $\cos \theta = \frac{3}{4}$; $270° < \theta < 360°$

44. $-\frac{\sqrt{5}}{3}$

45. $\frac{\sqrt{193}}{12}$

46. $-\frac{\sqrt{7}}{4}$

49. 3; 2π; $-\frac{\pi}{2}$

53. $\frac{\sqrt{6} + 2\sqrt{2}}{4}$

54. $\frac{2 - \sqrt{3}}{4}$

State the amplitude, period, and phase shift of each function. Then graph each function. *(Lesson 14-2)* **47–49. See pp. 811A–811N for graphs.**

47. $y = \cos (\theta - 30°)$ **1; 360°; 30°**

48. $y = \sin (\theta - 45°)$ **1; 360°; 45°**

49. $y = 3 \cos \left(\theta + \frac{\pi}{2}\right)$

50. What is the probability that an event occurs if the odds of the event occurring are 5:1? *(Lesson 12-4)* $\frac{5}{6}$

Getting Ready for the Next Lesson **PREREQUISITE SKILL** **Simplify each expression.**
*(To review **simplifying radical expressions**, see Lesson 5-6.)*

51. $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$ $\frac{\sqrt{6}}{4}$

52. $\frac{1}{2} \cdot \frac{\sqrt{2}}{2}$ $\frac{\sqrt{2}}{4}$

53. $\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2}$

54. $\frac{1}{2} - \frac{\sqrt{3}}{4}$

www.algebra2.com/self_check_quiz

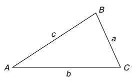
Answer

33. Sample answer: Consider a right triangle *ABC* with right angle at *C*. If an angle, say *A*, has a sine of *x*, then angle *B* must have a cosine of *x*. Since *A* and *B* are both in a right triangle and neither is the right angle, their sum must be $\frac{\pi}{2}$.

Enrichment, p. 860

Heron's Formula

Heron's formula can be used to find the area of a triangle if you know the lengths of the three sides. Consider any triangle *ABC*. Let *K* represent the area of △*ABC*. Then



$$K = \frac{1}{2}bc \sin A$$

$$K^2 = \frac{b^2c^2 \sin^2 A}{4}$$ Square both sides.

$$= \frac{b^2c^2(1 - \cos^2 A)}{4}$$

$$= \frac{b^2c^2(1 + \cos A)(1 - \cos A)}{4}$$

$$= \frac{b^2c^2}{4}\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)$$ Use the law of cosines.

$$= \frac{b + c + a}{2} \cdot \frac{b + c - a}{2} \cdot \frac{a + b - c}{2} \cdot \frac{a - b + c}{2}$$ Simplify.

Study Guide and Intervention, p. 855 (shown) and p. 856

Transform One Side of an Equation Use the basic trigonometric identities along with the definitions of the trigonometric functions to verify trigonometric identities. Often it is easier to begin with the more complicated side of the equation and transform that expression into the form of the simpler side.

Example **Verify that each of the following is an identity.**

a. $\frac{\sin \theta}{\cot \theta} - \sec \theta = -\cos \theta$

Transform the left side.

$$\frac{\sin \theta}{\cot \theta} - \sec \theta \stackrel{?}{=} -\cos \theta$$
$$\frac{\sin \theta}{\frac{\cos \theta}{\sin \theta}} - \frac{1}{\cos \theta} \stackrel{?}{=} -\cos \theta$$
$$\frac{\sin^2 \theta}{\cos \theta} - \frac{1}{\cos \theta} \stackrel{?}{=} -\cos \theta$$
$$\frac{\sin^2 - 1}{\cos \theta} \stackrel{?}{=} -\cos \theta$$
$$\frac{-\cos^2 \theta}{\cos \theta} \stackrel{?}{=} -\cos \theta$$
$$-\cos \theta = -\cos \theta$$

b. $\frac{\tan \theta}{\csc \theta} + \cos \theta = \sec \theta$

Transform the left side.

$$\frac{\tan \theta}{\csc \theta} + \cos \theta \stackrel{?}{=} \sec \theta$$
$$\frac{\frac{\sin \theta}{\cos \theta}}{\frac{1}{\sin \theta}} + \cos \theta \stackrel{?}{=} \sec \theta$$
$$\frac{\sin^2 \theta}{\cos \theta} + \cos \theta \stackrel{?}{=} \sec \theta$$
$$\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta} \stackrel{?}{=} \sec \theta$$
$$\frac{1}{\cos \theta} \stackrel{?}{=} \sec \theta$$
$$\sec \theta = \sec \theta$$

Exercises

Verify that each of the following is an identity.

1. $1 + \csc^2 \theta \cdot \cos^2 \theta = \csc^2 \theta$

$$1 + \frac{1}{\sin^2 \theta} \cdot \cos^2 \theta \stackrel{?}{=} \csc^2 \theta$$
$$\frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} \stackrel{?}{=} \csc^2 \theta$$
$$\frac{1}{\sin^2 \theta} \stackrel{?}{=} \csc^2 \theta$$
$$\csc^2 \theta = \csc^2 \theta$$

2. $\frac{\sin \theta}{1 - \cos \theta} - \frac{\cot \theta}{1 + \cos \theta} = \frac{1 - \cos^3 \theta}{\sin^3 \theta}$

$$\frac{\sin \theta(1 + \cos \theta) - \frac{\cos \theta}{\sin \theta}(1 - \cos \theta)}{(1 - \cos \theta)(1 + \cos \theta)} \stackrel{?}{=} \frac{1 - \cos^3 \theta}{\sin^3 \theta}$$
$$\frac{\sin \theta + \sin \theta \cdot \cos \theta - \frac{\cos \theta}{\sin \theta} + \frac{\cos^2 \theta}{\sin \theta}}{1 - \cos^2 \theta} \stackrel{?}{=} \frac{1 - \cos^3 \theta}{\sin^3 \theta}$$
$$\frac{\frac{\sin^2 \theta + \sin^2 \theta \cos \theta - \cos \theta + \cos^2 \theta}{\sin \theta}}{\sin^2 \theta} \stackrel{?}{=} \frac{1 - \cos^3 \theta}{\sin^3 \theta}$$
$$\frac{\sin^2 \theta + \cos^2 \theta + \cos \theta(\sin^2 \theta - 1)}{\sin^3 \theta} \stackrel{?}{=} \frac{1 - \cos^3 \theta}{\sin^3 \theta}$$
$$\frac{1 + \cos \theta(-\cos^2 \theta)}{\sin^3 \theta} \stackrel{?}{=} \frac{1 - \cos^3 \theta}{\sin^3 \theta}$$
$$\frac{1 - \cos^3 \theta}{\sin^3 \theta} = \frac{1 - \cos^3 \theta}{\sin^3 \theta}$$

Skills Practice, p. 857 and Practice, p. 858 (shown)

Verify that each of the following is an identity.

1. $\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} = \sec^2 \theta$

$$\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \sec^2 \theta$$
$$\frac{1}{\cos^2 \theta} \stackrel{?}{=} \sec^2 \theta$$
$$\sec^2 \theta = \sec^2 \theta$$

2. $\frac{\cos^2 \theta}{1 - \sin^2 \theta} = 1$

$$\frac{\cos^2 \theta}{1 - \sin^2 \theta} \stackrel{?}{=} 1$$
$$\frac{\cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} 1$$
$$1 = 1$$

3. $(1 + \sin \theta)(1 - \sin \theta) = \cos^2 \theta$

$$(1 + \sin \theta)(1 - \sin \theta) \stackrel{?}{=} \cos^2 \theta$$
$$1 - \sin^2 \theta \stackrel{?}{=} \cos^2 \theta$$
$$\cos^2 \theta = \cos^2 \theta$$

4. $\tan^4 \theta + 2 \tan^2 \theta + 1 = \sec^4 \theta$

$$\tan^4 \theta + 2 \tan^2 \theta + 1 \stackrel{?}{=} \sec^4 \theta$$
$$(\tan^2 \theta + 1)^2 \stackrel{?}{=} \sec^4 \theta$$
$$(\sec^2 \theta)^2 \stackrel{?}{=} \sec^4 \theta$$
$$\sec^4 \theta = \sec^4 \theta$$

5. $\cos^2 \theta \cot^2 \theta = \cot^2 \theta - \cos^2 \theta$

$$\cos^2 \theta \cot^2 \theta \stackrel{?}{=} \cot^2 \theta - \cos^2 \theta$$
$$\cos^2 \theta \cot^2 \theta \stackrel{?}{=} \frac{\cos^2 \theta}{\sin^2 \theta} - \cos^2 \theta$$
$$\cos^2 \theta \cot^2 \theta \stackrel{?}{=} \frac{\cos^2 \theta - \cos^2 \theta \sin^2 \theta}{\sin^2 \theta}$$
$$\cos^2 \theta \cot^2 \theta \stackrel{?}{=} \frac{(\cos^2 \theta)(1 - \sin^2 \theta)}{\sin^2 \theta}$$
$$\cos^2 \theta \cot^2 \theta \stackrel{?}{=} \frac{\cos^2 \theta}{1} \cdot \frac{\cos^2 \theta}{\sin^2 \theta}$$
$$\cos^2 \theta \cot^2 \theta = \cos^2 \theta \cot^2 \theta$$

6. $(\sin^2 \theta)(\csc^2 \theta + \sec^2 \theta) = \sec^2 \theta$

$$(\sin^2 \theta)(\csc^2 \theta + \sec^2 \theta) \stackrel{?}{=} \sec^2 \theta$$
$$(\sin^2 \theta)\left(\frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta}\right) \stackrel{?}{=} \sec^2 \theta$$
$$1 + \frac{\sin^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \sec^2 \theta$$
$$1 + \tan^2 \theta \stackrel{?}{=} \sec^2 \theta$$
$$\sec^2 \theta = \sec^2 \theta$$

7. **PROJECTILES** The square of the initial velocity of an object launched from the ground is $v^2 = \frac{2gh}{\sin^2 \theta}$, where θ is the angle between the ground and the initial path, h is the maximum height reached, and g is the acceleration due to gravity. Verify the identity $\frac{2gh}{\sin^2 \theta} = \frac{2gh \sec^2 \theta}{\sec^2 \theta - 1}$.

$$\frac{2gh}{\sin^2 \theta} = \frac{2gh}{1 - \cos^2 \theta} = \frac{2gh}{1 - \frac{1}{\sec^2 \theta}} = \frac{2gh}{\frac{\sec^2 \theta - 1}{\sec^2 \theta}} = \frac{2gh \sec^2 \theta}{\sec^2 \theta - 1}$$

8. **LIGHT** The intensity of a light source measured in candles is given by $I = ER^2 \sec \theta$, where E is the illuminance in foot candles on a surface, R is the distance in feet from the light source, and θ is the angle between the light beam and a line perpendicular to the surface. Verify the identity $ER^2(1 + \tan^2 \theta) \cos \theta = ER^2 \sec \theta$.

$$ER^2(1 + \tan^2 \theta) \cos \theta = ER^2 \sec^2 \theta \cos \theta = ER^2 \sec^2 \theta \cdot \frac{1}{\sec \theta} = ER^2 \sec \theta$$

Reading to Learn Mathematics, p. 859 ELL

Pre-Activity **How can you verify trigonometric identities?**

Read the introduction to Lesson 14-4 at the top of page 782 in your textbook.

For $\theta = -\pi$, 0, or π, $\sin \theta = \sin 2\theta$. Does this mean that $\sin \theta = \sin 2\theta$ is an identity? Explain your reasoning. **Sample answer: No; an identity is an equation that is true for *all* values of a variable for which the functions involved are defined, not just *some* values. If $\theta = \frac{\pi}{4}$, $\sin \theta = \frac{\sqrt{2}}{2}$, and $\sin 2\theta = 1$.**

Reading the Lesson

1. Determine whether each equation is an *identity* or *not an identity*.
   a. $\frac{1}{\sin^2 \theta} - \frac{1}{\tan^2 \theta} = 1$ **identity**
   b. $\frac{\cos \theta}{\sin \theta \tan \theta}$ **not an identity**
   c. $\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \cos \theta \sin \theta$ **not an identity**
   d. $\cos^2 \theta (\tan^2 \theta + 1) = 1$ **identity**
   e. $\frac{\sin^2 \theta}{\cos^2 \theta} + \sin \theta \csc \theta = \sec^2 \theta$ **identity**
   f. $\frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} = 2 \cos^2 \theta$ **not an identity**
   g. $\tan^2 \theta \cos^2 \theta = \frac{1}{\csc^2 \theta}$ **identity**
   h. $\frac{\sin \theta}{\sec \theta} = \frac{1}{\tan \theta} + \frac{1}{\cot \theta}$ **not an identity**

2. Which of the following is *not* permitted when verifying an identity? **B**
   A. simplifying one side of the identity to match the other side
   B. cross multiplying if the identity is a proportion
   C. simplifying each side of the identity separately to get the same expression on both sides

Helping You Remember

3. Many students have trouble knowing where to start in verifying a trigonometric identity. What is a simple rule that you can remember that you can always use if you don't see a quicker approach? **Sample answer: Write both sides in terms of sines and cosines. Then simplify each side as much as possible.**

14-5 Lesson Notes

1 Focus

5-Minute Check Transparency 14-5 Use as a quiz or review of Lesson 14-4.

Mathematical Background notes are available for this lesson on p. 760D.

How are the sum and difference formulas used to describe communication interference?

Ask students:

- Where else have you heard the term *interference*? **Sample answer: television and radio**
- At its peak, how does the amplitude of the combined wave compare to the amplitude of the initial two waves? **The amplitude of the combined wave is the sum of the amplitudes of the two initial waves.**
- Why does the combined wave cross the x-axis at a point where neither of the two initial waves are crossing the axis? **The combined wave is the sum of the other two waves. It crosses the x-axis at points where one of the initial waves is above the x-axis and the other wave is an equal distance below the x-axis.**

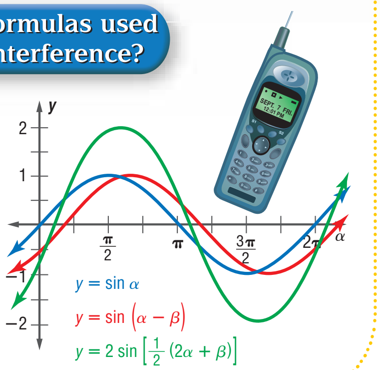
14-5 Sum and Difference of Angles Formulas

What You'll Learn

- Find values of sine and cosine involving sum and difference formulas.
- Verify identities by using sum and difference formulas.

How are the sum and difference formulas used to describe communication interference?

Have you ever been talking on a cell phone and temporarily lost the signal? Radio waves that pass through the same place at the same time cause interference. *Constructive interference* occurs when two waves combine to have a greater amplitude than either of the component waves. *Destructive interference* occurs when the component waves combine to have a smaller amplitude.



Study Tips

Reading Math

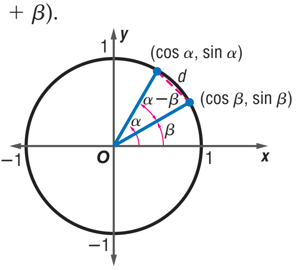
The Greek letter *beta*, β, can be used to denote the measure of an angle.

It is important to realize that $\sin(\alpha \pm \beta)$ is not the same as $\sin\alpha \pm \sin\beta$.

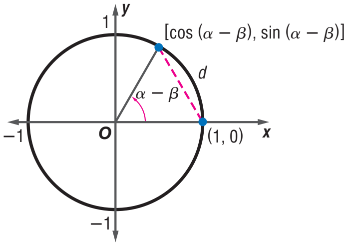
SUM AND DIFFERENCE FORMULAS

Notice that the third equation shown above involves the sum of α and β. It is often helpful to use formulas for the trigonometric values of the difference or sum of two angles. For example, you could find sin 15° by evaluating $\sin(60° - 45°)$. Formulas can be developed that can be used to evaluate expressions like $\sin(\alpha - \beta)$ or $\cos(\alpha + \beta)$.

The figure at the right shows two angles α and β in standard position on the unit circle. Use the Distance Formula to find d, where $(x_1, y_1) = (\cos\beta, \sin\beta)$ and $(x_2, y_2) = (\cos\alpha, \sin\alpha)$.



$$d = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$$
$$d^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$
$$d^2 = (\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta) + (\sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta)$$
$$d^2 = \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$
$$d^2 = 1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \quad \sin^2\alpha + \cos^2\alpha = 1 \text{ and}$$
$$d^2 = 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \quad \sin^2\beta + \cos^2\beta = 1$$



Now find the value of d^2 when the angle having measure $\alpha - \beta$ is in standard position on the unit circle, as shown in the figure at the left.

$$d = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2}$$
$$d^2 = [\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2$$
$$= [\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1] + \sin^2(\alpha - \beta)$$
$$= \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1$$
$$= 1 - 2\cos(\alpha - \beta) + 1$$
$$= 2 - 2\cos(\alpha - \beta)$$

Resource Manager

Workbook and Reproducible Masters

Chapter 14 Resource Masters

- Study Guide and Intervention, pp. 861–862
- Skills Practice, p. 863
- Practice, p. 864
- Reading to Learn Mathematics, p. 865
- Enrichment, p. 866

Transparencies

5-Minute Check Transparency 14-5
Answer Key Transparencies

Technology

Interactive Chalkboard

By equating the two expressions for d^2, you can find a formula for $\cos(\alpha - \beta)$.

$$\begin{aligned} d^2 &= d^2 \\ 2 - 2\cos(\alpha - \beta) &= 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \\ -1 + \cos(\alpha - \beta) &= -1 + \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{Divide each side by } -2. \\ \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{Add 1 to each side.} \end{aligned}$$

Use the formula for $\cos(\alpha - \beta)$ to find a formula for $\cos(\alpha + \beta)$.

$$\begin{aligned} \cos(\alpha - \beta) &= \cos[\alpha - (-\beta)] \\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta && \cos(-\beta) = \cos\beta;\ \sin(-\beta) = -\sin\beta \end{aligned}$$

You can use a similar method to find formulas for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$.

Key Concept — Sum and Difference of Angles Formulas

The following identities hold true for all values of α and β.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

Notice the symbol ∓ in the formula for $\cos(\alpha \pm \beta)$. It means "minus or plus." In the cosine formula, when the sign on the left side of the equation is plus, the sign on the right side is minus; when the sign on the left side is minus, the sign on the right side is plus. The signs match each other in the sine formula.

Example 1 Use Sum and Difference of Angles Formulas

Find the exact value of each expression.

a. cos 75°

Use the formula $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$\begin{aligned} \cos 75^\circ &= \cos(30^\circ + 45^\circ) && \alpha = 30^\circ,\ \beta = 45^\circ \\ &= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ \\ &= \left(\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2} \cdot \frac{\sqrt{2}}{2}\right) && \text{Evaluate each expression.} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} && \text{Multiply.} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} && \text{Simplify.} \end{aligned}$$

b. sin (−210°)

Use the formula $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

$$\begin{aligned} \sin(-210^\circ) &= \sin(60^\circ - 270^\circ) && \alpha = 60^\circ,\ \beta = 270^\circ \\ &= \sin 60^\circ \cos 270^\circ - \cos 60^\circ \sin 270^\circ \\ &= \left(\frac{\sqrt{3}}{2}\right)(0) - \left(\frac{1}{2}\right)(-1) && \text{Evaluate each expression.} \\ &= 0 - \left(-\frac{1}{2}\right) && \text{Multiply.} \\ &= \frac{1}{2} && \text{Simplify.} \end{aligned}$$

www.algebra2.com/extra_examples

2 Teach

Building on Prior Knowledge

The Distance Formula was first discussed and used back in Lesson 8-1. In this lesson, students will learn how to apply the Distance Formula to derive the Sum and Difference of Angles Formulas.

SUM AND DIFFERENCE FORMULAS

Teaching Tip Explain that sin 15° can be found by evaluating sin (60° − 45°) because the exact values of sin 60° and sin 45° are known. Stress that using a difference such as sin (90° − 75°) is ineffective because sin 75° is not easily computed or remembered.

Reading Tip Have the entire class read aloud the paragraph directly below the Key Concept box, about the minus and plus signs. Stress that its use in the formula for the cosine of a sum or difference indicates that the sign on the right side of the identity is the opposite of the sign on the left side.

In-Class Examples

PowerPoint®

1 Find the exact value of each expression.

a. sin 75° $\frac{\sqrt{2} + \sqrt{6}}{4}$

b. cos (−75°) $\frac{\sqrt{6} - \sqrt{2}}{4}$

2 Refer to Example 2 in the Student Edition. Use the difference of angles formula to determine the amount of light energy in Raleigh, North Carolina, located at a latitude of 35.8° N. **The maximum light energy per square foot is 0.9770*E*.**

DAILY INTERVENTION — Differentiated Instruction

Naturalist Have students apply what they know about geography, or have them conduct research, to find out whether the light energy per square foot would be increasing or decreasing as you travel toward the equator. Have students research the latitude of your city, or a city that interests them, and repeat Example 2 for that city.

VERIFY IDENTITIES

In-Class Example

Power Point®

3 Verify that each of the following is an identity.

a. $\cos(90° - \theta) = \sin \theta$

$$\begin{aligned} \cos(90° - \theta) &\overset{?}{=} \sin\theta \\ \cos 90° \cos\theta + \sin 90° \sin\theta &\overset{?}{=} \sin\theta \\ 0 \cos\theta + 1 \sin\theta &\overset{?}{=} \sin\theta \\ \sin\theta &= \sin\theta \end{aligned}$$

b. $\cos(180° - \theta) = -\cos \theta$

$$\begin{aligned} \cos(180° - \theta) &\overset{?}{=} -\cos\theta \\ \cos 180° \cos\theta + \sin 180° \sin\theta &\overset{?}{=} -\cos\theta \\ -1 \cos\theta + 0 \sin\theta &\overset{?}{=} -\cos\theta \\ -\cos\theta &= -\cos\theta \end{aligned}$$

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 14.
- record the sum and difference formulas.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises…

Organization by Objective

- **Sum and Difference Formulas:** 14–27
- **Verify Identities:** 28–39

Odd/Even Assignments

Exercises 14–39 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 15–25 odd, 29–37 odd, 40, 41–45 odd, 46–74

Average: 15–39 odd, 40, 41–45 odd, 46–74

Advanced: 14–40 even, 41–66 (optional: 67–74)

Study Tip

Reading Math

The symbol ϕ is the lowercase Greek letter *phi.*

Example 2 Use Sum and Difference Formulas to Solve a Problem

PHYSICS **On June 22, the maximum amount of light energy falling on a square foot of ground at a location in the northern hemisphere is given by $E \sin(113.5° - \phi)$, where ϕ is the latitude of the location and E is the amount of light energy when the Sun is directly overhead. Use the difference of angles formula to determine the amount of light energy in Rochester, New York, located at a latitude of 43.1° N.**

Use the difference formula for sine.

$$\begin{aligned} \sin(113.5° - \phi) &= \sin 113.5° \cos\phi - \cos 113.5° \sin\phi \\ &= \sin 113.5° \cos 43.1° - \cos 113.5° \sin 43.1° \\ &= 0.9171 \cdot 0.7301 - (-0.3987) \cdot 0.6833 \\ &= 0.9420 \end{aligned}$$

In Rochester, New York, the maximum light energy per square foot is $0.9420E$.

VERIFY IDENTITIES You can also use the sum and difference formulas to verify identities.

Example 3 Verify Identities

Verify that each of the following is an identity.

a. $\sin(180° + \theta) = -\sin\theta$

$\sin(180° + \theta) \overset{?}{=} -\sin\theta$	Original equation
$\sin 180° \cos\theta + \cos 180° \sin\theta \overset{?}{=} -\sin\theta$	Sum of angles formula
$0 \cos\theta + (-1) \sin\theta \overset{?}{=} -\sin\theta$	Evaluate each expression.
$-\sin\theta = -\sin\theta$	Simplify.

b. $\cos(180° + \theta) = -\cos\theta$

$\cos(180° + \theta) \overset{?}{=} -\cos\theta$	Original equation
$\cos 180° \cos\theta - \sin 180° \sin\theta \overset{?}{=} -\cos\theta$	Sum of angles formula
$(-1) \cos\theta - 0 \sin\theta \overset{?}{=} -\cos\theta$	Evaluate each expression.
$-\cos\theta = -\cos\theta$	Simplify.

Check for Understanding

Concept Check

1–2. See margin.

3. Sometimes; sample answer: The cosine function can equal 1.

1. **Determine** whether $\sin(\alpha + \beta) = \sin\alpha + \sin\beta$ is an identity.
2. **Describe** a method for finding the exact value of sin 105°. Then find the value.
3. **OPEN ENDED** Determine whether $\cos(\alpha - \beta) < 1$ is *sometimes, always,* or *never* true. Explain your reasoning.

Guided Practice

GUIDED PRACTICE KEY

Exercises	Examples
4–9, 13	1
10–12	3

Find the exact value of each expression.

4. sin 75° $\frac{\sqrt{6} + \sqrt{2}}{4}$
5. sin 165° $\frac{\sqrt{6} - \sqrt{2}}{4}$
6. cos 255° $\frac{\sqrt{2} - \sqrt{6}}{4}$
7. cos (−30°) $\frac{\sqrt{3}}{2}$
8. sin (−240°) $\frac{\sqrt{3}}{2}$
9. cos (−120°) $-\frac{1}{2}$

Verify that each of the following is an identity. 10–12. See pp 811A–811N.

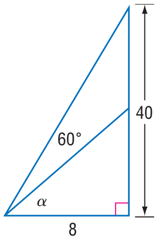
10. $\cos(270° - \theta) = -\sin\theta$
11. $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$
12. $\sin(\theta + 30°) + \cos(\theta + 60°) = \cos\theta$

Answers

1. $$\begin{aligned} \sin(\alpha + \beta) &\overset{?}{=} \sin\alpha + \sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta &\neq \sin\alpha + \sin\beta \end{aligned}$$

2. Use the formula $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. Since $\sin 105° = \sin(60° + 45°)$, replace α with 60° and β with 45° to get $\sin 60° \cos 45° + \cos 60° \sin 45°$. By finding the sum of the products of the values, the result is $\frac{\sqrt{6} + \sqrt{2}}{4}$ or about 0.9659.

Application **13. GEOMETRY** Determine the exact value of tan α. $\frac{5-\sqrt{3}}{1+5\sqrt{3}}$



★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
14–27	1
28–39	3
40, 41, 43–46	2

Find the exact value of each expression.

14. sin 135° $\frac{\sqrt{2}}{2}$
15. cos 105° $\frac{\sqrt{2}-\sqrt{6}}{4}$
16. sin 285° $\frac{-\sqrt{6}-\sqrt{2}}{4}$
17. cos 165°
18. cos 195°
19. sin 255°
20. cos 225° $-\frac{\sqrt{2}}{2}$
21. sin 315° $-\frac{\sqrt{2}}{2}$
22. sin (−15°) $\frac{\sqrt{2}-\sqrt{6}}{4}$
23. cos (−45°)
24. cos (−150°)
25. sin (−165°)

★ 26. What is the exact value of sin 75° − sin 15°? $\frac{\sqrt{2}}{2}$

★ 27. Find the exact value of cos 105° + cos 225°. $\frac{-\sqrt{6}-\sqrt{2}}{4}$

Verify that each of the following is an identity. 28–39. See pp. 811A–811N.

28. $\sin(270° - \theta) = -\cos\theta$
29. $\cos(90° + \theta) = -\sin\theta$
30. $\cos(90° - \theta) = \sin\theta$
31. $\sin(90° - \theta) = \cos\theta$
32. $\sin\left(\theta + \frac{3\pi}{2}\right) = -\cos\theta$
33. $\cos(\pi - \theta) = -\cos\theta$
34. $\cos(2\pi + \theta) = \cos\theta$
35. $\sin(\pi - \theta) = \sin\theta$
36. $\sin(60° + \theta) + \sin(60° - \theta) = \sqrt{3}\cos\theta$
37. $\sin\left(\theta + \frac{\pi}{3}\right) - \cos\left(\theta + \frac{\pi}{6}\right) = \sin\theta$

★ 38. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$

★ 39. $\cos(\alpha + \beta) = \frac{1 - \tan\alpha\tan\beta}{\sec\alpha\sec\beta}$

COMMUNICATION **For Exercises 40 and 41, use the following information.**
A radio transmitter sends out two signals, one for voice communication and another for data. Suppose the equation of the voice wave is $v = 10\sin(2t - 30°)$ and the equation of the data wave is $d = 10\cos(2t + 60°)$.

40. Draw a graph of the waves when they are combined. See margin.

41. Refer to the application at the beginning of the lesson. What type of interference results? Explain. Destructive; the resulting graph has a smaller amplitude than the two initial graphs.

PHYSICS **For Exercises 42–45, use the following information.**
On December 22, the maximum amount of light energy that falls on a square foot of ground at a certain location is given by $E\sin(113.5° + \phi)$, where ϕ is the latitude of the location. Use the sum of angles formula to find the amount of light energy, in terms of E, for each location. 42. 0.3681 E 43. 0.4179 E

42. Salem, OR (Latitude: 44.9° N)
43. Chicago, IL (Latitude: 41.8° N)
44. Charleston, SC (Latitude: 28.5°N) 0.6157 E
45. San Diego, CA (Latitude 32.7° N) 0.5563 E

46. **CRITICAL THINKING** Use the sum and difference formulas for sine and cosine to derive formulas for tan (α + β) and tan (α − β). See pp. 811A–811N.

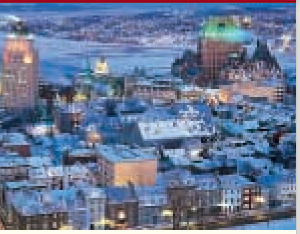
www.algebra2.com/self-check_quiz

Extra Practice

See page 860.

17. $\frac{-\sqrt{6}-\sqrt{2}}{4}$
18. $\frac{-\sqrt{6}-\sqrt{2}}{4}$
19. $\frac{-\sqrt{6}-\sqrt{2}}{4}$
23. $\frac{\sqrt{2}}{2}$
24. $-\frac{\sqrt{3}}{2}$
25. $\frac{\sqrt{2}-\sqrt{6}}{4}$

More About. . .

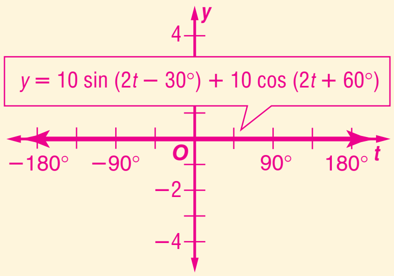


Physics

In the northern hemisphere, the day with the least number of hours of daylight is December 21 or 22, the first day of winter.

Source: www.infoplease.com

40.



Enrichment, p. 866

Identities for the Products of Sines and Cosines

By adding the identities for the sines of the sum and difference of the measures of two angles, a new identity is obtained.

$$\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \hline (i)\quad \sin(\alpha + \beta) + \sin(\alpha - \beta) &= 2\sin\alpha\cos\beta \end{aligned}$$

This new identity is useful for expressing certain products as sums.

Example **Write sin 3θ cos θ as a sum.**
In the identity let $\alpha = 3\theta$ and $\beta = \theta$ so that
$2\sin 3\theta\cos\theta = \sin(3\theta + \theta) + \sin(3\theta - \theta)$. Thus,
$\sin 3\theta\cos\theta = \frac{1}{2}\sin 4\theta + \frac{1}{2}\sin 2\theta$.

By subtracting the identities for sin (α + β) and sin (α − β), a similar identity for expressing a product as a difference is obtained.

Study Guide and Intervention, p. 861 (shown) and p. 862

Sum and Difference Formulas The following formulas are useful for evaluating an expression like sin 15° from the known values of sine and cosine of 60° and 45°.

Sum and Difference of Angles	The following identities hold true for all values of α and β. $\cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta$ $\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta$

Example **Find the exact value of each expression.**

a. cos 345°

$$\begin{aligned} \cos 345° &= \cos(300° + 45°) \\ &= \cos 300° \cdot \cos 45° - \sin 300° \cdot \sin 45° \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{3}}{2}\right) \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2}+\sqrt{6}}{4} \end{aligned}$$

b. sin (−105°)

$$\begin{aligned} \sin(-105°) &= \sin(45° - 150°) \\ &= \sin 45° \cdot \cos 150° - \cos 45° \cdot \sin 150° \\ &= \frac{\sqrt{2}}{2} \cdot \left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= -\frac{\sqrt{2}+\sqrt{6}}{4} \end{aligned}$$

Exercises

Find the exact value of each expression.

1. sin 105° $\frac{\sqrt{2}+\sqrt{6}}{4}$
2. cos 285° $\frac{\sqrt{6}-\sqrt{2}}{4}$
3. cos (−75°) $\frac{\sqrt{6}-\sqrt{2}}{4}$
4. cos (−165°) $-\frac{\sqrt{2}+\sqrt{6}}{4}$
5. sin 195° $\frac{\sqrt{2}-\sqrt{6}}{4}$
6. cos 420° $\frac{1}{2}$
7. sin (−75°) $-\frac{\sqrt{2}+\sqrt{6}}{4}$
8. cos 135° $-\frac{\sqrt{2}}{2}$
9. cos (−15°) $\frac{\sqrt{2}+\sqrt{6}}{4}$
10. sin 345° $\frac{\sqrt{2}-\sqrt{6}}{4}$
11. cos (−105°) $\frac{\sqrt{2}-\sqrt{6}}{4}$
12. sin 495° $\frac{\sqrt{2}}{2}$

Skills Practice, p. 863 and Practice, p. 864 (shown)

Find the exact value of each expression.

1. cos 75° $\frac{\sqrt{6}-\sqrt{2}}{4}$
2. cos 375° $\frac{\sqrt{6}+\sqrt{2}}{4}$
3. sin (−165°) $\frac{\sqrt{2}-\sqrt{6}}{4}$
4. sin (−105°) $\frac{-\sqrt{2}-\sqrt{6}}{4}$
5. sin 150° $\frac{1}{2}$
6. cos 240° $-\frac{1}{2}$
7. sin 225° $-\frac{\sqrt{2}}{2}$
8. sin (−75°) $\frac{-\sqrt{2}-\sqrt{6}}{4}$
9. sin 195° $\frac{\sqrt{2}-\sqrt{6}}{4}$

Verify that each of the following is an identity.

10. $\cos(180° - \theta) = -\cos\theta$

$$\begin{aligned} \cos(180° - \theta) &\stackrel{?}{=} -\cos\theta \\ \cos 180°\cos\theta + \sin 180°\sin\theta &\stackrel{?}{=} -\cos\theta \\ -1\cos\theta + 0\sin\theta &\stackrel{?}{=} -\cos\theta \\ -\cos\theta &= -\cos\theta \end{aligned}$$

11. $\sin(360° + \theta) = \sin\theta$

$$\begin{aligned} \sin(360° + \theta) &\stackrel{?}{=} \sin\theta \\ \sin 360°\cos\theta + \cos 360°\sin\theta &\stackrel{?}{=} \sin\theta \\ 0\cos\theta + 1\sin\theta &\stackrel{?}{=} \sin\theta \\ \sin\theta &= \sin\theta \end{aligned}$$

12. $\sin(45° + \theta) - \sin(45° - \theta) = \sqrt{2}\sin\theta$

$$\begin{aligned} &\sin(45° + \theta) - \sin(45° - \theta) \\ &\stackrel{?}{=} \sin 45°\cos\theta + \cos 45°\sin\theta - (\sin 45°\cos\theta - \cos 45°\sin\theta) \\ &\stackrel{?}{=} 2 \cdot \cos 45°\sin\theta \\ &\stackrel{?}{=} 2 \cdot \frac{\sqrt{2}}{2} \cdot \sin\theta \\ &= \sqrt{2}\sin\theta \end{aligned}$$

13. $\cos\left(x - \frac{\pi}{6}\right) + \sin\left(x - \frac{\pi}{3}\right) = \sin x$

$$\begin{aligned} &\cos\left(x - \frac{\pi}{6}\right) + \sin\left(x - \frac{\pi}{3}\right) \\ &\stackrel{?}{=} \cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} + \sin x\cos\frac{\pi}{3} - \cos x\sin\frac{\pi}{3} \\ &\stackrel{?}{=} \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x + \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x \\ &= \sin x \end{aligned}$$

14. **SOLAR ENERGY** On March 21, the maximum amount of solar energy that falls on a square foot of ground at a certain location is given by $E\sin(90° - \phi)$, where ϕ is the latitude of the location and E is a constant. Use the difference of angles formula to find the amount of solar energy, in terms of $\cos\phi$, for a location that has a latitude of ϕ. $E\cos\phi$

ELECTRICITY **In Exercises 15 and 16, use the following information.**
In a certain circuit carrying alternating current, the formula $i = 2\sin(120t)$ can be used to find the current i in amperes after t seconds.

15. Rewrite the formula using the sum of two angles. Sample answer: $i = 2\sin(90t + 30t)$
16. Use the sum of angles formula to find the exact current at $t = 1$ second. $\sqrt{3}$ amperes

Reading to Learn Mathematics, p. 865

ELL

Pre-Activity **How are the sum and difference formulas used to describe communication interference?**

Read the introduction to Lesson 14-5 at the top of page 786 in your textbook.

Consider the functions $y = \sin x$ and $y = 2\sin x$. Do the graphs of these two functions have *constructive* interference or *destructive* interference?
constructive

Reading the Lesson

1. Match each expression from the list on the left with an expression from the list on the right that is equal to it for all values of the variables. (Some of the expressions from the list on the right may be used more than once or not at all.)

a. sin (α − β) v	**i.** sin β
b. cos (α + β) vi	**ii.** sin α cos β + cos α sin β
c. sin (180° + β) vii	**iii.** −cos β
d. sin (180° − β) i	**iv.** cos α cos β + sin α sin β
e. cos (180° + β) iii	**v.** sin α cos β − cos α sin β
f. sin (α + β) ii	**vi.** cos α cos β − sin α sin β
g. cos (90° − β) i	**vii.** −sin β
h. cos (α − β) iv	**viii.** cos β

2. Which expressions are equal to sin 15°? (There may be more than one correct choice.)
   - **A.** sin 45° cos 30° + cos 45° sin 30°
   - **B.** sin 45° cos 30° − cos 45° sin 30° B and C
   - **C.** sin 60° cos 45° − cos 60° sin 45°
   - **D.** cos 60° cos 45° − sin 60° sin 45°

Helping You Remember

3. Some students have trouble remembering which signs to use on the right-hand sides of the sum and difference of angle formulas. What is an easy way to remember this?

Sample answer: In the *sine* identities, the signs are the *same* on both sides. In the *cosine* identities, the signs are *opposite* on the two sides.

4 Assess

Open-Ended Assessment

Modeling Show students a graph of the function $y = \sin x$. Have students point out on the graph which values would be most useful to use with the sum and difference of angles formulas, and ask them to explain their reasoning.

Getting Ready for Lesson 14-6

PREREQUISITE SKILL Students will find values using half-angle formulas in Lesson 14-6. The half-angle formulas include expressions within square root symbols, so students must be comfortable evaluating square roots. Use Exercises 67–74 to determine your students' familiarity with the Square Root Property.

Answers

47. Sample answer: To determine communication interference, you need to determine the sine or cosine of the sum or difference of two angles. Answers should include the following information.

- Interference occurs when waves pass through the same space at the same time. When the combined waves have a greater amplitude, constructive interference results and when the combined waves have a smaller amplitude, destructive interference results.

58. $\sin\theta = -\frac{3\sqrt{34}}{34}$, $\cos\theta = \frac{5\sqrt{34}}{34}$, $\tan\theta = -\frac{3}{5}$, $\csc\theta = -\frac{\sqrt{34}}{3}$, $\sec\theta = \frac{\sqrt{34}}{5}$, $\cot\theta = -\frac{5}{3}$

59. $\sin\theta = -\frac{4}{5}$, $\cos\theta = -\frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = -\frac{5}{3}$, $\cot\theta = \frac{3}{4}$

60. $\sin\theta = 1$, $\cos\theta = 0$, $\tan\theta =$ undefined, $\csc\theta = 1$, $\sec\theta =$ undefined, $\cot\theta = 0$

---

47. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

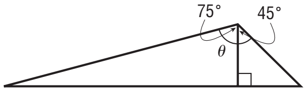
How are the sum and difference formulas used to describe communication interference?

Include the following in your answer:

- an explanation of the difference between constructive and destructive interference, and
- a description of how you would explain wave interference to a friend.

Standardized Test Practice

48. Find the exact value of $\sin\theta$. **A**



Ⓐ $\frac{\sqrt{3}}{2}$ Ⓑ $\frac{\sqrt{2}}{2}$

Ⓒ $\frac{1}{2}$ Ⓓ $\frac{\sqrt{3}}{3}$

49. Find the exact value of $\cos(-210°)$. **C**

Ⓐ $\frac{\sqrt{3}}{2}$ Ⓑ 0.5 Ⓒ $-\frac{\sqrt{3}}{2}$ Ⓓ -0.5

Maintain Your Skills

Mixed Review

50–53. See pp. 811A–811N.

Verify that each of the following is an identity. *(Lesson 14-4)*

50. $\cot\theta + \sec\theta = \frac{\cos^2\theta + \sin\theta}{\sin\theta\cos\theta}$

51. $\sin^2\theta + \tan^2\theta = (1 - \cos^2\theta) + \frac{\sec^2\theta}{\csc^2\theta}$

52. $\sin\theta(\sin\theta + \csc\theta) = 2 - \cos^2\theta$

53. $\frac{\sec\theta}{\tan\theta} = \csc\theta$

Simplify each expression. *(Lesson 14-3)*

54. $\frac{\tan\theta\csc\theta}{\sec\theta}$ **1**

55. $4\left(\sec^2\theta - \frac{\sin^2\theta}{\cos^2\theta}\right)$ **4**

56. $(\cot\theta + \tan\theta)\sin\theta$ **sec θ**

57. $\csc\theta\tan\theta + \sec\theta$ **2 sec θ**

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point. *(Lesson 13-3)* **58–60. See margin**

58. (5, −3)

59. (−3, −4)

60. (0, 2)

Evaluate each expression. *(Lesson 12-2)*

61. $P(6, 4)$ **360**

62. $P(12, 7)$ **3,991,680**

63. $C(8, 3)$ **56**

64. $C(10, 4)$ **210**

65. about 228 mi

65. **AVIATION** A pilot is flying from Chicago to Columbus, a distance of 300 miles. In order to avoid an area of thunderstorms, she alters her initial course by 15° and flies on this course for 75 miles. How far is she from Columbus? *(Lesson 13-5)*

73. $\pm\frac{\sqrt{\sqrt{6} - \sqrt{2}}}{2}$

66. Write $6y^2 - 34x^2 = 204$ in standard form. *(Lesson 8-5)* $\frac{y^2}{34} - \frac{x^2}{6} = 1$

Getting Ready for the Next Lesson

74. $\pm\frac{\sqrt{2 - 2\sqrt{2}}}{2}$

PREREQUISITE SKILL **Solve each equation.**

*(To review **solving equations using the Square Root Property**, see Lesson 6-4.)* **72.** $\pm\frac{3\sqrt{5}}{5}$

67. $x^2 = \frac{20}{16}$ $\pm\frac{2\sqrt{5}}{2}$

68. $x^2 = \frac{9}{25}$ $\pm\frac{3}{5}$

69. $x^2 = \frac{5}{25}$ $\pm\frac{\sqrt{5}}{5}$

70. $x^2 = \frac{18}{32}$ $\pm\frac{3}{4}$

71. $x^2 - 1 = \frac{1}{2}$ $\pm\frac{\sqrt{6}}{2}$

72. $x^2 - 1 = \frac{4}{5}$

73. $x^2 = \frac{\sqrt{3}}{2} - \frac{1}{2}$

74. $x^2 = \frac{\sqrt{2}}{2} - 1$

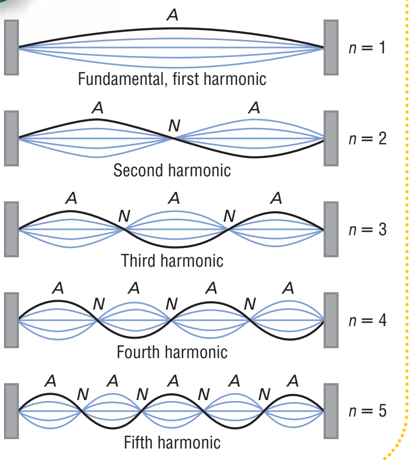
14-6 Double-Angle and Half-Angle Formulas

What You'll Learn

- Find values of sine and cosine involving double-angle formulas.
- Find values of sine and cosine involving half-angle formulas.

How can trigonometric functions be used to describe music?

Stringed instruments such as a piano, guitar, or violin rely on waves to produce the tones we hear. When the strings are struck or plucked, they vibrate. If the motion of the strings were observed in slow motion, you could see that there are places on the string, called *nodes*, that do not move under the vibration. Halfway between each pair of consecutive nodes are *antinodes* that undergo the maximum vibration. The nodes and antinodes form *harmonics*. These harmonics can be represented using variations of the equations $y = \sin 2\theta$ and $y = \sin \frac{1}{2}\theta$.



DOUBLE-ANGLE FORMULAS You can use the formula for $\sin(\alpha + \beta)$ to find the sine of twice an angle θ, $\sin 2\theta$, and the formula for $\cos(\alpha + \beta)$ to find the cosine of twice an angle θ, $\cos 2\theta$.

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta)\\ &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ &= 2\sin\theta\cos\theta\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

You can find alternate forms for $\cos 2\theta$ by making substitutions into the expression $\cos^2\theta - \sin^2\theta$.

$$\begin{aligned}\cos^2\theta - \sin^2\theta &= (1 - \sin^2\theta) - \sin^2\theta && \text{Substitute } 1 - \sin^2\theta \text{ for } \cos^2\theta.\\ &= 1 - 2\sin^2\theta && \text{Simplify.}\\ \cos^2\theta - \sin^2\theta &= \cos^2\theta - (1 - \cos^2\theta) && \text{Substitute } 1 - \cos^2\theta \text{ for } \sin^2\theta.\\ &= 2\cos^2\theta - 1 && \text{Simplify.}\end{aligned}$$

These formulas are called the **double-angle formulas**.

Key Concept *Double-Angle Formulas*

The following identities hold true for all values of θ.

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

14-6 Lesson Notes

1 Focus

5-Minute Check Transparency 14-6 Use as a quiz or review of Lesson 14-5.

Mathematical Background notes are available for this lesson on page 760D.

How can trigonometric functions be used to describe music?

Ask students:

- What do the values of n represent on the right side of the diagram? **the number of antinodes**
- What is occurring at an antinode? **maximum vibration**
- At what point do you think a guitar string would be more likely to break, a node or an antinode? **at an antinode**

2 Teach

DOUBLE-ANGLE FORMULAS

Teaching Tip Point out that the double-angle formulas are derived using the sum and difference of angles formulas presented in Lesson 14-5.

Resource Manager

Workbook and Reproducible Masters

Chapter 14 Resource Masters

- Study Guide and Intervention, pp. 867–868
- Skills Practice, p. 869
- Practice, p. 870
- Reading to Learn Mathematics, p. 871
- Enrichment, p. 872
- Assessment, p. 894

Transparencies

5-Minute Check Transparency 14-6
Answer Key Transparencies

Technology

Interactive Chalkboard

Teaching Tip Remind students of the identity $\sin^2 \theta + \cos^2 \theta = 1$, and its two variations that occur by subtracting either $\sin^2 \theta$ or $\cos^2 \theta$ from both sides. Explain that there are three versions of the formula for $\cos 2\theta$ because of the three variations of the identity $\sin^2 \theta + \cos^2 \theta = 1$.

In-Class Example

Power Point®

1 Find the exact value of each expression if $\sin \theta = \frac{3}{4}$ and θ is between 0° and 90°.

a. $\sin 2\theta$ $\frac{3\sqrt{7}}{8}$

b. $\cos 2\theta$ $-\frac{1}{8}$

Example 1 Double-Angle Formulas

Find the exact value of each expression if $\sin \theta = \frac{4}{5}$ and θ is between 90° and 180°.

a. $\sin 2\theta$

Use the identity $\sin 2\theta = 2 \sin \theta \cos \theta$.

First, find the value of $\cos \theta$.

$\cos^2 \theta = 1 - \sin^2 \theta$ $\cos^2 \theta + \sin^2 \theta = 1$

$\cos^2 \theta = 1 - \left(\frac{4}{5}\right)^2$ $\sin \theta = \frac{4}{5}$

$\cos^2 \theta = \frac{9}{25}$ Subtract.

$\cos \theta = \pm\frac{3}{5}$ Find the square root of each side.

Since θ is in the second quadrant, cosine is negative. Thus, $\cos \theta = -\frac{3}{5}$.

Now find $\sin 2\theta$.

$\sin 2\theta = 2 \sin \theta \cos \theta$ Double-angle formula

$\sin 2\theta = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)$ $\sin \theta = \frac{4}{5}$, $\cos \theta = -\frac{3}{5}$

$= -\frac{24}{25}$ The value of $\sin 2\theta$ is $-\frac{24}{25}$.

b. $\cos 2\theta$

Use the identity $\cos 2\theta = 1 - 2 \sin^2 \theta$.

$\cos 2\theta = 1 - 2 \sin^2 \theta$ Double-angle formula

$= 1 - 2\left(\frac{4}{5}\right)^2$ $\sin \theta = \frac{4}{5}$

$= -\frac{7}{25}$ The value of $\cos 2\theta$ is $-\frac{7}{25}$.

HALF-ANGLE FORMULAS You can derive formulas for the sine and cosine of half a given angle using the double-angle formulas.

Find $\sin \frac{\alpha}{2}$.

$1 - 2 \sin^2 \theta = \cos 2\theta$ Double-angle formula

$1 - 2 \sin^2 \frac{\alpha}{2} = \cos \alpha$ Substitute $\frac{\alpha}{2}$ for θ and α for 2θ.

$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2}$ Solve for $\sin^2 \frac{\alpha}{2}$.

$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$ Take the square root of each side.

Find $\cos \frac{\alpha}{2}$.

$2 \cos^2 \theta - 1 = \cos 2\theta$ Double-angle formula

$2 \cos^2 \frac{\alpha}{2} - 1 = \cos \alpha$ Substitute $\frac{\alpha}{2}$ for θ and α for 2θ.

$\cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2}$ Solve for $\cos^2 \frac{\alpha}{2}$.

$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$ Take the square root of each side.

These are called the **half-angle formulas**. The signs are determined by the function of $\frac{\alpha}{2}$.

Key Concept *Half-Angle Formulas*

The following identities hold true for all values of α.

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Example 2 Half-Angle Formulas

Find $\cos\frac{\alpha}{2}$ if $\sin\alpha = -\frac{3}{4}$ and α is in the third quadrant.

Since $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$, we must find $\cos\alpha$ first.

$\cos^2\alpha = 1 - \sin^2\alpha$ — $\cos^2\alpha + \sin^2\alpha = 1$

$\cos^2\alpha = 1 - \left(-\frac{3}{4}\right)^2$ — $\sin\alpha = -\frac{3}{4}$

$\cos^2\alpha = \frac{7}{16}$ — Simplify.

$\cos\alpha = \pm\frac{\sqrt{7}}{4}$ — Take the square root of each side.

Since α is in the third quadrant, $\cos\alpha = -\frac{\sqrt{7}}{4}$.

$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$ — Half-angle formula

$= \pm\sqrt{\frac{1-\frac{\sqrt{7}}{4}}{2}}$ — $\cos\alpha = -\frac{\sqrt{7}}{4}$

$= \pm\sqrt{\frac{4-\sqrt{7}}{8}}$ — Simplify the radicand.

$= \pm\frac{\sqrt{4-\sqrt{7}}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ — Rationalize.

$= \pm\frac{\sqrt{8-2\sqrt{7}}}{4}$ — Multiply.

Since α is between 180° and 270°, $\frac{\alpha}{2}$ is between 90° and 135°. Thus, $\cos\frac{\alpha}{2}$ is negative and equals $-\frac{\sqrt{8-2\sqrt{7}}}{4}$.

Study Tip

Choosing the Sign

You may want to determine the quadrant in which the terminal side of $\frac{\alpha}{2}$ will lie in the first step of the solution. Then you can use the correct sign from the beginning.

Example 3 Evaluate Using Half-Angle Formulas

Find the exact value of each expression by using the half-angle formulas.

a. sin 105°

$\sin 105° = \sin\frac{210°}{2}$

$= \sqrt{\frac{1-\cos 210°}{2}}$ — $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$

(continued on the next page)

HALF-ANGLE FORMULAS

Tips for New Teachers

Intervention Stress the Study Tip provided in the margin next to Example 2. Determining the proper sign for the answer at the beginning of the computation will help some students avoid forgetting this step at the end of their computations.

In-Class Example Power Point®

2 Find $\cos\frac{\alpha}{2}$ if $\sin\alpha = \frac{4}{5}$ and α is in the second quadrant. $\frac{\sqrt{5}}{5}$

DAILY INTERVENTION

Differentiated Instruction

Auditory/Musical If possible, ask a music teacher at your school to talk to students about harmonics. Students playing stringed instruments may also be willing to share what they have learned about harmonics and waves. If a music teacher is not available, a physics teacher may also be able to demonstrate harmonics or bring a device that creates standing waves to class.

In-Class Examples

Power Point®

3 Find the exact value of each expression by using the half-angle formulas.

a. $\sin 165°$ $\frac{\sqrt{2-\sqrt{3}}}{2}$

b. $\cos \frac{9\pi}{8}$ $-\frac{\sqrt{2+\sqrt{2}}}{2}$

4 Verify that $\sin\theta(\cos^2\theta - \cos 2\theta) = \sin^3\theta$ is an identity.

$$\begin{aligned}&\sin\theta(\cos^2\theta - \cos 2\theta)\\ &\stackrel{?}{=} \sin\theta[\cos^2\theta - (\cos^2\theta - \sin^2\theta)]\\ &\stackrel{?}{=} \sin\theta(\cos^2\theta - \cos^2\theta + \sin^2\theta)\\ &\stackrel{?}{=} \sin\theta(\sin^2\theta)\\ &= \sin^3\theta\end{aligned}$$

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 14.
- record the double-angle and half-angle formulas.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

$$= \sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}} \qquad \cos 210° = -\frac{\sqrt{3}}{2}$$

$$= \sqrt{\frac{2+\sqrt{3}}{4}} \qquad \text{Simplify the radicand.}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2} \qquad \text{Simplify the denominator.}$$

b. $\cos\frac{\pi}{8}$

$$\cos\frac{\pi}{8} = \frac{\frac{\pi}{4}}{2}$$

$$= \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} \qquad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}} \qquad \text{Simplify the radicand.}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{2} \qquad \text{Simplify the denominator.}$$

Recall that you can use the sum and difference formulas to verify identities. Double- and half-angle formulas can also be used to verify identities.

Example 4 Verify Identities

Verify that $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$ is an identity.

$$(\sin\theta + \cos\theta)^2 \stackrel{?}{=} 1 + \sin 2\theta \qquad \text{Original equation}$$

$$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \stackrel{?}{=} 1 + \sin 2\theta \qquad \text{Multiply.}$$

$$1 + 2\sin\theta\cos\theta \stackrel{?}{=} 1 + \sin 2\theta \qquad \sin^2\theta + \cos^2\theta = 1$$

$$1 + \sin 2\theta = 1 + \sin 2\theta \qquad \text{Double-angle formula}$$

Check for Understanding

Concept Check

1–3. See margin.

1. **Explain** how to find $\cos\frac{x}{2}$ if x is in the third quadrant.
2. **Find a counterexample** to show that $\cos 2\theta = 2\cos\theta$ is not an identity.
3. **OPEN ENDED** Describe the conditions under which you would use each of the three identities for $\cos 2\theta$.

Guided Practice

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin\frac{\theta}{2}$, and $\cos\frac{\theta}{2}$ for each of the following.

4. $\cos\theta = \frac{3}{5}$; $0° < \theta < 90°$
5. $\cos\theta = -\frac{2}{3}$; $180° < \theta < 270°$
6. $\sin\theta = \frac{1}{2}$; $0° < \theta < 90°$
7. $\sin\theta = -\frac{3}{4}$; $270° < \theta < 360°$

4–7. See margin.

Find the exact value of each expression by using the half-angle formulas.

8. $\sin 195°$ $-\frac{\sqrt{2-\sqrt{3}}}{2}$
9. $\cos\frac{19\pi}{12}$ $\frac{\sqrt{2-\sqrt{3}}}{2}$

GUIDED PRACTICE KEY

Exercises	Examples
4–7	1–2
8, 9, 12	3
10, 11	4

Answers

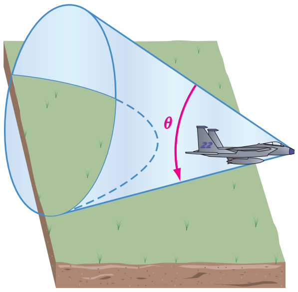
1. Sample answer: If x is in the third quadrant, then $\frac{x}{2}$ is between 90° and 135°. Use the half-angle formula for cosine knowing that the value is negative.
2. Sample answer: 45°; $\cos 2(45°) = \cos 90°$ or 0, $2\cos 45° = 2 \cdot \frac{\sqrt{2}}{2}$ or $\sqrt{2}$
3. Sample answer: The identity used for $\cos 2\theta$ depends on whether you know the value of $\sin\theta$, $\cos\theta$, or both values.
4. $\frac{24}{25}, -\frac{7}{25}, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}$
5. $\frac{4\sqrt{5}}{9}, -\frac{1}{9}, \frac{\sqrt{30}}{6}, -\frac{\sqrt{6}}{6}$
6. $\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{2-\sqrt{3}}}{2}, \frac{\sqrt{2+\sqrt{3}}}{2}$
7. $-\frac{3\sqrt{7}}{8}, -\frac{1}{8}, \frac{\sqrt{8-2\sqrt{7}}}{4}, -\frac{\sqrt{8+2\sqrt{7}}}{4}$

Verify that each of the following is an identity. **10–11. See margin.**

10. $\cot x = \frac{\sin 2x}{1 - \cos 2x}$

11. $\cos^2 2x + 4 \sin^2 x \cos^2 x = 1$

Application



12. AVIATION When a jet travels at speeds greater than the speed of sound, a sonic boom is created by the sound waves forming a cone behind the jet. If θ is the measure of the angle at the vertex of the cone, then the Mach number M can be determined using the formula $\sin \frac{\theta}{2} = \frac{1}{M}$. Find the Mach number of a jet if a sonic boom is created by a cone with a vertex angle of 75°. **1.64**

Practice and Apply

Homework Help

For Exercises	See Examples
13–24, 38, 39	1, 2
25–30, 37	3
31–36	4

Extra Practice

See page 861.

13–24. See margin.

Find the exact values of sin 2θ, cos 2θ, sin $\frac{\theta}{2}$, and cos $\frac{\theta}{2}$ for each of the following.

13. $\sin \theta = \frac{5}{13}$; $90° < \theta < 180°$

14. $\cos \theta = \frac{1}{5}$; $270° < \theta < 360°$

15. $\cos \theta = -\frac{1}{3}$; $180° < \theta < 270°$

16. $\sin \theta = -\frac{3}{5}$; $180° < \theta < 270°$

17. $\sin \theta = -\frac{3}{8}$; $270° < \theta < 360°$

18. $\cos \theta = -\frac{1}{4}$; $90° < \theta < 180°$

19. $\cos \theta = \frac{1}{6}$; $0° < \theta < 90°$

20. $\cos \theta = -\frac{12}{13}$; $180° < \theta < 270°$

21. $\sin \theta = -\frac{1}{3}$; $270° < \theta < 360°$

22. $\sin \theta = -\frac{1}{4}$; $180° < \theta < 270°$

23. $\cos \theta = \frac{2}{3}$; $0° < \theta < 90°$

24. $\sin \theta = \frac{2}{5}$; $90° < \theta < 180°$

Find the exact value of each expression by using the half-angle formulas.

25. $\cos 165°$ $-\frac{\sqrt{2 + \sqrt{3}}}{2}$

26. $\sin 22\frac{1}{2}°$ $\frac{\sqrt{2 - \sqrt{2}}}{2}$

27. $\cos 157\frac{1}{2}°$ $-\frac{\sqrt{2 + \sqrt{2}}}{2}$

28. $\sin 345°$ $-\frac{\sqrt{2 - \sqrt{3}}}{2}$

29. $\sin \frac{7\pi}{8}$ $\frac{\sqrt{2 - \sqrt{2}}}{2}$

30. $\cos \frac{7\pi}{12}$ $-\frac{\sqrt{2 - \sqrt{3}}}{2}$

Verify that each of the following is an identity. **31–36. See pp. 811A–811N.**

31. $\sin 2x = 2 \cot x \sin^2 x$

32. $2 \cos^2 \frac{x}{2} = 1 + \cos x$

33. $\sin^4 x - \cos^4 x = 2 \sin^2 x - 1$

34. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

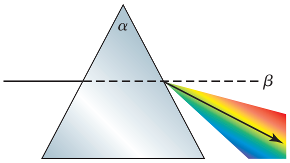
35. $\tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}$

36. $\frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x} = \tan x$

More About. . .

Optics

A rainbow appears when the sun shines through water droplets that act as a prism.



37. OPTICS If a glass prism has an apex angle of measure α and an angle of deviation of measure β, then the index of refraction n of the prism is given by $n = \frac{\sin\left[\frac{1}{2}(\alpha + \beta)\right]}{\sin \frac{\alpha}{2}}$.

What is the angle of deviation of a prism with an apex angle of 40° and an index of refraction of 2? **46.3°**

www.algebra2.com/self_check_quiz

About the Exercises...

Organization by Objective

- **Double–Angle Formulas:** 13–24, 31
- **Half–Angle Formulas:** 13–30, 32, 35

Odd/Even Assignments

Exercises 13–36 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 13–37 odd, 38, 39, 41–65

Average: 13–37 odd, 38, 39, 41–65

Advanced: 14–38 even, 39–59 (optional: 60–65)

All: Practice Quiz 2 (1–10)

Answers

15. $\frac{4\sqrt{2}}{9}, -\frac{7}{9}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}$

16. $\frac{24}{25}, \frac{7}{25}, \frac{3\sqrt{10}}{10}, -\frac{\sqrt{10}}{10}$

17. $-\frac{3\sqrt{55}}{32}, \frac{23}{32}, \frac{\sqrt{8 - \sqrt{55}}}{4},$
$-\frac{\sqrt{8 + \sqrt{55}}}{4}$

18. $-\frac{\sqrt{15}}{8}, -\frac{7}{8}, \frac{\sqrt{10}}{4}, \frac{\sqrt{6}}{4}$

19. $\frac{\sqrt{35}}{18}, -\frac{17}{18}, \frac{\sqrt{15}}{6}, \frac{\sqrt{21}}{6}$

20. $\frac{120}{169}, \frac{119}{169}, \frac{5\sqrt{26}}{26}, -\frac{\sqrt{26}}{26}$

21. $-\frac{4\sqrt{2}}{9}, \frac{7}{9}, \frac{\sqrt{18 - 12\sqrt{2}}}{6},$
$-\frac{\sqrt{18 - 12\sqrt{2}}}{6}$

22. $\frac{\sqrt{15}}{8}, \frac{7}{8}, \frac{\sqrt{8 + 2\sqrt{15}}}{4},$
$-\frac{\sqrt{8 - 2\sqrt{15}}}{4}$

23. $\frac{4\sqrt{5}}{9}, -\frac{1}{9}, \frac{\sqrt{6}}{6}, \frac{\sqrt{30}}{6}$

24. $-\frac{4\sqrt{21}}{5}, \frac{17}{25}, \frac{\sqrt{5\sqrt{2} + 10\sqrt{21}}}{10},$
$\frac{\sqrt{5\sqrt{10} - 10\sqrt{21}}}{10}$

Answers

10. $\cot x \stackrel{?}{=} \frac{\sin 2x}{1 - \cos 2x}$
$\stackrel{?}{=} \frac{2 \sin x \cos x}{1 - (1 - 2\sin^2 x)}$
$\stackrel{?}{=} \frac{2 \sin x \cos x}{2 \sin^2 x}$
$\stackrel{?}{=} \frac{\cos x}{\sin x}$
$= \cot x$

11. $\cos^2 2x + 4 \sin^2 x \cos^2 x \stackrel{?}{=} 1$
$\cos^2 2x + \sin^2 2x \stackrel{?}{=} 1$
$1 = 1$

13. $-\frac{120}{169}, \frac{119}{169}, \frac{5\sqrt{26}}{26}, \frac{\sqrt{26}}{26}$

14. $-\frac{4\sqrt{6}}{25}, -\frac{23}{25}, \frac{\sqrt{10}}{5}, -\frac{\sqrt{15}}{5}$

Study Guide and Intervention, p. 867 (shown) and p. 868

Double-Angle Formulas

Double-Angle Formulas	The following identities hold true for all values of θ. $\sin 2\theta = 2\sin\theta\cdot\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\cos 2\theta = 1 - 2\sin^2\theta$ $\cos 2\theta = 2\cos^2\theta - 1$

Example **Find the exact values of sin 2θ and cos 2θ if $\sin\theta = -\frac{9}{10}$ and $180° < \theta < 270°$.**

First, find the value of cos θ.

$$\cos^2\theta = 1 - \sin^2\theta \qquad \cos^2\theta + \sin^2\theta = 1$$
$$\cos^2\theta = 1 - \left(-\frac{9}{10}\right)^2 \qquad \sin\theta = -\frac{9}{10}$$
$$\cos^2\theta = \frac{19}{100}$$
$$\cos\theta = \pm\frac{\sqrt{19}}{10}$$

Since θ is in the third quadrant, cos θ is negative. Thus $\cos\theta = -\frac{\sqrt{19}}{10}$.

To find sin 2θ, use the identity $\sin 2\theta = 2\sin\theta\cdot\cos\theta$.

$$\sin 2\theta = 2\sin\theta\cdot\cos\theta = 2\left(-\frac{9}{10}\right)\left(-\frac{\sqrt{19}}{10}\right) = \frac{9\sqrt{19}}{50}$$

The value of sin 2θ is $\frac{9\sqrt{19}}{50}$.

To find cos 2θ, use the identity $\cos 2\theta = 1 - 2\sin^2\theta$.

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(-\frac{9}{10}\right)^2 = -\frac{31}{50}.$$

The value of cos 2θ is $-\frac{31}{50}$.

Exercises

Find the exact values of sin 2θ and cos 2θ for each of the following.

1. $\sin\theta = \frac{1}{4}, 0° < \theta < 90°$ $\frac{\sqrt{15}}{8}, \frac{7}{8}$

2. $\sin\theta = -\frac{1}{8}, 270° < \theta < 360°$ $-\frac{3\sqrt{7}}{32}, \frac{31}{32}$

3. $\cos\theta = -\frac{3}{5}, 180° < \theta < 270°$ $\frac{24}{25}, -\frac{7}{25}$

4. $\cos\theta = -\frac{4}{5}, 90° < \theta < 180°$ $-\frac{24}{25}, \frac{7}{25}$

5. $\sin\theta = -\frac{3}{5}, 270° < \theta < 360°$ $-\frac{24}{25}, \frac{7}{25}$

6. $\cos\theta = -\frac{2}{3}, 90° < \theta < 180°$ $-\frac{4\sqrt{5}}{9}, -\frac{1}{9}$

Skills Practice, p. 869 and Practice, p. 870 (shown)

Find the exact values of sin 2θ, cos 2θ, sin $\frac{\theta}{2}$, and cos $\frac{\theta}{2}$ for each of the following.

1. $\cos\theta = \frac{5}{13}, 0° < \theta < 90°$ $\frac{120}{169}, -\frac{119}{169}, \frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}$

2. $\sin\theta = \frac{8}{17}, 90° < \theta < 180°$ $-\frac{240}{289}, \frac{161}{289}, \frac{4\sqrt{17}}{17}, \frac{\sqrt{17}}{17}$

3. $\cos\theta = \frac{1}{4}, 270° < \theta < 360°$ $-\frac{\sqrt{15}}{8}, -\frac{7}{8}, \frac{\sqrt{6}}{4}, \frac{\sqrt{10}}{4}$

4. $\sin\theta = -\frac{2}{3}, 180° < \theta < 270°$ $\frac{4\sqrt{5}}{9}, \frac{1}{9}, \frac{\sqrt{18+6\sqrt{5}}}{6}, -\frac{\sqrt{18-6\sqrt{5}}}{6}$

Find the exact value of each expression by using the half-angle formulas.

5. tan 105° $-2 - \sqrt{3}$

6. tan 15° $2 - \sqrt{3}$

7. cos 67.5° $\frac{\sqrt{2-\sqrt{2}}}{2}$

8. $\sin\left(-\frac{\pi}{8}\right)$ $-\frac{\sqrt{2-\sqrt{2}}}{2}$

Verify that each of the following is an identity.

9. $\sin^2\frac{\theta}{2} = \frac{\tan\theta - \sin\theta}{2\tan\theta}$

$$\left(\pm\sqrt{\frac{1-\cos\theta}{2}}\right)^2 \overset{?}{=} \frac{\tan\theta - \sin\theta}{2\tan\theta};\quad \frac{1-\cos\theta}{2} \overset{?}{=} \frac{\frac{\tan\theta}{\tan\theta} - \frac{\sin\theta}{\tan\theta}}{2\frac{\tan\theta}{\tan\theta}};\quad \frac{1-\cos\theta}{2} = \frac{1-\cos\theta}{2}$$

10. $\sin 4\theta = 4\cos 2\theta\sin\theta\cos\theta$

$$\sin 4\theta \overset{?}{=} 4\cos 2\theta\sin\theta\cos\theta$$
$$\sin 2(2\theta) \overset{?}{=} 4\cos 2\theta\sin\theta\cos\theta$$
$$2\sin 2\theta\cos 2\theta \overset{?}{=} 4\cos 2\theta\sin\theta\cos\theta$$
$$2(2\sin\theta\cos\theta)(\cos 2\theta) \overset{?}{=} 4\cos 2\theta\sin\theta\cos\theta$$
$$4\cos 2\theta\sin\theta\cos\theta = 4\cos 2\theta\sin\theta\cos\theta$$

11. AERIAL PHOTOGRAPHY In aerial photography, there is a reduction in film exposure for any point X not directly below the camera. The reduction E_θ is given by $E_\theta = E_0\cos^4\theta$, where θ is the angle between the perpendicular line from the camera to the ground and the line from the camera to point X, and E_0 is the exposure for the point directly below the camera. Using the identity $2\sin^2\theta = 1 - \cos 2\theta$, verify that $E_0\cos^4\theta = E_0\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right)^2$.

$$E_0\cos^4\theta = E_0(\cos^2\theta)^2 = E_0(1 - \sin^2\theta)^2 = E_0\left(1 - \frac{2\sin^2\theta}{2}\right)^2 = E_0\left(1 - \frac{1-\cos 2\theta}{2}\right)^2 = E_0\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right)^2$$

12. IMAGING A scanner takes thermal images from altitudes of 300 to 12,000 meters. The width W of the swath covered by the image is given by $W = 2H'\tan\theta$, where H' is the height and θ is half the scanner's field of view. Verify that $\frac{2H'\sin 2\theta}{1+\cos 2\theta} = 2H'\tan\theta$.

$$\frac{2H'\sin 2\theta}{1+\cos 2\theta} = \frac{4H'\sin\theta\cos\theta}{1+(2\cos^2\theta - 1)} = \frac{4H'\sin\theta\cos\theta}{2\cos^2\theta} = \frac{2H'\sin\theta}{\cos\theta} = 2H'\tan\theta$$

Reading to Learn Mathematics, p. 871

ELL

Pre-Activity **How can trigonometric functions be used to describe music?**

Read the introduction to Lesson 14-6 at the top of page 791 in your textbook.

Suppose that the equation for the second harmonic is $y = \sin a\theta$. Then what would be the equations for the fundamental tone (first harmonic), third harmonic, fourth harmonic, and fifth harmonic?
$y = \sin 0.5a\theta$; $y = \sin 1.5a\theta$; $y = \sin 2a\theta$; $y = \sin 2.5a\theta$

Reading the Lesson

1. Match each expression from the list on the left with *all* expressions from the list on the right that are equal to it for all values of β.

a. $\sin\frac{\beta}{2}$ v	**i.** $2\sin\beta\cos\beta$
b. $\cos 2\beta$ ii and iii	**ii.** $1 - 2\sin^2\beta$
c. $\cos\frac{\beta}{2}$ iv	**iii.** $\cos^2\beta - \sin^2\beta$
d. $\sin 2\beta$ i	**iv.** $\pm\sqrt{\frac{1+\cos\beta}{2}}$
	v. $\pm\sqrt{\frac{1-\cos\beta}{2}}$

2. Determine whether you would use the *positive* or *negative* square root in the half-angle identities for $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$ in each of the following situations. (Do not actually calculate $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$.)

a. $\sin\frac{\alpha}{2}$, if $\cos\alpha = \frac{2}{5}$ and α is in Quadrant I positive

b. $\cos\frac{\alpha}{2}$, if $\cos\alpha = -0.9$ and α is in Quadrant II positive

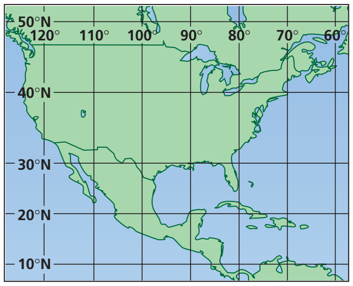
c. $\cos\frac{\alpha}{2}$, if $\sin\alpha = -0.75$ and α is in Quadrant III negative

d. $\sin\frac{\alpha}{2}$, if $\sin\alpha = -0.8$ and α is in Quadrant IV positive

Helping You Remember

3. Many students find it difficult to remember a large number of identities. How can you obtain all three of the identities for cos 2θ by remembering only one of them and using a Pythagorean identity?
Sample answer: Just remember the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$. Using the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$, you can substitute either $1 - \sin^2\theta$ for $\cos^2\theta$ or $1 - \cos^2\theta$ for $\sin^2\theta$ to get the other two identities for cos 2θ.

GEOGRAPHY **For Exercises 38 and 39, use the following information.**
A Mercator projection map uses a flat projection of Earth in which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection uses the expression $\tan\left(45° + \frac{L}{2}\right)$, where L is the latitude of the point.



38. Write this expression in terms of a trigonometric function of L.

$$38.\ \frac{1 \pm \sqrt{\frac{1-\cos L}{1+\cos L}}}{1 \mp \sqrt{\frac{1-\cos L}{1+\cos L}}}$$

39. Find the exact value of the expression if $L = 60°$. $2 + \sqrt{3}$

PHYSICS **For Exercises 40 and 41, use the following information.**
An object is propelled from ground level with an initial velocity of v at an angle of elevation θ. **40. See pp. 811A–811N.**

40. The horizontal distance d it will travel can be determined using $d = \frac{v^2\sin 2\theta}{g}$, where g is acceleration due to gravity. Verify that this expression is the same as $\frac{2}{g}v^2(\tan\theta - \tan\theta\sin^2\theta)$.

41. The maximum height h the object will reach can be determined using the formula $h = \frac{v^2\sin^2\theta}{2g}$. Find the ratio of the maximum height attained to the horizontal distance traveled. $\frac{1}{4}\tan\theta$

CRITICAL THINKING **For Exercises 42–46, use the following information.**
Consider the functions $f(x) = \sin 2x$, $g(x) = \sin^2 x$, $h(x) = -\cos^2 x$, and $k(x) = -\frac{1}{2}\cos 2x$. **42–46. See pp. 811A–811N.**

42. Draw the graphs of $y = g(x)$, $y = h(x)$, and $y = k(x)$ on the same coordinate plane on the interval from $x = -2\pi$ to $x = 2\pi$. What do you notice about the graphs?

43. Where do the maxima and minima of g, h, and k occur?

44. Draw the graph of $y = f(x)$ on a separate coordinate plane.

45. What is the behavior of the graph of $f(x)$ at the locations found in Exercise 43?

46. Use what you know about transformations to determine c and d so that $g(x) = h(x) + c = k(x) + d$.

47. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson. **See margin.**

How can trigonometric functions be used to describe music?

Include the following in your answer:

- a description of what happens to the graph of the function of a vibrating string as it moves from one harmonic to the next, and
- an explanation of what happens to the period of the function as you move from the nth harmonic to the $(n + 1)$th harmonic.

Standardized Test Practice

Ⓐ Ⓑ Ⓒ Ⓓ

48. Find the exact value of $\cos 2\theta$ if $\sin\theta = \frac{-\sqrt{5}}{3}$ and $180° < \theta < 270°$. **D**

Ⓐ $\frac{-\sqrt{6}}{6}$ Ⓑ $\frac{-\sqrt{30}}{6}$ Ⓒ $\frac{-4\sqrt{5}}{9}$ Ⓓ $\frac{-1}{9}$

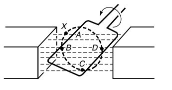
49. Find the exact value of $\sin\frac{\theta}{2}$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $0° < \theta < 90°$. **B**

Ⓐ $\frac{\sqrt{3}}{2}$ Ⓑ $\frac{\sqrt{2-\sqrt{3}}}{2}$ Ⓒ $\frac{\sqrt{2+\sqrt{3}}}{2}$ Ⓓ $\frac{1}{2}$

Enrichment, p. 872

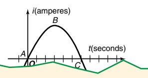
Alternating Current

The figure at the right represents an alternating current generator. A rectangular coil of wire is suspended between the poles of a magnet. As the coil of wire is rotated, it passes through the magnetic field and generates current.



As point X on the coil passes through the points A and C, its motion is along the direction of the magnetic field between the poles. Therefore, no current is generated. However, through points $B$and D, the motion of X is perpendicular to the magnetic field. This induces maximum current in the coil. Between A and B, B and C, C and D, and D and A, the current in the coil will have an intermediate value. Thus, the graph of the current of an alternating current generator is closely related to the sine curve.

The maximum current may have a positive or negative value.



The actual current, i, in a household current is given

Maintain Your Skills

Mixed Review

52. $-\frac{\sqrt{2}}{2}$

53. $-\frac{\sqrt{3}}{2}$

Find the exact value of each expression. *(Lesson 14-5)*

50. $\cos 15°$ $\frac{\sqrt{6}+\sqrt{2}}{4}$

51. $\sin 15°$ $\frac{\sqrt{6}-\sqrt{2}}{4}$

52. $\sin(-135°)$

53. $\cos 150°$

54. $\sin 105°$ $\frac{\sqrt{6}+\sqrt{2}}{4}$

55. $\cos(-300°)$ $\frac{1}{2}$

Verify that each of the following is an identity. *(Lesson 14-4)*

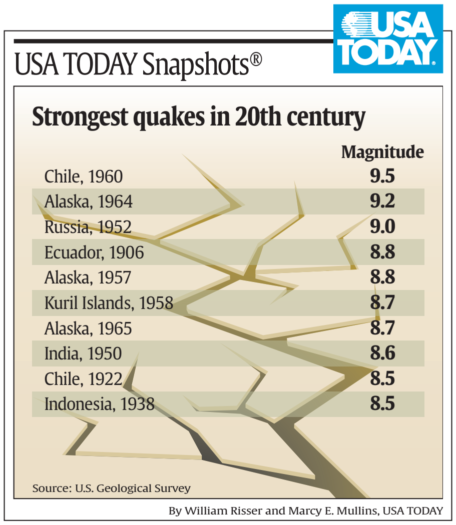
56. $\cot^2\theta - \sin^2\theta = \frac{\cos^2\theta\csc^2\theta - \sin^2\theta}{\sin^2\theta\csc^2\theta}$ **See pp. 811A–811N.**

57. $\cos\theta(\cos\theta + \cot\theta) = \cot\theta\cos\theta(\sin\theta + 1)$ **See pp. 811A–811N.**

EARTHQUAKE **For Exercises 58 and 59, use the following information.**
The magnitude of an earthquake M measured on the Richter scale is given by $M = \log_{10}x$, where x represents the amplitude of the seismic wave causing ground motion. *(Lesson 10-2)* **58. 10^1 or 10**

58. How many times as great was the 1960 Chile earthquake as the 1938 Indonesia earthquake?

59. The largest aftershock of the 1964 Alaskan earthquake was 6.7 on the Richter scale. How many times as great was the main earthquake as this aftershock? **$10^{2.5}$ or about 316 times as great**



USA TODAY Snapshots®

Strongest quakes in 20th century

	Magnitude
Chile, 1960	9.5
Alaska, 1964	9.2
Russia, 1952	9.0
Ecuador, 1906	8.8
Alaska, 1957	8.8
Kuril Islands, 1958	8.7
Alaska, 1965	8.7
India, 1950	8.6
Chile, 1922	8.5
Indonesia, 1938	8.5

Source: U.S. Geological Survey

By William Risser and Marcy E. Mullins, USA TODAY

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation.**
*(To review **solving equations using the Zero Product Property**, see Lesson 6-3.)*

60. $(x + 6)(x - 5) = 0$ **−6, 5**

61. $(x - 1)(x + 1) = 0$ **1, −1**

62. $x(x + 2) = 0$ **0, −2**

63. $(2x - 5)(x + 2) = 0$ $\frac{5}{2}, -2$

64. $(2x + 1)(2x - 1) = 0$ $-\frac{1}{2}, \frac{1}{2}$

65. $x^2(2x + 1) = 0$ $0, -\frac{1}{2}$

Practice Quiz 2

Lessons 14-4 through 14-6

Verify that each of the following is an identity. *(Lessons 14-5)* **1–3. See pp. 811A–811N.**

1. $\sin\theta\sec\theta = \tan\theta$

2. $\sec\theta - \cos\theta = \sin\theta\tan\theta$

3. $\sin\theta + \tan\theta = \frac{\sin\theta(\cos\theta + 1)}{\cos\theta}$

Verify that each of the following is an identity. *(Lessons 14-4 and 14-5)* **4–6. See pp. 811A–811N.**

4. $\sin(90° + \theta) = \cos\theta$

5. $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

6. $\sin(\theta + 30°) + \cos(\theta + 60°) = \cos\theta$

Find the exact value of each expression by using the double-angle or half-angle formulas. *(Lesson 14-6)*

7. $\sin 2\theta$ if $\cos\theta = -\frac{\sqrt{3}}{2}$; $180° < \theta < 270°$ $\frac{\sqrt{3}}{2}$

8. $\cos\frac{\theta}{2}$ if $\sin\theta = -\frac{9}{41}$; $270° \le \theta < 360°$ $-\frac{9\sqrt{82}}{82}$

9. $\sin 165°$ $\frac{\sqrt{2-\sqrt{3}}}{2}$

10. $\cos\frac{5\pi}{8}$ $-\frac{\sqrt{2-\sqrt{2}}}{2}$

4 Assess

Open-Ended Assessment

Writing Have students write their own problems like Examples 2 and 3, and have them write an explanation of how to use a double-angle or half-angle formula to solve their examples.

Getting Ready for Lesson 14-7

PREREQUISITE SKILL In Lesson 14-7, students will solve trigonometric equations using the Zero Product Property. Use Exercises 60–65 to determine your students' familiarity with the Zero Product Property.

Assessment Options

Practice Quiz 2 The quiz provides students with a brief review of the concepts and skills in Lessons 14-4 through 14-6. Lesson numbers are given to the right of the exercises or instruction lines so students can review concepts not yet mastered.

Quiz (Lessons 14-5 and 14-6) is available on p. 894 of the *Chapter 14 Resource Masters*.

Answer (p. 796)

47. Sample answer: The sound waves associated with music can be modeled using trigonometric functions. Answers should include the following information.

- **In moving from one harmonic to the next, the number of vibrations that appear as sine waves increases by 1.**
- **The period of the function as you move from the nth harmonic to the $(n + 1)$th harmonic decreases from $\frac{2\pi}{n}$ to $\frac{2\pi}{n+1}$.**

Online Lesson Plans

USA TODAY Education's Online site offers resources and interactive features connected to each day's newspaper. *Experience TODAY*, USA TODAY's daily lesson plan, is available on the site and delivered daily to subscribers. This plan provides instruction for integrating USA TODAY graphics and key editorial features into your mathematics classroom. Log on to www.education.usatoday.com.

Graphing Calculator Investigation

A Preview of Lesson 14-7

Getting Started

Approximate Solutions In Example 1, approximate solutions can also be found by using the Trace feature. In most situations however, the Intersect feature will give more accurate solutions.

Teach

- You might wish to demonstrate the technique shown in Example 1 by first using a simple quadratic equation like $x^2 = 9$, whose solutions students will readily know. Graph $y = x^2$ and $y = 9$ to see that they intersect at two points, where $x = -3$ and $x = 3$, just as students will expect.
- Remind students that the solutions of the equation are the x values of the points of intersection, not the y values.
- If the expression on the right side of an equation is just 0 (as in Exercises 3 and 6), you can graph the function for the left side of the equation and then just use the Zero feature on the CALC menu to find approximate solutions.

Assess

In **Exercises 3–4**, check to see that students can explain why there are no real solutions.

In **Exercise 6**, make sure students found all four possible values. Students who found fewer than four are not using the correct domain for x.

Graphing Calculator Investigation

A Preview of Lesson 14-7

Solving Trigonometric Equations

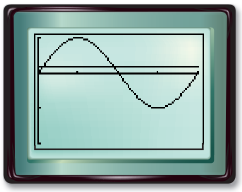
The graph of a trigonometric function is made up of points that represent all values that satisfy the function. To solve a trigonometric equation, you need to find all values of the variable that satisfy the equation. You can use a TI-83 Plus to solve trigonometric equations by graphing each side of the equation as a function and then locating the points of intersection.

Example 1 Use a graphing calculator to solve $\sin x = 0.2$ if $0° \leq x < 360°$.

Rewrite the equation as two functions, $y = \sin x$ and $y = 0.2$. Then graph the two functions. Look for the point of intersection.

Make sure that your calculator is in degree mode to get the correct viewing window.

KEYSTROKES: MODE ▼ ▼ ▶ ENTER WINDOW 0 ENTER 360 ENTER 90 ENTER −2 ENTER 1 ENTER 1 ENTER Y= SIN X,T,θ,n ENTER 0.2 ENTER GRAPH



[0, 360] scl: 90 by [−2, 1] scl: 1

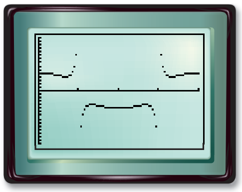
Based on the graph, you can see that there are two points of intersection in the interval $0° \leq x < 360°$. Use Zoom or 2nd [CALC] 5 to approximate the solutions. The approximate solutions are 168.5° and 11.5°.

Like other equations you have studied, some trigonometric equations have no real solutions. Carefully examine the graphs over their respective periods for points of intersection. If there are no points of intersection, then the trigonometric equation has no real solutions.

Example 2 Use a graphing calculator to solve $\tan^2 x \cos x + 5 \cos x = 0$ if $0° \leq x < 360°$.

Because the tangent function is not continuous, place the calculator in Dot mode. The related functions to be graphed are $y = \tan^2 x \cos x + 5 \cos x$ and $y = 0$.

These two functions do not intersect. Therefore, the equation $\tan^2 x \cos x + 5 \cos x = 0$ has no real solutions.



[0, 360] scl: 90 by [−15, 15] scl: 1

1–6. See pp. 811A–811N for graphs.

Exercises 1. 53.1°, 126.9° 3. no real solution 4. no real solution

Use a graphing calculator to solve each equation for the values of x indicated.

1. $\sin x = 0.8$ if $0° \leq x < 360°$

2. $\tan x = \sin x$ if $0° \leq x < 360°$ 0°, 180°

3. $2 \cos x + 3 = 0$ if $0° \leq x < 360°$

4. $0.5 \cos x = 1.4$ if $-720° \leq x < 720°$

5. $\sin 2x = \sin x$ if $0° \leq x < 360°$ 60°, 180°, 300°

6. $\sin 2x - 3 \sin x = 0$ if $-360° \leq x < 360°$ −360°, −180°, 0°, 180°

www.algebra2.com/other_calculator_keystrokes

14-7 Solving Trigonometric Equations

What You'll Learn

- Solve trigonometric equations.
- Use trigonometric equations to solve real-world problems.

Vocabulary

- trigonometric equation

How can trigonometric equations be used to predict temperature?

The average daily high temperature for a region can be described by a trigonometric function. For example, the average daily high temperature for each month in Orlando, Florida, can be modeled by the function $T = 11.56 \sin (0.4516x - 1.641) + 80.89$, where T represents the average daily high temperature in degrees Fahrenheit and x represents the month of the year. This equation can be used to predict the months in which the average temperature in Orlando will be at or above a desired temperature.

SOLVE TRIGONOMETRIC EQUATIONS You have seen that trigonometric identities are true for *all* values of the variable for which the equation is defined. However, most **trigonometric equations** like some algebraic equations, are true for *some* but not *all* values of the variable.

Example 1 Solve Equations for a Given Interval

Find all solutions of each equation for the given interval.

a. $\cos^2 \theta = 1; 0° \leq \theta < 360°$

$$\begin{aligned} \cos^2 \theta &= 1 && \text{Original equation} \\ \cos^2 \theta - 1 &= 0 && \text{Solve for 0.} \\ (\cos \theta + 1)(\cos \theta - 1) &= 0 && \text{Factor.} \end{aligned}$$

Now use the Zero Product Property.

$$\begin{aligned} \cos \theta + 1 &= 0 \quad \text{or} & \cos \theta - 1 &= 0 \\ \cos \theta &= -1 & \cos \theta &= 1 \\ \theta &= 180° & \theta &= 0° \end{aligned}$$

The solutions are 0° and 180°.

b. $\sin 2\theta = 2 \cos \theta; 0 \leq \theta < 2\pi$

$$\begin{aligned} \sin 2\theta &= 2 \cos \theta && \text{Original equation} \\ 2 \sin \theta \cos \theta &= 2 \cos \theta && \sin 2\theta = 2 \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta - 2 \cos \theta &= 0 && \text{Solve for 0.} \\ 2 \cos \theta (\sin \theta - 1) &= 0 && \text{Factor.} \end{aligned}$$

(continued on the next page)

14-7 Lesson Notes

1 Focus

5-Minute Check Transparency 14-7 Use as a quiz or review of Lesson 14-6.

Mathematical Background notes are available for this lesson on page 760D.

How can trigonometric equations be used to predict temperature?

Ask students:

- Why would temperature not be modeled by a quadratic function? **Sample answer: Temperature varies periodically and does not continue upward or downward to infinity. So temperature should be modeled by a trigonometric, not a quadratic, function.**
- What can you tell about the range of temperatures by studying the function T? **The temperature varies 11.56 degrees (the amplitude of the function) above and below a temperature of 80.89° F (the midline of the function).**
- What can you tell about the maximum temperature by studying the function? **The maximum temperature is 11.56 + 80.89, or 92.45° F.**

Resource Manager

Workbook and Reproducible Masters

Chapter 14 Resource Masters

- Study Guide and Intervention, pp. 873–874
- Skills Practice, p. 875
- Practice, p. 876
- Reading to Learn Mathematics, p. 877
- Enrichment, p. 878
- Assessment, p. 894

School-to-Career Masters, p. 28

Science and Mathematics Lab Manual, pp. 145–148

Teaching Algebra With Manipulatives Masters, p. 304

Transparencies

5-Minute Check Transparency 14-7
Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

Building on Prior Knowledge

In Lesson 6-3, students learned to use the Zero Product Property to solve equations. In this lesson, students will use the Zero Product Property to solve trigonometric equations.

SOLVE TRIGONOMETRIC EQUATIONS

In-Class Examples

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1 Find all solutions of each equation for the given interval.

a. $2\cos^2\theta - 1 = \sin\theta$; $0° < \theta \le 360°$ **30°, 150°, 270°**

b. $\sin\theta = \sin 2\theta$; $0 < \theta \le 2\pi$ **$\frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$**

Teaching Tip In Example 2, show students how to look for patterns in the solution of part a. Students should look for pairs of solutions that differ by exactly π or 2π.

2

a. Solve $2\sin\theta\cos\theta = \cos\theta$ for all values of θ if θ is measured in radians. **$\frac{\pi}{6} + 2k\pi, \frac{\pi}{2} + 2k\pi, \frac{5\pi}{6} + 2k\pi, \frac{3\pi}{2} + 2k\pi$, where k is any integer**

b. Solve $\cos\theta = -\cos 2\theta$ for all values of θ if θ is measured in degrees. **$60° + k \cdot 360°$, $180° + k \cdot 360°$, $300° + k \cdot 360°$, where k is any integer**

Use the Zero Product Property.

$$2\cos\theta = 0 \quad \text{or} \quad \sin\theta - 1 = 0$$
$$\cos\theta = 0 \qquad \sin\theta = 1$$
$$\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \qquad \theta = \frac{\pi}{2}$$

The solutions are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Trigonometric equations are usually solved for values of the variable between 0° and 360° or 0 radians and 2π radians. There are solutions outside that interval. These other solutions differ by integral multiples of the period of the function.

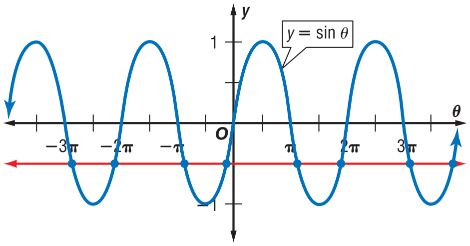
Example 2 Solve Trigonometric Equations

a. Solve $2\sin\theta = -1$ for all values of θ if θ is measured in radians.

$2\sin\theta = -1$ Original equation

$\sin\theta = -\frac{1}{2}$ Divide each side by 2.

Look at the graph of $y = \sin\theta$ to find solutions of $\sin\theta = -\frac{1}{2}$.



The solutions are $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$, and so on, and $-\frac{7\pi}{6}, -\frac{11\pi}{6}, -\frac{19\pi}{6}, -\frac{23\pi}{6}$, and so on. The only solutions in the interval 0 to 2π are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The period of the sine function is 2π radians. So the solutions can be written as $\frac{7\pi}{6} + 2k\pi$ and $\frac{11\pi}{6} + 2k\pi$, where k is any integer.

b. Solve $\cos 2\theta + \cos\theta + 1 = 0$ for all values of θ if θ is measured in degrees.

$\cos 2\theta + \cos\theta + 1 = 0$ Original equation

$2\cos^2\theta - 1 + \cos\theta + 1 = 0$ $\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta + \cos\theta = 0$ Simplify.

$\cos\theta\,(2\cos\theta + 1) = 0$ Factor.

Solve for θ in the interval 0° to 360°.

$$\cos\theta = 0 \quad \text{or} \quad 2\cos\theta + 1 = 0$$
$$\theta = 90° \text{ or } 270° \qquad 2\cos\theta = -1$$
$$\cos\theta = -\frac{1}{2}$$
$$\theta = 120° \text{ or } 240°$$

The solutions are $90° + k \cdot 180°$, $120° + k \cdot 360°$, and $240° + k \cdot 360°$.

Study Tip

Expressing Solutions as Multiples

The expression $90° + k \cdot 180°$ includes 270° and its multiples, so it is not necessary to list them separately.

If an equation cannot be solved easily by factoring, try rewriting the expression using trigonometric identities. However, using identities and some algebraic operations, such as squaring, may result in extraneous solutions. So, it is necessary to check your solutions using the original equation.

Example 3 Solve Trigonometric Equations Using Identities

Solve $\cos\theta\tan\theta - \sin^2\theta = 0$.

$\cos\theta\tan\theta - \sin^2\theta = 0$ Original equation

$\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right) - \sin^2\theta = 0$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta - \sin^2\theta = 0$ Multiply.

$\sin\theta(1 - \sin\theta) = 0$ Factor.

$\sin\theta = 0$ or $1 - \sin\theta = 0$

$\theta = 0°, 180°, \text{or } 360°$ $\quad\sin\theta = 1$

$\theta = 90°$

CHECK

$\cos\theta\tan\theta - \sin^2\theta = 0$

$\cos 0°\tan 0° - \sin^2 0° \stackrel{?}{=} 0$ $\theta = 0°$

$1 \cdot 0 - 0 \stackrel{?}{=} 0$

$0 = 0$ ✓

$\cos\theta\tan\theta - \sin^2\theta = 0$

$\cos 180°\tan 180° - \sin^2 180° \stackrel{?}{=} 0$ $\theta = 180°$

$-1 \cdot 0 - 0 \stackrel{?}{=} 0$

$0 = 0$ ✓

$\cos\theta\tan\theta - \sin^2\theta = 0$

$\cos 360°\tan 360° - \sin^2 360° \stackrel{?}{=} 0$ $\theta = 360°$

$1 \cdot 0 - 0 \stackrel{?}{=} 0$

$0 = 0$ ✓

$\cos\theta\tan\theta - \sin^2\theta = 0$

$\cos 90°\tan 90° - \sin^2 90° \stackrel{?}{=} 0$ $\theta = 90°$

tan 90° is undefined.

Thus, 90° is not a solution.

The solution is $0° + k \cdot 180°$.

Some trigonometric equations have no solution. For example, the equation $\cos x = 4$ has no solution since all values of $\cos x$ are between -1 and 1, inclusive. Thus, the solution set for $\cos x = 4$ is empty.

Example 4 Determine Whether a Solution Exists

Solve $3\cos 2\theta - 5\cos\theta = 1$.

$3\cos 2\theta - 5\cos\theta = 1$ Original equation

$3(2\cos^2\theta - 1) - 5\cos\theta = 1$ $\cos 2\theta = 2\cos^2\theta - 1$

$6\cos^2\theta - 3 - 5\cos\theta = 1$ Multiply.

$6\cos^2\theta - 5\cos\theta - 4 = 0$ Subtract 1 from each side.

$(3\cos\theta - 4)(2\cos\theta + 1) = 0$ Factor.

$3\cos\theta - 4 = 0$ or $2\cos\theta + 1 = 0$

$3\cos\theta = 4$ $\quad 2\cos\theta = -1$

$\cos\theta = \frac{4}{3}$ $\quad\cos\theta = -\frac{1}{2}$

Not possible since $\cos\theta$ cannot be greater than 1. $\quad\theta = 120° \text{ or } 240°$

Thus, the solutions are $120° + k \cdot 360°$ and $240° + k \cdot 360°$.

www.algebra2.com/extra_examples

In-Class Examples

Power Point®

3 Solve $\sin\theta\cot\theta = \cos^2\theta$. **$90° + k \cdot 180°$, where k is any integer**

4 Solve $\sin^2\theta = \frac{1}{4} + \cos\theta$. **$60° + k \cdot 360°$, $300° + k \cdot 360°$, where k is any integer**

DAILY INTERVENTION

Differentiated Instruction

Interpersonal As students work through this lesson, have them create a class list on the chalkboard that identifies common errors they made. Encourage students to add suggestions for how to avoid their errors. For example, one common error is having one's calculator set to degrees when it needs to be set to radians for a problem, and vice versa.

USE TRIGONOMETRIC EQUATIONS

In-Class Example

Power Point®

Teaching Tip As students work Example 5, some of them may calculate $\sin^{-1} 0.392$ as 23.079. Point out that this result occurs when their calculator is set to degrees instead of radians.

5 **GARDENING** Refer to Example 5 in the Student Edition. If Rhonda decides to wait until there are 15 hours of daylight, on what day can she plant her flowers? **around the 114th day of the year, or around April 24**

3 Practice/Apply

Study Notebook

Have students—

- complete the definitions/examples for the remaining terms on their Vocabulary Builder worksheets for Chapter 14.
- summarize some techniques they used for solving trigonometric equations.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective

- **Solve Trigonometric Equations:** 15–40
- **Use Trigonometric Equations:** 41–44

Odd/Even Assignments

Exercises 15–40 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 15–39 odd, 41–55

Average: 15–39 odd, 41–55

Advanced: 16–40 even, 41–55

USE TRIGONOMETRIC EQUATIONS

Trigonometric equations are often used to solve real-world situations.

Example 5 Use a Trigonometric Equation

GARDENING **Rhonda wants to wait to plant her flowers until there are at least 14 hours of daylight. The number of hours of daylight H in her town can be represented by $H = 11.45 + 6.5 \sin (0.0168d - 1.333)$, where d is the day of the year and angle measures are in radians. On what day is it safe for Rhonda to plant her flowers?**

$H = 11.45 + 6.5 \sin (0.0168d - 1.333)$ Original equation

$14 = 11.45 + 6.5 \sin (0.0168d - 1.333)$ $H = 14$

$2.55 = 6.5 \sin (0.0168d - 1.333)$ Subtract 11.45 from each side.

$0.392 = \sin (0.0168d - 1.333)$ Divide each side by 6.5.

$0.403 = 0.0168d - 1.333$ $\text{Sin}^{-1}\ 0.392 = 0.403$

$1.736 = 0.0168d$ Add 1.333 to each side.

$103.333 = d$ Divide each side by 0.0168.

Rhonda can safely plant her flowers around the 104th day of the year, or around April 14.

Check for Understanding

Concept Check

1–3. See margin.

1. **Tell** why the equation $\sec \theta = 0$ has no solutions.
2. **Explain** why the number of solutions to the equation $\sin \theta = \frac{\sqrt{3}}{2}$ is infinite.
3. **OPEN ENDED** Write an example of a trigonometric equation that has no solution.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4–7	1
8–11	2
12, 13	3
14	5

Find all solutions of each equation for the given interval. **6.** $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

4. $4 \cos^2 \theta = 1; 0° \leq \theta < 360°$ **60°, 120°, 240°, 300°**
5. $2 \sin^2 \theta - 1 = 0; 90° < \theta < 270°$ **135°, 225°**
6. $\sin 2\theta = \cos \theta; 0 \leq \theta < 2\pi$
7. $3 \sin^2 \theta - \cos^2 \theta = 0; 0 \leq \theta < \frac{\pi}{2}$ $\frac{\pi}{6}$

Solve each equation for all values of θ if θ is measured in radians.

8. $\cos 2\theta = \cos \theta$ $0 + \frac{2k\pi}{3}$
9. $\sin \theta + \sin \theta \cos \theta = 0$ $0 + k\pi$

Solve each equation for all values of θ if θ is measured in degrees.

10. $\sin \theta = 1 + \cos \theta$ $90° + k \cdot 360°, 180° + k \cdot 360°$
11. $2 \cos^2 \theta + 2 = 5 \cos \theta$ $60° + k \cdot 360°, 300 + k \cdot 360°$

Solve each equation for all values of θ.

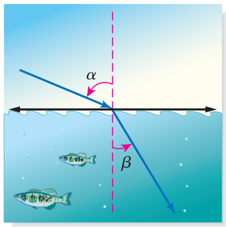
12. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$
13. $2 \cos^2 \theta + 3 \sin \theta - 3 = 0$

12. $\frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi$ or $210° + k \cdot 360°, 330° + k \cdot 360°$

Application

13. $\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi, \frac{\pi}{2} + 2k\pi$ or $30° + k \cdot 360°, 150° + k \cdot 360°, 90° + k \cdot 360°$

14. **PHYSICS** According to Snell's law, the angle at which light enters water α is related to the angle at which light travels in water β by the equation $\sin \alpha = 1.33 \sin \beta$. At what angle does a beam of light enter the water if the beam travels at an angle of 23° through the water? **31.3°**



Answers

1. Sample answer: If $\sec \theta = 0$ then $\frac{1}{\cos \theta} = 0$. Since no value of θ makes $\frac{1}{\cos \theta} = 0$, there are no solutions.
2. Sample answer: The function is periodic with two solutions in each of its infinite number of periods.
3. Sample answer: $\sin \theta = 2$

23. $\frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi$

24. $\pi + 2k\pi, \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi$

25 $\frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi$

26. $0 + 2k\pi$

27. $\frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi$

28. $0 + k\pi, \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi$

Practice and Apply

Homework Help

For Exercises	See Examples
15–22	1
23–34	2
35–40	3, 4
41–43	5

Extra Practice

See page 861.

15. 60°, 300°
16. 240°, 300°
17. 210°, 330°
18. 30°, 150°, 210°, 330°
19. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
20. $\frac{\pi}{2}$
21. $\frac{7\pi}{6}, \frac{11\pi}{6}$
22. $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$

Find all solutions of each equation for the given interval.

15. $2 \cos \theta - 1 = 0;\ 0° \leq \theta < 360°$
16. $2 \sin \theta = -\sqrt{3};\ 180° < \theta < 360°$
17. $4 \sin^2 \theta = 1;\ 180° < \theta < 360°$
18. $4 \cos^2 \theta = 3;\ 0° \leq \theta < 360°$
19. $2 \cos^2 \theta = \sin \theta + 1;\ 0 \leq \theta < 2\pi$
20. $\sin^2 \theta - 1 = \cos^2 \theta;\ 0 \leq \theta < \pi$
21. $2 \sin^2 \theta + \sin \theta = 0;\ \pi < \theta < 2\pi$
22. $2 \cos^2 \theta = -\cos \theta;\ 0 \leq \theta < 2\pi$

Solve each equation for all values of θ if θ is measured in radians.

23. $\cos 2\theta + 3 \cos \theta - 1 = 0$
24. $2 \sin^2 \theta - \cos \theta - 1 = 0$
25. $\cos^2 \theta - \frac{5}{2} \cos \theta - \frac{3}{2} = 0$
26. $\cos \theta = 3 \cos \theta - 2$
27. $4 \cos^2 \theta - 4 \cos \theta + 1 = 0$
28. $\cos 2\theta = 1 - \sin \theta$

23–28. See margin.

Solve each equation for all values of θ if θ is measured in degrees.

29. $\sin \theta = \cos \theta$
30. $\tan \theta = \sin \theta$
31. $\sin^2 \theta - 2 \sin \theta - 3 = 0$
32. $4 \sin^2 \theta - 4 \sin \theta + 1 = 0$
33. $\tan^2 \theta - \sqrt{3} \tan \theta = 0$
34. $\cos^2 \theta - \frac{7}{2} \cos \theta - 2 = 0$

29–34. See margin.

Solve each equation for all values of θ. 35–40. See pp. 811A–811N.

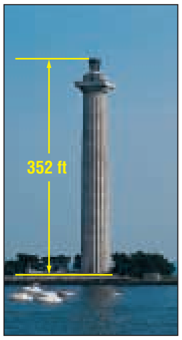
35. $\sin^2 \theta + \cos 2\theta - \cos \theta = 0$
36. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$
37. $\sin^2 \theta = \cos^2 \theta - 1$
38. $2 \cos^2 \theta + \cos \theta = 0$
39. $\sin \frac{\theta}{2} + \cos \theta = 1$
40. $\sin \frac{\theta}{2} + \cos \frac{\theta}{2} = \sqrt{2}$

41. $S = \frac{352}{\tan \theta}$ or $S = 352 \cot \theta$

LIGHT For Exercises 41 and 42, use the information shown.

41. The length of the shadow S of the International Peace Memorial at Put-In-Bay, Ohio, depends upon the angle of inclination of the Sun, θ. Express S as a function of θ.

42. Find the angle of inclination θ that will produce a shadow 560 feet long. about 32°



More About. . .

Waves

In the oceans, the height and period of water waves are determined by wind velocity, the duration of the wind, and the distance the wind has blown across the water.

Source: www.infoplease.com

WAVES For Exercises 43 and 44, use the following information.

For a short time after a wave is created by a boat, the height of the wave can be modeled using $y = \frac{1}{2}h + \frac{1}{2}h \sin \frac{2\pi t}{P}$, where h is the maximum height of the wave in feet, P is the period in seconds, and t is the propagation of the wave in seconds.

43. If $h = 3$ and $P = 2$ seconds, write the equation for the wave and draw its graph over a 10-second interval. See pp. 811A–811N.

44. How many times over the first 10 seconds does the graph predict the wave to be one foot high? 10

www.algebra2.com/self_check_quiz

Answers

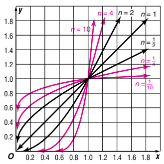
29. $45° + k \cdot 180°$
30. $0° + k \cdot 180°$
31. $270° + k \cdot 360°$
32. $30° + k \cdot 360°,\ 150° + k \cdot 360°$
33. $0° + k \cdot 180°,\ 60° + k \cdot 180°$
34. $120° + k \cdot 360°,\ 240° + k \cdot 360°$

Enrichment, p. 878

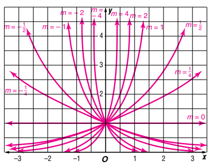
Families of Curves

Use these graphs for the problems below.

The Family $y = x^n$



The Family $y = e^{mx}$



Study Guide and Intervention, p. 873 (shown) and p. 874

Solve Trigonometric Equations You can use trigonometric identities to solve trigonometric equations, which are true for only certain values of the variable.

Example 1 **Find all solutions of $4 \sin^2 \theta - 1 = 0$ for the interval $0° < \theta < 360°$.**

$4 \sin^2 \theta - 1 = 0$
$4 \sin^2 \theta = 1$
$\sin^2 \theta = \frac{1}{4}$
$\sin \theta = \pm\frac{1}{2}$
$\theta = 30°, 150°, 210°, 330°$

Example 2 **Solve $\sin 2\theta + \cos \theta = 0$ for all values of θ. Give your answer in both radians and degrees.**

$\sin 2\theta + \cos \theta = 0$
$2 \sin \theta \cos \theta + \cos \theta = 0$
$\cos \theta\ (2 \sin \theta + 1) = 0$
$\cos \theta = 0$ or $2 \sin \theta + 1 = 0$
$\sin \theta = -\frac{1}{2}$
$\theta = 90° + k \cdot 180°;$ $\theta = 210° + k \cdot 360°,$
$\theta = \frac{\pi}{2} + k \cdot \pi$ $330° + k \cdot 360°;$
$\theta = \frac{7\pi}{6} + k \cdot 2\pi,$
$\frac{11\pi}{6} + k \cdot 2\pi$

Exercises

Find all solutions of each equation for the given interval.

1. $2 \cos^2 \theta + \cos \theta = 1, 0 \leq \theta < 2\pi$ $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$
2. $\sin^2 \theta \cos^2 \theta = 0, 0 \leq \theta < 2\pi$ $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$
3. $\cos 2\theta = \frac{\sqrt{3}}{2}, 0° \leq \theta < 360°$ 15°, 165°, 195°, 345°
4. $2 \sin \theta - \sqrt{3} = 0, 0 \leq \theta < 2\pi$ $\frac{\pi}{3}, \frac{2\pi}{3}$

Solve each equation for all values of θ if θ is measured in radians.

5. $4 \sin^2 \theta - 3 = 0$ $\frac{\pi}{3} + k \cdot \pi, \frac{2\pi}{3} + k \cdot \pi$
6. $2 \cos \theta \sin \theta + \cos \theta = 0$ $\frac{\pi}{2} + k \cdot 2\pi, \frac{3\pi}{2} + k \cdot 2\pi, \frac{7\pi}{6} + k \cdot 2\pi, \frac{11\pi}{6} + k \cdot 2\pi$

Solve each equation for all values of θ if θ is measured in degrees.

7. $\cos 2\theta + \sin^2 \theta = \frac{1}{2}$ $45° + k \cdot 90°$
8. $\tan 2\theta = -1$ $67.5° + k \cdot 360°, 157.5° + k \cdot 360°$

Skills Practice, p. 875 and Practice, p. 876 (shown)

Find all solutions of each equation for the given interval.

1. $\sin 2\theta = \cos \theta, 90° \leq \theta < 180°$ 90°, 150°
2. $\sqrt{2} \cos \theta = \sin 2\theta\ , 0° \leq \theta < 360°$ 45°, 90°, 135°, 270°
3. $\cos 4\theta = \cos 2\theta, 180° \leq \theta < 360°$ 180°, 240°, 300°
4. $\cos \theta + \cos (90 - \theta) = 0, 0 \leq \theta < 2\pi$ $\frac{3\pi}{4}, \frac{7\pi}{4}$
5. $2 + \cos \theta = 2 \sin^2 \theta, \pi \leq \theta \leq \frac{3\pi}{2}$ $\frac{4\pi}{3}, \frac{3\pi}{2}$
6. $\tan^2 \theta + \sec \theta = 1, \frac{\pi}{2} \leq \theta < \pi$ $\frac{2\pi}{3}$

Solve each equation for all values of θ if θ is measured in radians.

7. $\cos^2 \theta = \sin^2 \theta$ $\frac{\pi}{4} + k\frac{\pi}{2}$
8. $\cot \theta = \cot^3 \theta$ $\frac{\pi}{2} + k\pi$ and $\frac{\pi}{4} + k\frac{\pi}{2}$
9. $\sqrt{2} \sin^3 \theta = \sin^2 \theta$ $k\pi, \frac{\pi}{4} + 2k\pi$, and $\frac{3\pi}{4} + 2k\pi$
10. $\cos^2 \theta \sin \theta = \sin \theta$ $k\pi$
11. $2 \cos 2\theta = 1 - 2 \sin^2 \theta$ $\frac{\pi}{4} + k\frac{\pi}{2}$
12. $\sec^2 \theta = 2$ $\frac{\pi}{4} + k\frac{\pi}{2}$

Solve each equation for all values of θ if θ is measured in degrees.

13. $\sin^2 \theta \cos \theta = \cos \theta$ $90° + k \cdot 180°$
14. $\csc^2 \theta - 3 \csc \theta + 2 = 0$ $30° + k \cdot 360°, 90° + k \cdot 360°$, and $150° + k \cdot 360°$
15. $\frac{3}{1 + \cos \theta} = 4(1 - \cos \theta)$ $60° + k \cdot 180°$ and $120° + k \cdot 180°$
16. $\sqrt{2} \cos^2 \theta = \cos^2 \theta$ $90° + k \cdot 180°$ and $450° + k \cdot 360°$

Solve each equation for all values of θ.

17. $4 \sin^2 \theta = 3$ $\frac{\pi}{3} + k\pi$ and $\frac{2\pi}{3} + k\pi$, or $60° + k \cdot 180°$ and $120° + k \cdot 180°$
18. $4 \sin^2 \theta - 1 = 0$ $\frac{\pi}{6} + k\pi$ and $\frac{5\pi}{6} + k\pi$, or $30° + k \cdot 180°$ and $150° + k \cdot 180°$
19. $2 \sin^2 \theta - 3 \sin \theta = -1$ $\frac{\pi}{6} + \frac{\pi k}{3}$, or $30° + k \cdot 60°$
20. $\cos 2\theta + \sin \theta - 1 = 0$ $k\pi$ and $\frac{\pi}{6} + 2k\pi$, or $k \cdot 180°$ and $30° + k \cdot 360°$
21. **WAVES** Waves are causing a buoy to float in a regular pattern in the water. The vertical position of the buoy can be described by the equation $h = 2 \sin x$. Write an expression that describes the position of the buoy when its height is at its midline. $k\pi$ or $k \cdot 180°$
22. **ELECTRICITY** The electric current in a certain circuit with an alternating current can be described by the formula $i = 3 \sin 240t$, where i is the current in amperes and t is the time in seconds. Write an expression that describes the times at which there is no current. $0.75kt$

Reading to Learn Mathematics, p. 877 ELL

Pre-Activity **How can trigonometric equations be used to predict temperature?**

Read the introduction to Lesson 14-7 at the top of page 799 in your textbook.

Describe how you could use a graphing calculator to determine the months in which the average daily high temperature is above 80°F. (Assume that $x = 1$ represents January.) Specify the graphing window that you would use. Sample answer: Graph the functions $y = 11.56 \sin (0.4516x - 1.641) + 80.89$ (using radian mode) and $y = 80$ on the same screen. Use the window [1, 12] by [60, 100] with Xscl = 1 and Yscl = 4. Note the x values for which the curve is above the horizontal line.

Reading the Lesson

1. Identify which equations have no solution. C, E, and G

A. $\sin \theta = 1$
B. $\tan \theta = 0.001$
C. $\sec \theta = \frac{1}{2}$
D. $\csc \theta = -3$
E. $\cos \theta = 1.01$
F. $\cot \theta = -1000$
G. $\cos \theta + 2 = -1$
H. $\sec \theta - 1.5 = 0$
I. $\sin \theta - 0.009 = 0.99$

2. Use a trigonometric identity to write the first step in the solution of each trigonometric equation. (Do not complete the solution.)

a. $\tan \theta = \cos^2 \theta + \sin^2 \theta, 0 \leq \theta < 2\pi$ $\tan \theta = 1$
b. $\sin^2 \theta - 2 \sin \theta + 1 = 0, 0° \leq \theta < 360°$ $(\sin \theta -1)^2 = 0$
c. $\cos 2\theta = \sin \theta, 0° \leq \theta < 360°$ $1 - 2 \sin^2 \theta = \sin \theta$
d. $\sin 2\theta = \cos \theta, 0 \leq \theta < 2\pi$ $2 \sin \theta \cos \theta = \cos \theta$
e. $2 \cos 2\theta + 3 \cos \theta = -1, 0° \leq \theta < 360°$ $2(2 \cos^2 \theta - 1) + 3 \cos \theta = -1$
f. $3 \tan^2 \theta + 5 \tan \theta - 2 = 0$ $(3 \tan \theta - 1)(\tan \theta + 2) = 0$

Helping You Remember

3. A good way to remember something is to explain it to someone else. How would you explain to a friend the difference between verifying a trigonometric identity and solving a trigonometric equation. Sample answer: Verifying a trigonometric identity means showing that the two sides are equal for *all* values of the variable for which the functions involved are defined. This is done by transforming one or both sides until the same expression is obtained on both sides. Solving a trigonometric equation means finding the values of the variable for which both sides are equal. This process may require simplifying trigonometric expressions, but it also requires finding the angles for which a trigonometric function has a particular value.

4 Assess

Open-Ended Assessment

Speaking Have students show you a trigonometric equation they solved and explain step by step how they performed each step of their computation.

Assessment Options

Quiz (Lesson 14-7) is available on p. 894 of the *Chapter 14 Resource Masters*.

Answer

46. Sample answer: Temperatures are cyclic and can be modeled by trigonometric functions. Answers should include the following information.

- **A temperature could occur twice in a given period such as when the temperature rises in the spring and falls in autumn.**

45. **CRITICAL THINKING** Computer games often use transformations to distort images on the screen. In one such transformation, an image is rotated counterclockwise using the equations $x' = x \cos \theta - y \sin \theta$ and $y' = x \sin \theta + y \cos \theta$. If the coordinates of an image point are (3, 4) after a 60° rotation, what are the coordinates of the preimage point? **(4.964, −0.598)**

46. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

How can trigonometric equations be used to predict temperature?

Include the following in your answer:

- an explanation of why the sine function can be used to model the average daily temperature, and
- an explanation of why, during one period, you might find a specific average temperature twice.

Standardized Test Practice

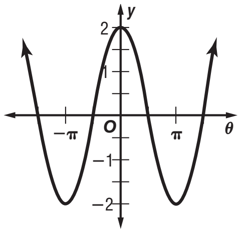
47. Which of the following is *not* a possible solution of $0 = \sin \theta + \cos \theta \tan^2 \theta$? **D**

Ⓐ $\frac{3\pi}{4}$ Ⓑ $\frac{7\pi}{4}$ Ⓒ 2π Ⓓ $\frac{5\pi}{2}$

48. The graph of the equation $y = 2 \cos \theta$ is shown. Which is a solution for $2 \cos \theta = 1$? **B**

Ⓐ $\frac{8\pi}{3}$ Ⓑ $\frac{13\pi}{3}$

Ⓒ $\frac{10\pi}{3}$ Ⓓ $\frac{15\pi}{3}$



Maintain Your Skills

Mixed Review

Find the exact value of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following. *(Lesson 14-6)*

49. $\sin \theta = \frac{3}{5}$; $0° < \theta < 90°$

50. $\cos \theta = \frac{1}{2}$; $0° < \theta < 90°$

51. $\cos \theta = \frac{5}{6}$; $0° < \theta < 90°$

52. $\sin \theta = \frac{4}{5}$; $0° < \theta < 90°$

49. $\frac{24}{25}, \frac{7}{25}, \frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10}$

50. $\frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}$

51. $\frac{5\sqrt{11}}{18}, \frac{7}{18}, \frac{\sqrt{3}}{6}, \frac{\sqrt{33}}{6}$

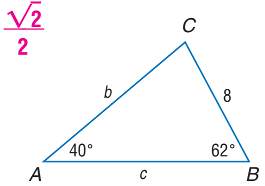
52. $\frac{24}{25}, -\frac{7}{25}, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}$

Find the exact value of each expression. *(Lesson 14-5)*

53. $\sin 240°$ $-\frac{\sqrt{3}}{2}$

54. $\cos 315°$ $\frac{\sqrt{2}}{2}$

55. Solve $\triangle ABC$. Round measures of sides and angles to the nearest tenth. *(Lesson 13-4)*
$b = 11.0$, $c = 12.2$, $m\angle C = 78$



WebQuest Internet Project

Trig Class Angles for Lessons in Lit

It is time to complete your project. Use the information and data you have gathered about the applications of trigonometry to prepare a poster, report, or Web page. Be sure to include graphs, tables, or diagrams in the presentation.

www.algebra2.com/webquest

Chapter 14 Study Guide and Review

Vocabulary and Concept Check

amplitude (p. 763)
double-angle formula (p. 791)
half-angle formula (p. 793)
midline (p. 771)
phase shift (p. 769)
trigonometric equation (p. 799)
trigonometric identity (p. 777)
vertical shift (p. 771)

Choose the correct letter that best matches each phrase.

1. horizontal translation of a trigonometric function **h**
2. a reference line about which a graph oscillates **b**
3. vertical translation of a trigonometric function **d**
4. the formula used to find $\cos 22\frac{1}{2}°$ **f**
5. $\sin 2\theta = 2 \sin \theta \cos \theta$ **e**
6. a measure of how long it takes for a graph to repeat itself **c**
7. $\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ **g**
8. the absolute value of half the difference between the maximum and minimum values of a periodic function **a**

a. amplitude
b. midline
c. period
d. vertical shift
e. double-angle formula
f. half-angle formula
g. difference of angles formula
h. phase shift

Lesson-by-Lesson Review

14-1 *Graphing Trigonometric Functions*

See pages 762–768.

Concept Summary

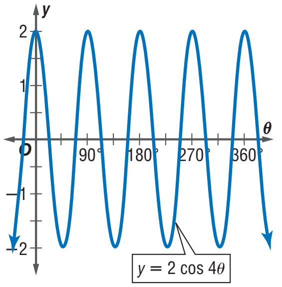
- For trigonometric functions of the form $y = a \sin b\theta$ and $y = a \cos b\theta$, the amplitude is $|a|$, and the period is $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$.
- The period of $y = a \tan b\theta$ is $\frac{180°}{|b|}$ or $\frac{\pi}{|b|}$.

Example **Find the amplitude and period of $y = 2 \cos 4\theta$. Then graph the function.**

The amplitude is $|2|$ or 2.

The period is $\frac{360°}{|4|}$ or 90°.

Use the amplitude and period to graph the function.



Exercises **Find the amplitude, if it exists, and period of each function. Then graph each function.** *See Example 1 on page 765.* **9–14. See pp. 811A–811N.**

9. $y = -\frac{1}{2} \cos \theta$
10. $y = 4 \sin 2\theta$
11. $y = \sin \frac{1}{2}\theta$
12. $y = 5 \sec \theta$
13. $y = \frac{1}{2} \csc \frac{2}{3}\theta$
14. $y = \tan 4\theta$

www.algebra2.com/vocabulary_review

FOLDABLES™ Study Organizer

For more information about Foldables, see *Teaching Mathematics with Foldables.*

Have students look through the chapter to make sure they have included notes and examples of graphs for each lesson in this chapter in their Foldable.

Encourage students to refer to their Foldables while completing the Study Guide and Review and to use them in preparing for the Chapter Test.

Chapter 14 Study Guide and Review

Vocabulary and Concept Check

- This alphabetical list of vocabulary terms in Chapter 14 includes a page reference where each term was introduced.
- **Assessment** A vocabulary test/review for Chapter 14 is available on p. 892 of the *Chapter 14 Resource Masters*.

Lesson-by-Lesson Review

For each lesson,

- the main ideas are summarized,
- additional examples review concepts, and
- practice exercises are provided.

Vocabulary PuzzleMaker

ELL The Vocabulary PuzzleMaker software improves students' mathematics vocabulary using four puzzle formats—crossword, scramble, word search using a word list, and word search using clues. Students can work on a computer screen or from a printed handout.

MindJogger Videoquizzes

ELL MindJogger Videoquizzes provide an alternative review of concepts presented in this chapter. Students work in teams in a game show format to gain points for correct answers. The questions are presented in three rounds.

Round 1 Concepts (5 questions)
Round 2 Skills (4 questions)
Round 3 Problem Solving (4 questions)

14-2 Translations of Trigonometric Graphs

See pages 769–776.

Concept Summary

- For trigonometric functions of the form $y = a \sin b(\theta - h)$, $y = a \cos(\theta - h)$, and $y = a \tan(\theta - h)$, the phase shift is to the right when $h > 0$ and to the left when $h < 0$.
- For trigonometric functions of the form $y = a \sin b(\theta - h) + k$, $y = a \cos(\theta - h) + k$, and $y = a \tan(\theta - h) + k$, the vertical shift is up when $k > 0$ and down when $k < 0$.

Example **State the vertical shift, amplitude, period, and phase shift of $y = 3 \sin\left[2\left(\theta - \frac{\pi}{2}\right)\right] - 2$. Then graph the function.**

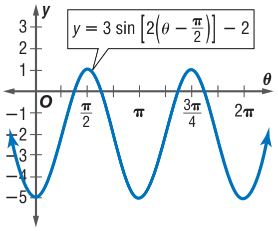
Identify the values of k, a, b, and h.

$k = -2$, so the vertical shift is –2.

$a = 3$, so the amplitude is 3.

$b = 2$, so the period is $\frac{2\pi}{|2|}$ or π.

$h = \frac{\pi}{2}$, so the phase shift is $\frac{\pi}{2}$ to the right.



Exercises **State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.** *See Example 3 on page 772.* **15–18. See margin.**

15. $y = \frac{1}{2} \sin[2(\theta - 60°)] - 1$

16. $y = 2 \tan\left[\frac{1}{4}(\theta - 90°)\right] + 3$

17. $y = 3 \sec\left[\frac{1}{2}\left(\theta + \frac{\pi}{4}\right)\right] + 1$

18. $y = \frac{1}{3} \cos\left[\frac{1}{3}\left(\theta - \frac{2\pi}{3}\right)\right] - 2$

14-3 Trigonometric Identities

See pages 777–781.

Concept Summary

- Quotient Identities: $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\cot \theta = \frac{\cos \theta}{\sin \theta}$
- Reciprocal Identities: $\csc \theta = \frac{1}{\sin \theta}$, $\sec \theta = \frac{1}{\cos \theta}$, $\cot \theta = \frac{1}{\tan \theta}$
- Pythagorean Identities: $\cos^2 \theta + \sin^2 \theta = 1$, $\tan^2 \theta + 1 = \sec^2 \theta$, $\cot^2 \theta + 1 = \sec^2 \theta$

Example **Simplify $\sin \theta \cot \theta \cos \theta$.**

$$\sin \theta \cot \theta \cos \theta = \frac{\sin \theta}{1} \cdot \frac{\cos \theta}{\sin \theta} \cdot \frac{\cos \theta}{1} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$= \cos^2 \theta \qquad \text{Multiply.}$$

Exercises **Find the value of each expression.** *See Example 1 on page 778.*

19. $\cot \theta$, if $\csc \theta = -\frac{5}{3}$; $270° < \theta < 360°$ $-\frac{4}{3}$

20. $\sec \theta$, if $\sin \theta = \frac{1}{2}$; $0° \le \theta < 90°$ $\frac{2\sqrt{3}}{3}$

Simplify each expression. *See Example 2 on page 778.* **21. $\sin^2 \theta$ 22. $\cot \theta$ 23. $\sec \theta$**

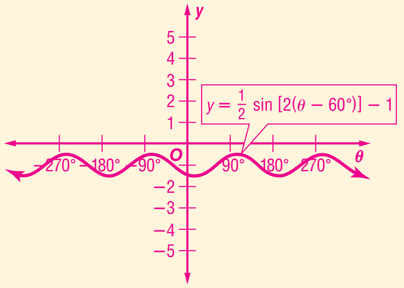
21. $\sin \theta \csc \theta - \cos^2 \theta$

22. $\cos^2 \theta \sec \theta \csc \theta$

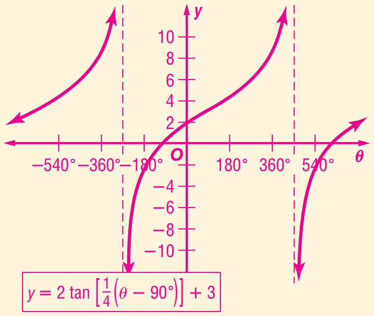
23. $\cos \theta + \sin \theta \tan \theta$

Answers

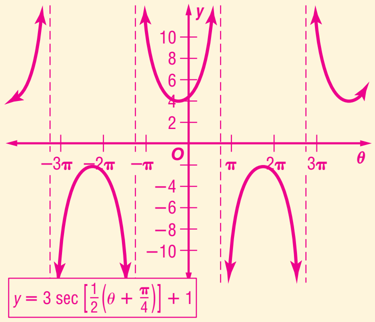
15. $-1, \frac{1}{2}, 180°, 60°$



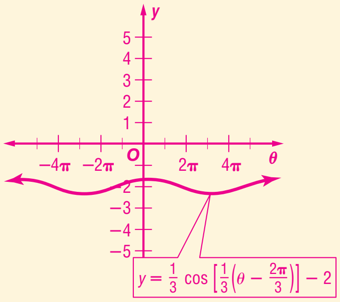
16. 3, does not exist, 720°, 90°



17. 1, does not exist, 4π, $-\frac{\pi}{4}$



18. $-2, \frac{1}{3}, 6\pi, \frac{2\pi}{3}$



14-4 Verifying Trigonometric Identities

See pages 782–785.

Concept Summary

- Use the basic trigonometric identities to transform one or both sides of a trigonometric equation into the same form.

Example Verify that $\tan\theta + \cot\theta = \sec\theta\csc\theta$.

$\tan\theta + \cot\theta \stackrel{?}{=} \sec\theta\csc\theta$ Original equation

$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \stackrel{?}{=} \sec\theta\csc\theta$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \stackrel{?}{=} \sec\theta\csc\theta$ Rewrite using the LCD, $\cos\theta\sin\theta$.

$\frac{1}{\cos\theta\sin\theta} \stackrel{?}{=} \sec\theta\csc\theta$ $\sin^2\theta + \cos^2\theta = 1$

$\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta} \stackrel{?}{=} \sec\theta\csc\theta$ Rewrite as the product of two expressions.

$\sec\theta\csc\theta = \sec\theta\csc\theta$ $\frac{1}{\cos\theta} = \sec\theta$, $\frac{1}{\sin\theta} = \csc\theta$

Exercises **Verify that each of the following is an identity.**
See Examples 1–3 on pages 782–783. **24–27. See margin.**

24. $\frac{\sin\theta}{\tan\theta} + \frac{\cos\theta}{\cot\theta} = \cos\theta + \sin\theta$
25. $\frac{\sin\theta}{1 - \cos\theta} = \csc\theta + \cot\theta$
26. $\cot^2\theta\sec^2\theta = 1 + \cot^2\theta$
27. $\sec\theta(\sec\theta - \cos\theta) = \tan^2\theta$

14-5 Sum and Difference of Angles Formulas

See pages 786–790.

Concept Summary

- For all values of α and β : $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
  $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

Example Find the exact value of sin 195°.

$\sin 195° = \sin(150° + 45°)$ $195° = 150° + 45°$

$= \sin 150°\cos 45° + \cos 150°\sin 45°$ $\alpha = 150°, \beta = 45°$

$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$ Evaluate each expression.

$= \frac{\sqrt{2} - \sqrt{6}}{4}$ Simplify.

Exercises **Find the exact value of each expression.** *See Example 1 on page 787.*

28. cos 15° $\frac{\sqrt{6}+\sqrt{2}}{4}$
29. cos 285° $\frac{\sqrt{6}-\sqrt{2}}{4}$
30. sin 150° $\frac{1}{2}$
31. sin 195°
32. cos (−210°) $-\frac{\sqrt{3}}{2}$
33. sin (−105°) $\frac{-\sqrt{6}-\sqrt{2}}{4}$

Verify that each of the following is an identity. *See Example 3 on page 788.*

34. $\cos(90° + \theta) = -\sin\theta$
35. $\sin(30° - \theta) = \cos(60° + \theta)$
36. $\sin(\theta + \pi) = -\sin\theta$
37. $-\cos\theta = \cos(\pi + \theta)$

31. $\frac{\sqrt{2}-\sqrt{6}}{4}$

34–37. See pp. 811A–811N.

Answers

25.
$$\frac{\sin\theta}{1-\cos\theta} \stackrel{?}{=} \csc\theta + \cot\theta$$
$$\frac{\sin\theta}{1-\cos\theta} \stackrel{?}{=} \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$
$$\frac{\sin\theta}{1-\cos\theta} \stackrel{?}{=} \frac{1+\cos\theta}{\sin\theta}$$
$$\frac{\sin\theta}{1-\cos\theta} \stackrel{?}{=} \frac{1+\cos\theta}{\sin\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}$$
$$\frac{\sin\theta}{1-\cos\theta} \stackrel{?}{=} \frac{1-\cos^2\theta}{\sin\theta(1-\cos\theta)}$$
$$\frac{\sin\theta}{1-\cos\theta} \stackrel{?}{=} \frac{\sin^2\theta}{\sin\theta(1-\cos\theta)}$$
$$\frac{\sin\theta}{1-\cos\theta} = \frac{\sin\theta}{1-\cos\theta}$$

26.
$$\cot^2\theta\sec^2\theta \stackrel{?}{=} 1 + \cot^2\theta$$
$$\frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\cos^2\theta} \stackrel{?}{=} 1 + \cot^2\theta$$
$$\frac{1}{\sin^2\theta} \stackrel{?}{=} 1 + \cot^2\theta$$
$$\csc^2\theta \stackrel{?}{=} 1 + \cot^2\theta$$
$$1 + \cot^2\theta = 1 + \cot^2\theta$$

27.
$$\sec\theta(\sec\theta - \cos\theta) \stackrel{?}{=} \tan^2\theta$$
$$\frac{1}{\cos\theta}\left(\frac{1}{\cos\theta} - \cos\theta\right) \stackrel{?}{=} \tan^2\theta$$
$$\frac{1}{\cos^2\theta} - 1 \stackrel{?}{=} \tan^2\theta$$
$$\sec^2\theta - 1 \stackrel{?}{=} \tan^2\theta$$
$$\tan^2\theta = \tan^2\theta$$

Answer

24.
$$\frac{\sin\theta}{\tan\theta} + \frac{\cos\theta}{\cot\theta} \stackrel{?}{=} \cos\theta + \sin\theta$$
$$\frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} + \frac{\cos\theta}{\frac{\cos\theta}{\sin\theta}} \stackrel{?}{=} \cos\theta + \sin\theta$$
$$\sin\theta\cdot\frac{\cos\theta}{\sin\theta} + \cos\theta\cdot\frac{\sin\theta}{\cos\theta} \stackrel{?}{=} \cos\theta + \sin\theta$$
$$\cos\theta + \sin\theta = \cos\theta + \sin\theta$$

Chapter 14 For More ...
- Extra Practice, see pages 859–861.
- Mixed Problem Solving, see page 875.

14-6 Double-Angle and Half-Angle Formulas

See pages 791–797.

Concept Summary

- Double-angle formulas: $\sin 2\theta = 2 \sin \theta \cos \theta$, $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$, $\cos 2\theta = 1 - 2 \sin^2 \theta$, $\cos 2\theta = 2 \cos^2 \theta - 1$
- Half-angle formulas: $\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$, $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$

Example **Verify that $\csc 2\theta = \frac{\sec \theta}{2 \sin \theta}$ is an identity.**

$\csc 2\theta \overset{?}{=} \frac{\sec \theta}{2 \sin \theta}$ Original equation

$\frac{1}{\sin 2\theta} \overset{?}{=} \frac{\frac{1}{\cos \theta}}{2 \sin \theta}$ $\csc \theta = \frac{1}{\sin \theta}$, $\sec \theta = \frac{1}{\cos \theta}$

$\frac{1}{\sin 2\theta} \overset{?}{=} \frac{1}{2 \sin \theta \cos \theta}$ Simplify the complex fraction.

$\frac{1}{\sin 2\theta} = \frac{1}{\sin 2\theta}$ $2 \sin \theta \cos \theta = \sin 2\theta$

Exercises **Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.** *See Examples 1 and 2 on pages 792 and 793.* **38–41. See margin.**

38. $\sin \theta = \frac{1}{4}$; $0° < \theta < 90°$

39. $\sin \theta = -\frac{5}{13}$; $180° < \theta < 270°$

40. $\cos \theta = -\frac{5}{17}$; $90° < \theta < 180°$

41. $\cos \theta = \frac{12}{13}$; $270° < \theta < 360°$

14-7 Solving Trigonometric Equations

See pages 799–804.

Concept Summary

- Solve trigonometric equations by factoring or by using trigonometric identities.

Example **Solve $\sin 2\theta + \sin \theta = 0$ if $0° \leq \theta < 360°$.**

$\sin 2\theta + \sin \theta = 0$ Original equation

$2 \sin \theta \cos \theta + \sin \theta = 0$ $\sin 2\theta = 2 \sin \theta \cos \theta$

$\sin \theta\,(2 \cos \theta + 1) = 0$ Factor.

$\sin \theta = 0$ or $2 \cos \theta + 1 = 0$

$\theta = 0°$ or $180°$ $\theta = 120°$ or $240°$

Exercises **Find all solutions of each equation for the interval $0° \leq \theta < 360°$.** *See Example 1 on page 799.*

42. $2 \sin 2\theta = 1$ **15°, 75°, 195°, 255°**

43. $2 \cos^2 \theta + \sin^2 \theta = 2 \cos \theta$ **0°**

Solve each equation for all values of θ if θ is measured in radians. *See Example 2 on page 800.* **44. $\frac{7\pi}{6} + 2k\pi$, $\frac{11\pi}{6} + 2k\pi$ 45. $\frac{\pi}{6} + 2k\pi$, $\frac{5\pi}{6} + 2k\pi$**

44. $6 \sin^2 \theta - 5 \sin \theta - 4 = 0$

45. $2 \cos^2 \theta = 3 \sin \theta$

Answers

38. $\frac{\sqrt{15}}{8}, \frac{7}{8}, \frac{\sqrt{8 - 2\sqrt{15}}}{4}, \frac{\sqrt{8 + 2\sqrt{15}}}{4}$

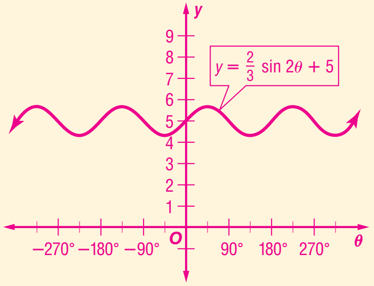
39. $\frac{120}{169}, \frac{119}{169}, \frac{\sqrt{26}}{26}, -\frac{5\sqrt{26}}{26}$

40. $-\frac{20\sqrt{66}}{289}, -\frac{239}{289}, \frac{\sqrt{187}}{17}, \frac{\sqrt{102}}{17}$

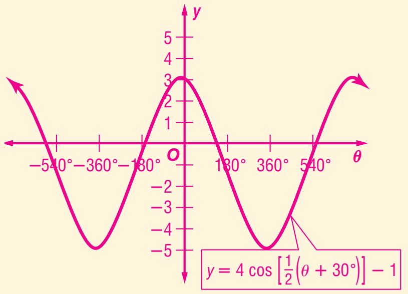
41. $-\frac{120}{169}, \frac{119}{169}, \frac{\sqrt{26}}{26}, -\frac{5\sqrt{26}}{26}$

Answers (p. 809)

5. 5, $\frac{2}{3}$, 180°, no phase shift



6. −1, 4, 720°, −30°



Chapter 14 Practice Test

Vocabulary and Concepts

Choose the correct term to complete each sentence.

1. The (*period*, *phase shift*) of $y = 3 \sin 2(\theta - 60°) + 2$ is 120°.
2. A midline is used with a (*phase shift*, *vertical shift*) of a trigonometric function.
3. The amplitude of $y = \frac{1}{3} \cos [3(\theta + 4)] - 1$ is $\left(\frac{1}{3}, 3\right)$.
4. The (*cosine*, *cosecant*) has no amplitude.

Skills and Applications

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function. **5–6. See margin.**

5. $y = \frac{2}{3} \sin 2\theta + 5$

6. $y = 4 \cos \left[\frac{1}{2}(\theta + 30°)\right] - 1$

Find the value of each expression.

7. $\tan \theta$, if $\sin \theta = \frac{1}{2}$; $90° < \theta < 180°$ $-\frac{\sqrt{3}}{3}$

8. $\sec \theta$, if $\cot \theta = \frac{3}{4}$; $180° < \theta < 270°$ $-\frac{5}{3}$

Verify that each of the following is an identity. **9–12. See pp. 811A–811N.**

9. $(\sin \theta - \cos \theta)^2 = 1 - \sin 2\theta$

10. $\frac{\cos \theta}{1 - \sin^2 \theta} = \sec \theta$

11. $\frac{\sec \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = \cot \theta$

12. $\frac{1 + \tan^2 \theta}{\cos^2 \theta} = \sec^4 \theta$

Find the exact value of each expression. **13.** $\frac{\sqrt{6} - \sqrt{2}}{4}$ **14.** $\frac{\sqrt{2} - \sqrt{6}}{4}$ **15.** $\frac{\sqrt{2}}{2}$

13. $\cos 285°$

14. $\sin 345°$

15. $\sin (-225°)$

16. $\cos 480°$ $-\frac{1}{2}$

17. $\cos 67.5°$ $\frac{\sqrt{2 - \sqrt{2}}}{2}$

18. $\sin 75°$ $\frac{\sqrt{6} + \sqrt{2}}{4}$

Solve each equation for all values of θ if θ is measured in degrees.

19. $\sec \theta = 1 + \tan \theta$ $0° + k \cdot 360°$

20. $\cos 2\theta = \cos \theta$ $0° + k \cdot 360°, 120° + k \cdot 360°, 240° + k \cdot 360°$

21. $\cos 2\theta + \sin \theta = 1$ $0° + k \cdot 180°, 30° + k \cdot 360°, 150° + k \cdot 360°$

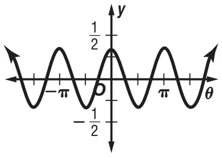
22. $\sin \theta = \tan \theta$ $0° + k \cdot 180°$

GOLF **For Exercises 23 and 24, use the following information.**
A golf ball is hit with an initial velocity of 100 feet per second. The distance the ball travels is found by the formula $d = \frac{{v_0}^2}{g} \sin 2\theta$, where v_0 is the initial velocity, g is the acceleration due to gravity, 32 feet per second squared, and θ is the measurement of the angle that the path of the ball makes with the ground.

23. Find the distance that the ball travels if the angle between the path of the ball and the ground measures 60°. **270.6 ft**

24. If a ball travels 312.5 feet, what was the angle the path of the ball made with the ground to the nearest degree? **45°**

25. **STANDARDIZED TEST PRACTICE** Identify the equation of the graphed function. **B**



Ⓐ $y = 3 \cos 2\theta$ Ⓑ $y = \frac{1}{3} \cos 2\theta$

Ⓒ $y = 3 \cos \frac{1}{2}\theta$ Ⓓ $y = \frac{1}{3} \cos \frac{1}{2}\theta$

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Portfolio Suggestion

Introduction In mathematics, trigonometric functions can be used to model real-world problems.

Ask Students Find a real-world problem modeled in this chapter that interests you and show how you solved it. Explain how the function models the real-world problem and what could be gained by understanding the real-world problem better. Place your work in your portfolio.

Chapter 14 Practice Test

Assessment Options

Vocabulary Test A vocabulary test/review for Chapter 14 can be found on p. 892 of the *Chapter 14 Resource Masters*.

Chapter Tests There are six Chapter 14 Tests and an Open-Ended Assessment task available in the *Chapter 14 Resource Masters*.

Chapter 14 Tests			
Form	**Type**	**Level**	**Pages**
1	MC	basic	879–880
2A	MC	average	881–882
2B	MC	average	883–884
2C	FR	average	885–886
2D	FR	average	887–888
3	FR	advanced	889–890

MC = multiple-choice questions
FR = free-response questions

Open-Ended Assessment Performance tasks for Chapter 14 can be found on p. 891 of the *Chapter 14 Resource Masters*. A sample scoring rubric for these tasks appears on p. A28.

Unit 5 Test A unit test/review can be found on pp. 899–900 of the *Chapter 14 Resource Masters*.

End-of-Year Tests A Second Semester Test for Chapters 8–14 and a Final Test for Chapters 1–14 can be found on pp. 901–910 of the *Chapter 14 Resource Masters*.

TestCheck and Worksheet Builder

This **networkable software** has three modules for assessment.

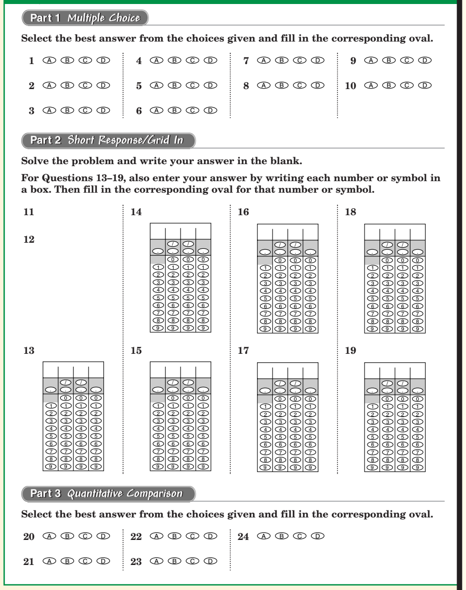
- **Worksheet Builder** to make worksheets and tests.
- **Student Module** to take tests on-screen.
- **Management System** to keep student records.

Chapter 14 Standardized Test Practice

These two pages contain practice questions in the various formats that can be found on the most frequently given standardized tests.

A practice answer sheet for these two pages can be found on p. A1 of the *Chapter 14 Resource Masters*.

Standardized Test Practice
Student Recording Sheet, p. A1



Part 1 Multiple Choice

Select the best answer from the choices given and fill in the corresponding oval.

1 Ⓐ Ⓑ Ⓒ Ⓓ
2 Ⓐ Ⓑ Ⓒ Ⓓ
3 Ⓐ Ⓑ Ⓒ Ⓓ
4 Ⓐ Ⓑ Ⓒ Ⓓ
5 Ⓐ Ⓑ Ⓒ Ⓓ
6 Ⓐ Ⓑ Ⓒ Ⓓ
7 Ⓐ Ⓑ Ⓒ Ⓓ
8 Ⓐ Ⓑ Ⓒ Ⓓ
9 Ⓐ Ⓑ Ⓒ Ⓓ
10 Ⓐ Ⓑ Ⓒ Ⓓ

Part 2 Short Response/Grid In

Solve the problem and write your answer in the blank.

For Questions 13–19, also enter your answer by writing each number or symbol in a box. Then fill in the corresponding oval for that number or symbol.

11
12
13
14
15
16
17
18
19

Part 3 Quantitative Comparison

Select the best answer from the choices given and fill in the corresponding oval.

20 Ⓐ Ⓑ Ⓒ Ⓓ
21 Ⓐ Ⓑ Ⓒ Ⓓ
22 Ⓐ Ⓑ Ⓒ Ⓓ
23 Ⓐ Ⓑ Ⓒ Ⓓ
24 Ⓐ Ⓑ Ⓒ Ⓓ

Additional Practice

See pp. 897–898 in the *Chapter 14 Resource Masters* for additional standardized test practice.

Chapter 14 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. Which of the following is *not* equal to 3.5×10^{-2}? **D**

(A) $\frac{35}{1000}$
(B) 0.035
(C) $\frac{7}{200}$
(D) (0.5)(0.007)

2. The sum of five consecutive odd integers is 55. What is the sum of the greatest and least of these integers? **B**

(A) 11
(B) 22
(C) 26
(D) 30

3. If 8 bananas cost a cents and 6 oranges cost b cents, what is the cost of 2 bananas and 2 oranges in terms of a and b? **D**

(A) $\frac{ab}{12}$
(B) $3a + \frac{b}{3}$
(C) $3a + 4b$
(D) $\frac{3a + 4b}{12}$

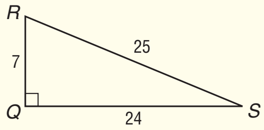
4. A bag contains 16 peppermint candies, 10 butterscotch candies, and 8 cherry candies. Emma chooses one piece at random, puts it in her pocket, and then repeats the process. If she has chosen 3 peppermint candies, 2 butterscotch candies, and 1 cherry candy, what is the probability that the next piece of candy she chooses will be cherry? **C**

(A) $\frac{7}{34}$
(B) $\frac{8}{34}$
(C) $\frac{1}{4}$
(D) $\frac{3}{4}$

5. What is the value of $\dfrac{\sin\frac{\pi}{6}}{\cos\frac{2\pi}{3}}$? **B**

(A) $-\sqrt{3}$
(B) -1
(C) $-\frac{\sqrt{3}}{3}$
(D) 1

6. In right triangle QRS, what is the value of $\tan R$? **D**



(A) $\frac{7}{25}$
(B) $\frac{7}{24}$
(C) $\frac{25}{24}$
(D) $\frac{24}{7}$

7. What is the value of $\sin\left(\cos^{-1}\frac{1}{3}\right)$? **B**

(A) $\frac{2}{3}$
(B) $\frac{2\sqrt{2}}{3}$
(C) $\frac{\sqrt{2}}{3}$
(D) $\frac{\sqrt{6}}{3}$

8. What is the least positive value for x where $y = \sin 2x$ reaches its minimum? **C**

(A) $\frac{\pi}{2}$
(B) π
(C) $\frac{3\pi}{4}$
(D) $\frac{3\pi}{2}$

9. Which of the following is equivalent to $\frac{\sin^2\theta + \cos^2\theta}{\sec^2\theta}$? **A**

(A) $\cos^2\theta$
(B) $\sin^2\theta$
(C) $\tan^2\theta$
(D) $\sin^2\theta + 1$

10. If $\cos\theta = -\frac{1}{2}$ and θ is in Quadrant II, what is the value of $\sin 2\theta$? **D**

(A) $\frac{1}{2}$
(B) $-\frac{1}{2}$
(C) $\frac{\sqrt{3}}{2}$
(D) $-\frac{\sqrt{3}}{2}$

Log On **for Test Practice**

The Princeton Review offers additional test-taking tips and practice problems at their web site. Visit www.princetonreview.com or www.review.com

TestCheck and Worksheet Builder

Special banks of standardized test questions similar to those on the SAT, ACT, TIMSS 8, NAEP 8, and Algebra 1 End-of-Course tests can be found on this CD-ROM.

Aligned and verified by The Princeton Review

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. If k is a positive integer, and $7k + 3$ equals a prime number that is less than 50, then what is one possible value of $7k + 3$? **17 or 31**

12. It costs \$8 to make a book. The selling price will include an additional 200%. What will be the selling price? **\$24**

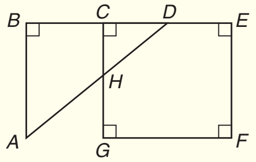
13. The mean of seven numbers is 0. The sum of three of the numbers is −9. What is the sum of the remaining four numbers? **9**

14. If $4a - 6b = 0$ and $c = 9b$, what is the ratio of a to c? **1/6**

15. What is the value of x if $\frac{3^3 \cdot 3}{\sqrt{81}} = 3^x$? **2**

16. The ages of children at a party are 6, 7, 6, 6, 7, 7, 8, 6, 7, 8, 9, 7, and 7. Let N represent the median of their ages and m represent the mode. What is $N - m$? **0**

17. In the figure below, $CEFG$ is a square, ABD is a right triangle, D is the midpoint of side CE, H is the midpoint of side CG, and C is the midpoint of side BD. $BCDE$ is a line segment, and AHD is a line segment. If the measure of the area of square $CEFG$ is 16, what is the measure of the area of quadrilateral $ABCH$? **6**



The Princeton Review Test-Taking Tip

Always write down your calculations on scrap paper or in the test booklet, even if you think you can do the calculations in your head. Writing down your calculations will help you avoid making simple mistakes.

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18. A line with a slope of $\frac{3}{8}$ passes through points $(6, 4n)$ and $(0, n)$. What is the value of n? **3/4**

19. If $\sin 60° = \frac{\sqrt{3}}{2}$, what is the value of $\sin^2 30° + \cos^2 30°$? **1**

Part 3 Quantitive Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- Ⓐ **the quantity in Column A is greater,**
- Ⓑ **the quantity in Column B is greater,**
- Ⓒ **the two quantities are equal, or**
- Ⓓ **the relationship cannot be determined from the information given.**

20. **A**

Column A	Column B
the length of a diagonal of a square whose area is 100	the length of a diagonal of a 6 × 8 rectangle

21. $\langle c \rangle = c^2 + \frac{2}{c} - 2$

B

$\langle 1 \rangle$	$\langle 2 \rangle$

22. $w = 2x,\ x = \frac{1}{2}w$

C

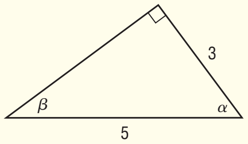
w	x

23. $(a + b)^2 = a^2 + b^2$

C

$(a - b)^2$	$a^2 + b^2$

24.

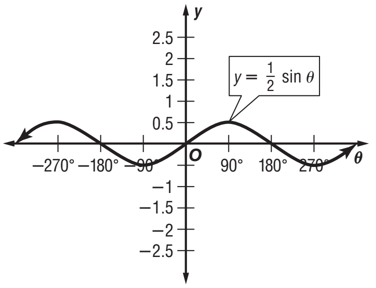


B

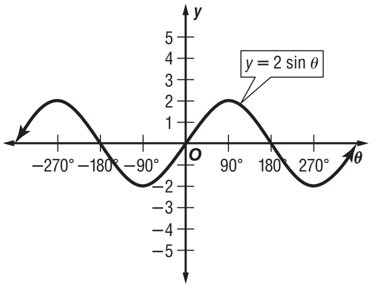
$\tan \beta$	$\sin \alpha$

Pages 767–768, Lesson 14-1

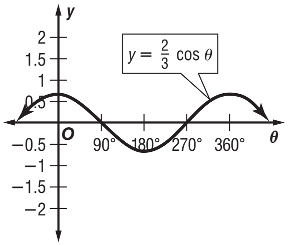
4. amplitude: $\frac{1}{2}$; period 360° or 2π



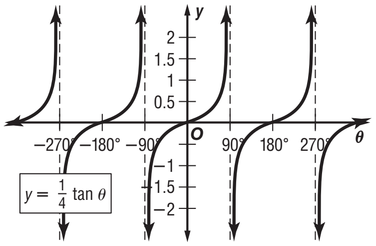
5. amplitude: 2; period: 360° or 2π



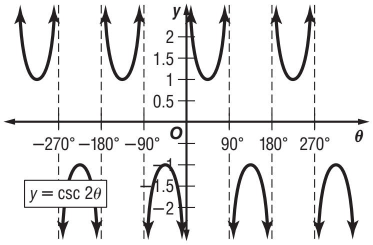
6. amplitude: $\frac{2}{3}$; period: 360° or 2π



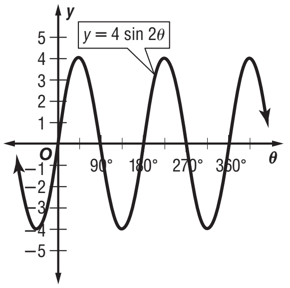
7. amplitude: does not exist; period: 180° or π



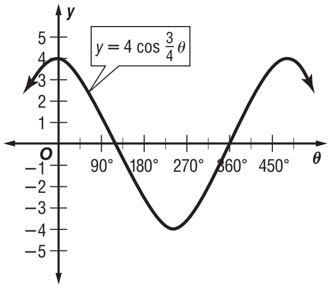
8. amplitude: does not exist; period: 180° or π



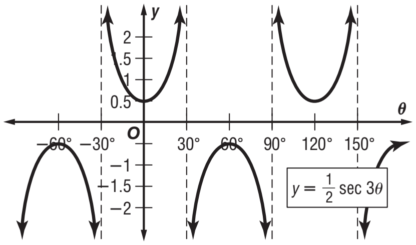
9. amplitude: 4; period: 180° or π



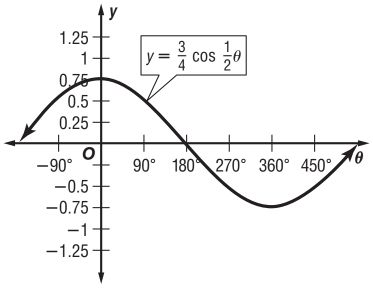
10. amplitude: 4; period: 480° or $\frac{8\pi}{3}$



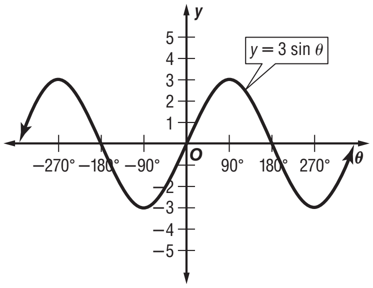
11. amplitude: does not exist; period: 120° or $\frac{2\pi}{3}$



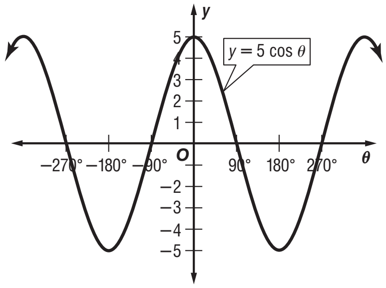
12. amplitude: $\frac{3}{4}$; period: 720° or 4π



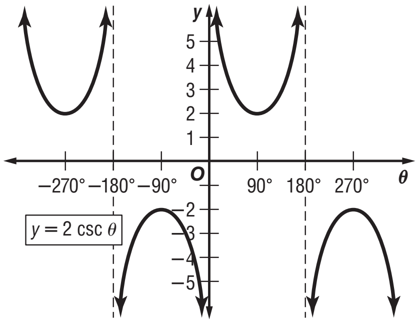
15. amplitude: 3; period: 360° or 2π



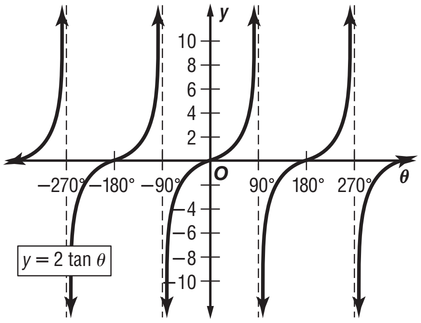
16. amplitude: 5; period: 360° or 2π



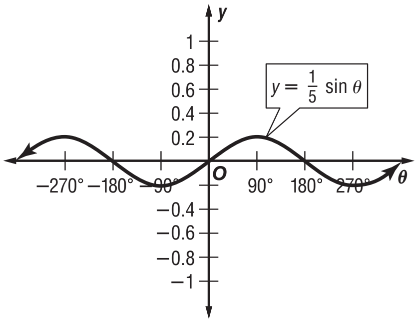
17. amplitude: does not exist; period: 360° or 2π



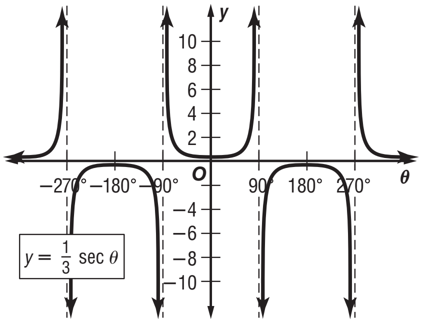
18. amplitude: does not exist; period: 180° or π



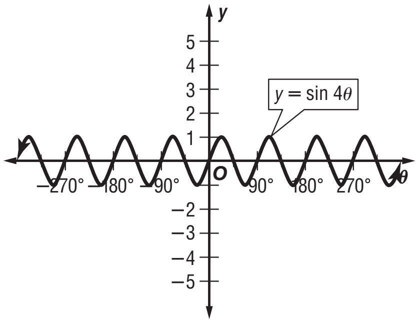
19. amplitude: $\frac{1}{5}$; period: 360° or 2π



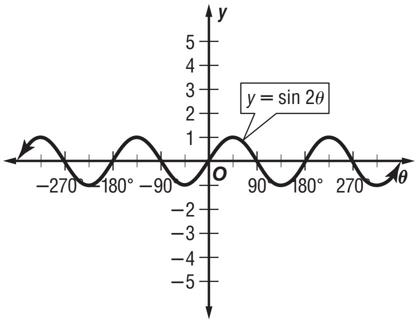
20. amplitude: does not exist; period: 360° or 2π



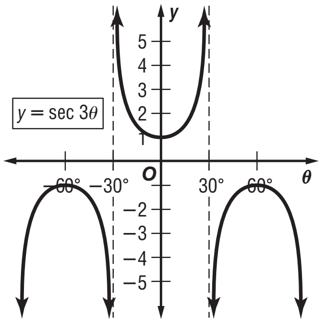
21. amplitude: 1; period 90° or $\frac{\pi}{2}$



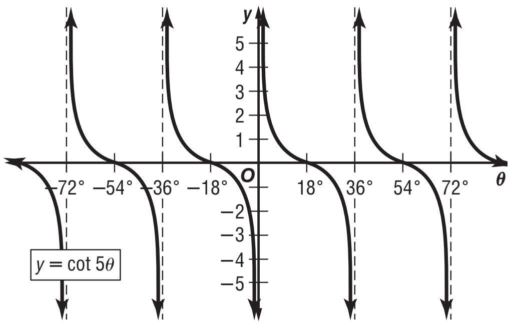
22. amplitude: 1; period: 180° or π



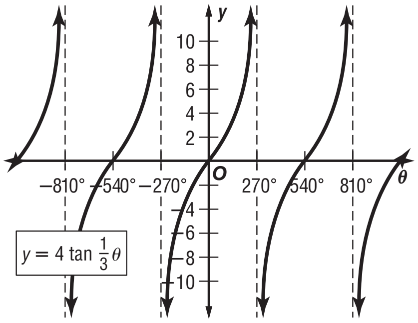
23. amplitude: does not exist; period: 120° or $\frac{2\pi}{3}$



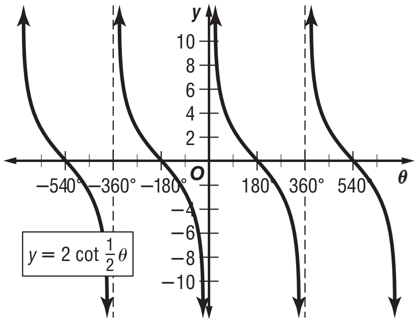
24. amplitude: does not exist; period: 36° or $\frac{\pi}{5}$



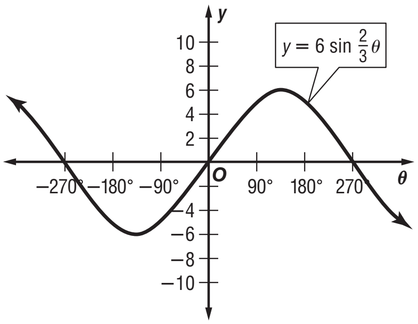
25. amplitude: does not exist; period: 540° or 3π



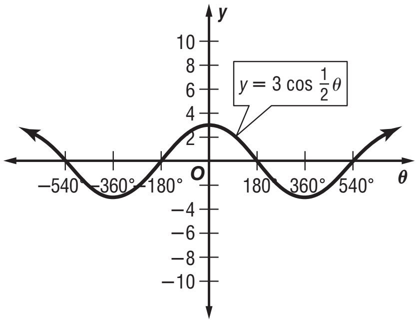
26. amplitude: does not exist; period: 360° or 2π



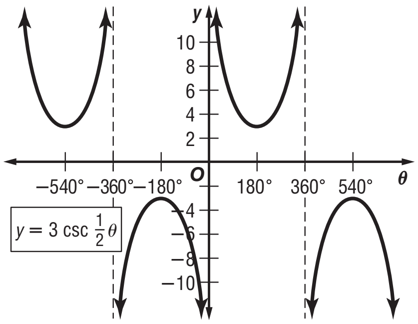
27. amplitude: 6; period: 540° or 3π



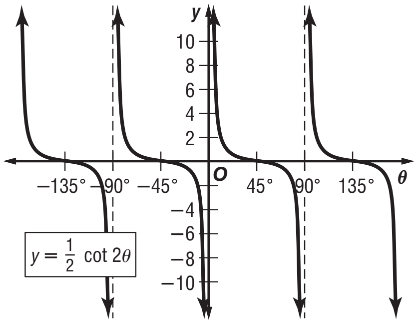
28. amplitude: 3; period: 720° or 4π



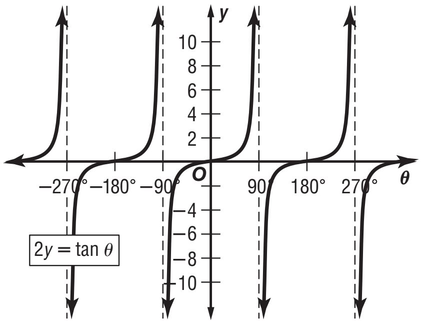
29. amplitude: does not exist; period: 720° or 4π



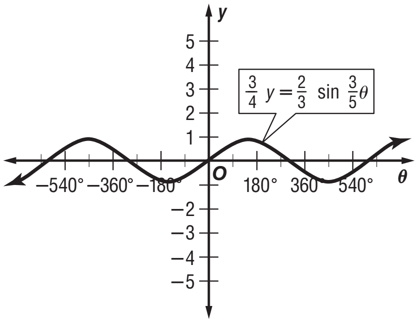
30. amplitude: does not exist; period: 90° or $\frac{\pi}{2}$



31. amplitude: does not exist; period: 180° or π



32. amplitude: $\frac{8}{9}$; period: 600° or $\frac{10\pi}{3}$



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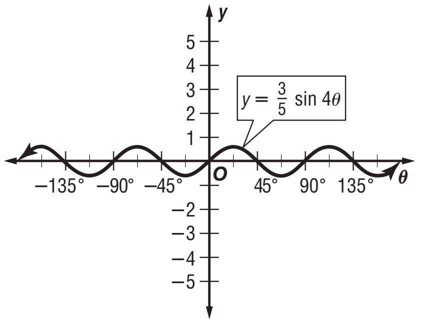

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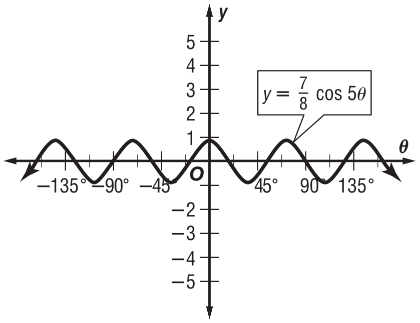

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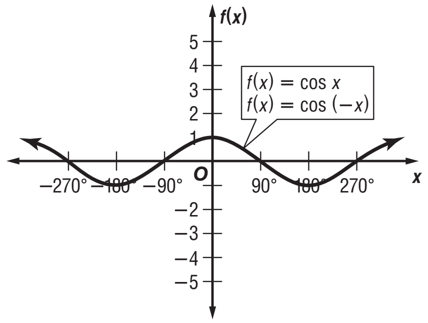

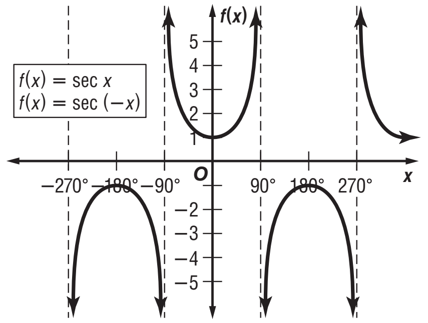

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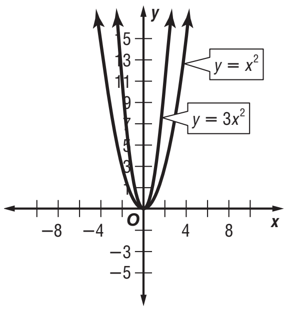

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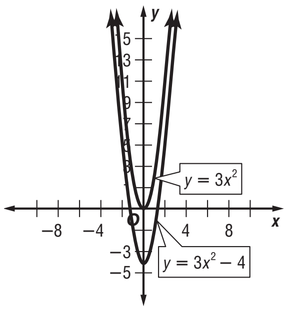

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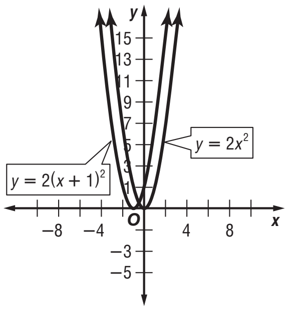

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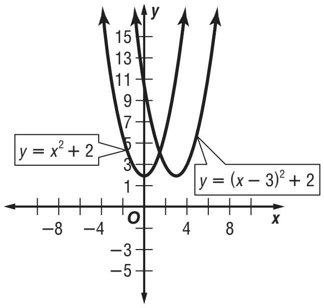

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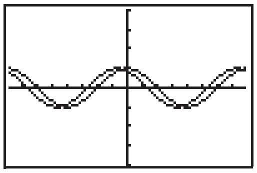
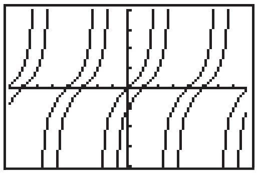
Graphing Calculator Investigation

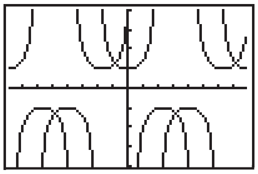
4.



[−360, 360] scl: 90 by [−5, 5] scl: 1



[−360, 360] scl: 90 by [−5, 5] scl: 1



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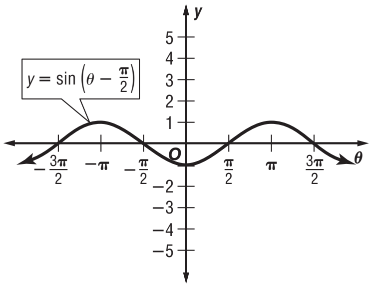
Pages 774–776, Lesson 14-2

4.



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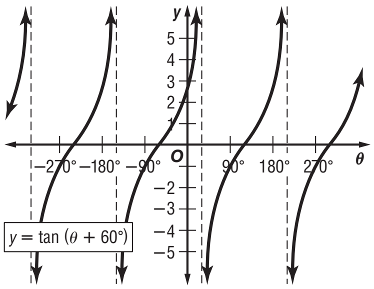

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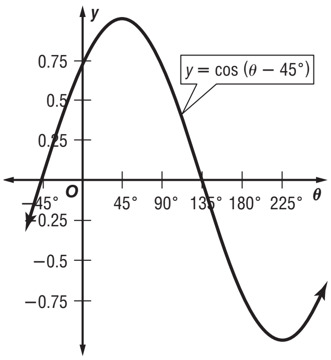

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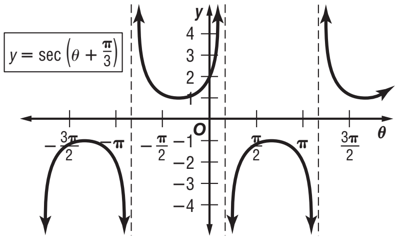

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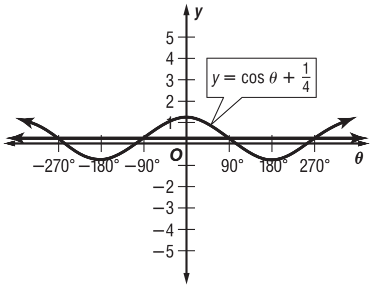

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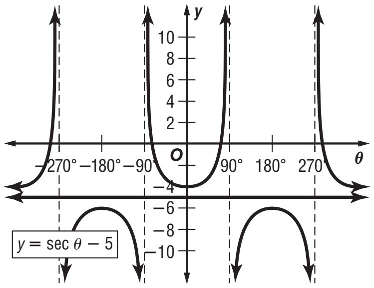

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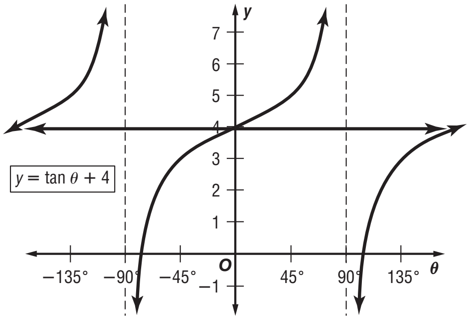

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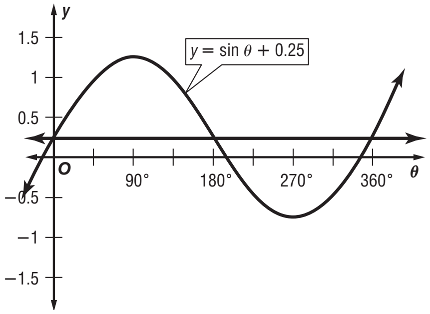

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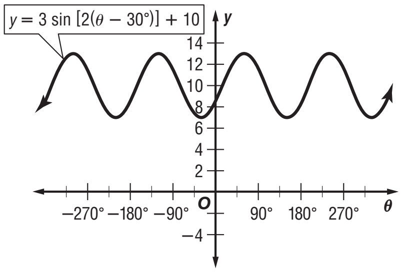

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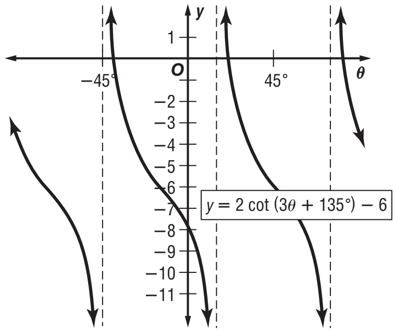

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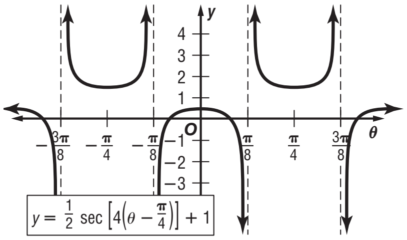

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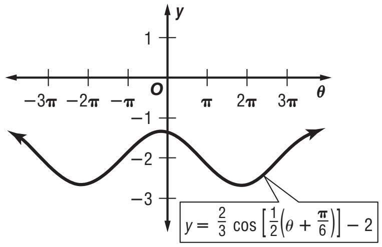

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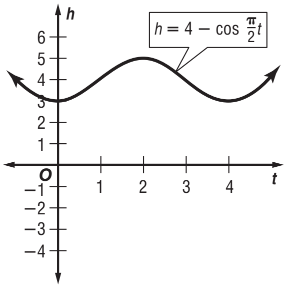

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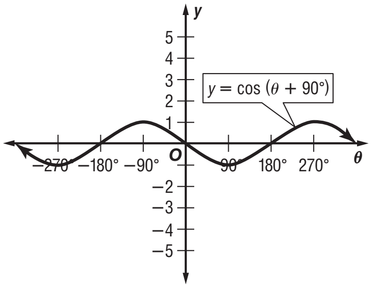

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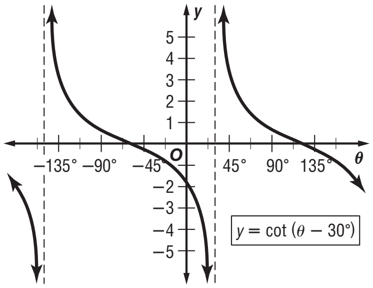

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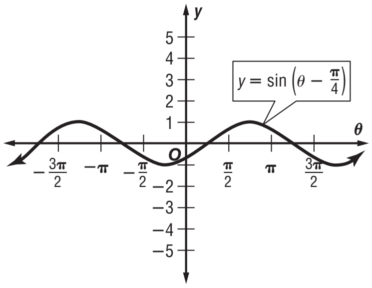

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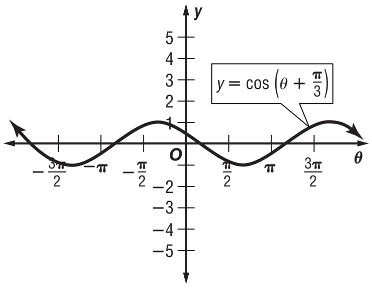

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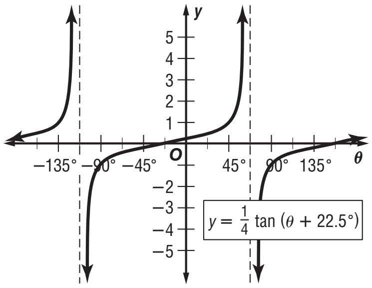

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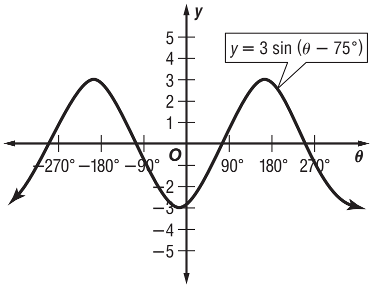

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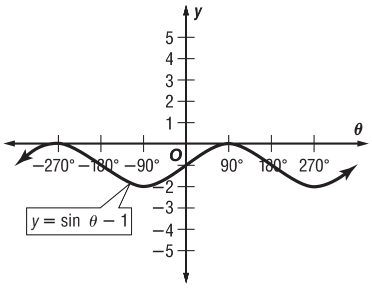

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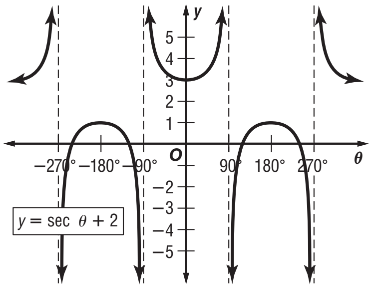

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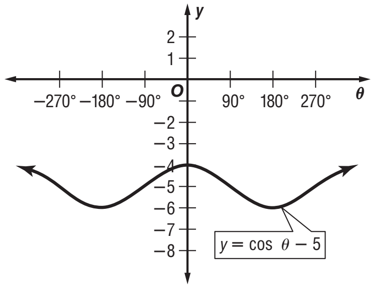

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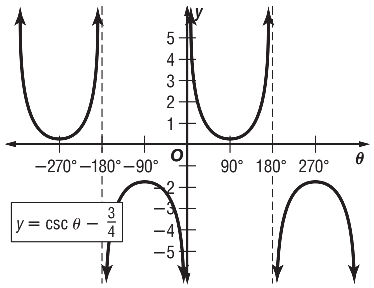

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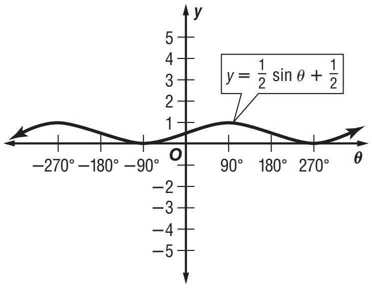

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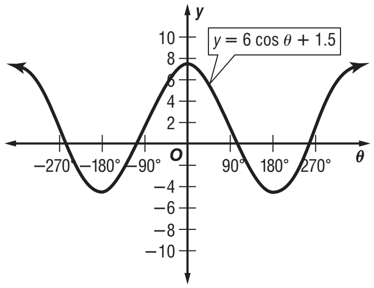
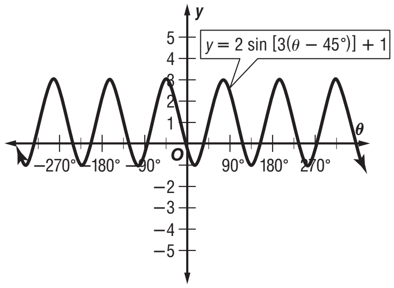
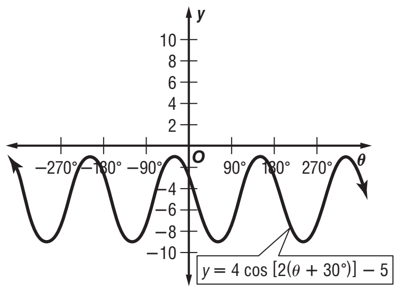


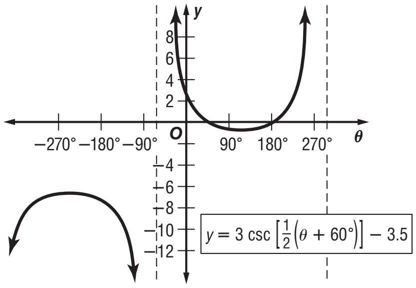
33. 1; 2; 120°; 45°



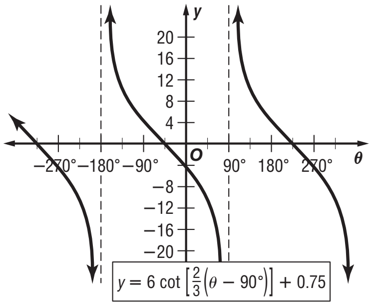
34. −5; 4; 180°; −30°



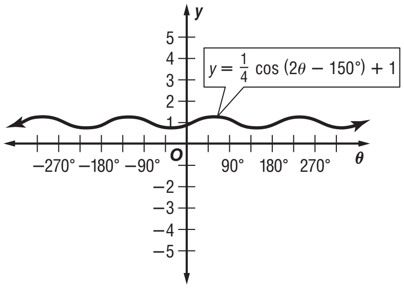
35. −3.5; does not exist; 720°; −60°



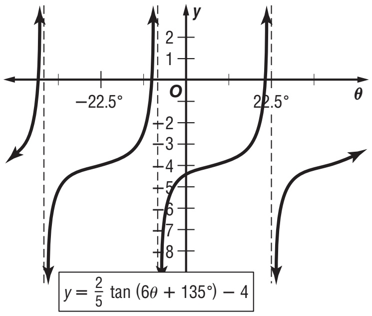
36. 0.75; does not exist; 270°; 90°



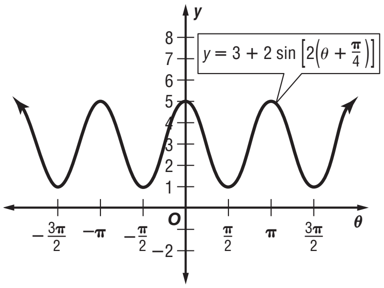
37. 1; $\frac{1}{4}$; 180°; 75°



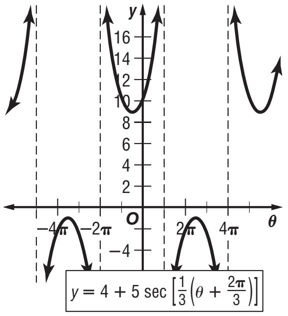
38. −4; does not exist; 30°; −22.5°



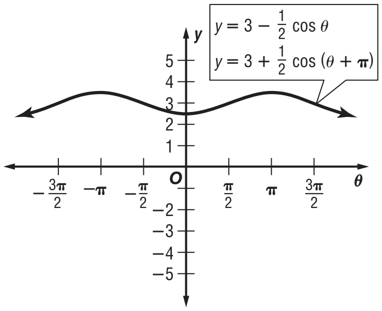
39. 3; 2; π; $-\frac{\pi}{4}$



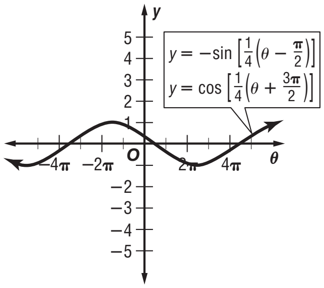
40. 4; does not exist; 6π; $-\frac{2\pi}{3}$



41. The graphs are identical.



42. The graphs are identical.

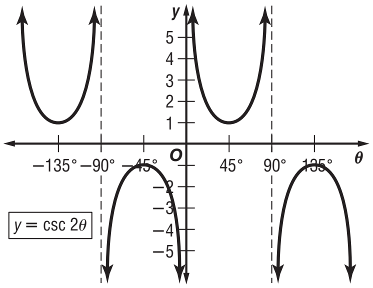


49. Sample answer: You can use changes in amplitude and period along with vertical and horizontal shifts to show an animal population's starting point and display changes to that population over a period of time. Answers should include the following information.

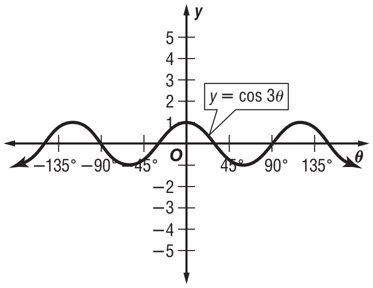
- The equation shows a rabbit population that begins at 1200, increases to a maximum of 1450 then decreases to a minimum of 950 over a period of 4 years.
- Relative to $y = a\cos bx$, $y = a\cos bx + k$ would have a vertical shift of k units, while $y = a\cos[b(x - h)]$ has a horizontal shift of h units.

Page 781, Lesson 14-3

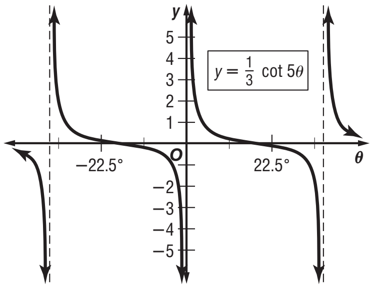
50. amplitude: does not exist; period: 180° or π



51. amplitude: 1; period: 120° or $\frac{2\pi}{3}$

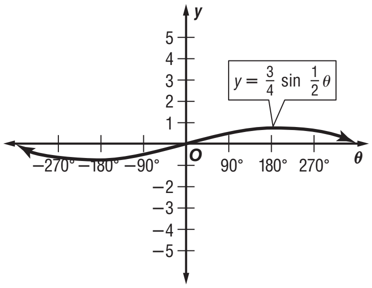


52. amplitude: does not exist; period: 36° or $\frac{\pi}{5}$

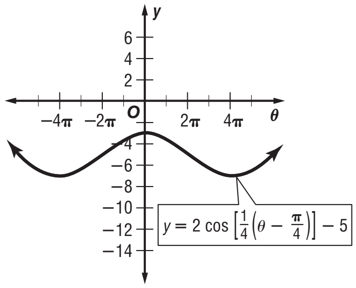


Page 781, Practice Quiz 1

1.



2.



Pages 784–785, Lesson 14-4

1. $\sin\theta\tan\theta \stackrel{?}{=} \sec\theta - \cos\theta$

$\sin\theta\tan\theta \stackrel{?}{=} \frac{1}{\cos\theta} - \cos\theta$ $\quad$ $\sec\theta = \frac{1}{\cos\theta}$

$\sin\theta\tan\theta \stackrel{?}{=} \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}$ $\quad$ Multiply by the LCD, $\cos\theta$.

$\sin\theta\tan\theta \stackrel{?}{=} \frac{1 - \cos^2\theta}{\cos\theta}$ $\quad$ Subtract.

$\sin\theta\tan\theta \stackrel{?}{=} \frac{\sin^2\theta}{\cos\theta}$ $\quad$ $1 - \cos^2\theta = \sin^2\theta$

$\sin\theta\tan\theta \stackrel{?}{=} \sin\theta \cdot \frac{\sin\theta}{\cos\theta}$ $\quad$ Factor.

$\sin\theta\tan\theta = \sin\theta\tan\theta$ $\quad$ $\frac{\sin\theta}{\cos\theta} = \tan\theta$

2. Sample answer: Use various identities, multiply or divide terms to form an equivalent expression, factor, and simplify rational expressions.

3. Sample answer: $\sin^2\theta = 1 + \cos^2\theta$; it is not an identity because $\sin^2\theta = 1 - \cos^2\theta$.

4.
$$\begin{aligned}
\tan\theta\,(\cot\theta + \tan\theta) &\stackrel{?}{=} \sec^2\theta\\
1 + \tan^2\theta &\stackrel{?}{=} \sec^2\theta\\
\sec^2\theta &= \sec^2\theta
\end{aligned}$$

5.
$$\begin{aligned}
\tan^2\theta\cos^2\theta &\stackrel{?}{=} 1 - \cos^2\theta\\
\frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta &\stackrel{?}{=} \sin^2\theta\\
\sin^2\theta &= \sin^2\theta
\end{aligned}$$

6.
$$\begin{aligned}
\frac{\cos^2\theta}{1-\sin\theta} &\stackrel{?}{=} 1 + \sin\theta\\
\frac{1-\sin^2\theta}{1-\sin\theta} &\stackrel{?}{=} 1 + \sin\theta\\
\frac{(1-\sin\theta)(1+\sin\theta)}{1-\sin\theta} &\stackrel{?}{=} 1 + \sin\theta\\
1 + \sin\theta &= 1 + \sin\theta
\end{aligned}$$

7.
$$\begin{aligned}
\frac{1+\tan^2\theta}{\csc^2\theta} &\stackrel{?}{=} \tan^2\theta\\
\frac{\sec^2\theta}{\csc^2\theta} &\stackrel{?}{=} \tan^2\theta\\
\frac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}} &\stackrel{?}{=} \tan^2\theta\\
\frac{1}{\cos^2\theta}\cdot\sin^2\theta &\stackrel{?}{=} \tan^2\theta\\
\tan^2\theta &= \tan^2\theta
\end{aligned}$$

8.
$$\begin{aligned}
\frac{\sin\theta}{\sec\theta} &\stackrel{?}{=} \frac{1}{\tan\theta + \cot\theta}\\
\frac{\sin\theta}{\sec\theta} &\stackrel{?}{=} \frac{1}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}\\
\frac{\sin\theta}{\sec\theta} &\stackrel{?}{=} \frac{1}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}\\
\frac{\sin\theta}{\sec\theta} &\stackrel{?}{=} \frac{\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta}\\
\frac{\sin\theta}{\sec\theta} &\stackrel{?}{=} \frac{\sin\theta\cos\theta}{1}\\
\frac{\sin\theta}{\sec\theta} &= \frac{\sin\theta}{\sec\theta}
\end{aligned}$$

9.
$$\begin{aligned}
\frac{\sec\theta + 1}{\tan\theta} &\stackrel{?}{=} \frac{\tan\theta}{\sec\theta - 1}\\
\frac{\sec\theta + 1}{\tan\theta} &\stackrel{?}{=} \frac{\tan\theta}{\sec\theta - 1}\cdot\frac{\sec\theta + 1}{\sec\theta + 1}\\
\frac{\sec\theta + 1}{\tan\theta} &\stackrel{?}{=} \frac{\tan\theta\cdot(\sec\theta + 1)}{\sec^2\theta - 1}\\
\frac{\sec\theta + 1}{\tan\theta} &\stackrel{?}{=} \frac{\tan\theta\cdot(\sec\theta + 1)}{\tan^2\theta}\\
\frac{\sec\theta + 1}{\tan\theta} &= \frac{\sec\theta + 1}{\tan\theta}
\end{aligned}$$

11.
$$\begin{aligned}
\cos^2\theta + \tan^2\theta\cos^2\theta &\stackrel{?}{=} 1\\
\cos^2\theta + \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta &\stackrel{?}{=} 1\\
\cos^2\theta + \sin^2\theta &\stackrel{?}{=} 1\\
1 &= 1
\end{aligned}$$

12.
$$\begin{aligned}
\cot\theta\,(\cot\theta + \tan\theta) &\stackrel{?}{=} \csc^2\theta\\
\cot^2\theta + \cot\theta\tan\theta &\stackrel{?}{=} \csc^2\theta\\
\cot^2\theta + \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} &\stackrel{?}{=} \csc^2\theta\\
\cot^2\theta + 1 &\stackrel{?}{=} \csc^2\theta\\
\csc^2\theta &= \csc^2\theta
\end{aligned}$$

13.
$$\begin{aligned}
1 + \sec^2\theta\sin^2\theta &\stackrel{?}{=} \sec^2\theta\\
1 + \frac{1}{\cos^2\theta}\cdot\sin^2\theta &\stackrel{?}{=} \sec^2\theta\\
1 + \tan^2\theta &\stackrel{?}{=} \sec^2\theta\\
\sec^2\theta &= \sec^2\theta
\end{aligned}$$

14.
$$\begin{aligned}
\sin\theta\sec\theta\cot\theta &\stackrel{?}{=} 1\\
\sin\theta\cdot\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} &\stackrel{?}{=} 1\\
1 &= 1
\end{aligned}$$

15.
$$\begin{aligned}
\frac{1-\cos\theta}{1+\cos\theta} &\stackrel{?}{=} (\csc\theta - \cot\theta)^2\\
\frac{1-\cos\theta}{1+\cos\theta} &\stackrel{?}{=} \csc^2\theta - 2\cot\theta\csc\theta + \cot^2\theta\\
\frac{1-\cos\theta}{1+\cos\theta} &\stackrel{?}{=} \frac{1}{\sin^2\theta} - 2\cdot\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta} + \frac{\cos^2\theta}{\sin^2\theta}\\
\frac{1-\cos\theta}{1+\cos\theta} &\stackrel{?}{=} \frac{1}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}\\
\frac{1-\cos\theta}{1+\cos\theta} &\stackrel{?}{=} \frac{1 - 2\cos\theta + \cos^2\theta}{\sin^2\theta}\\
\frac{1-\cos\theta}{1+\cos\theta} &\stackrel{?}{=} \frac{(1-\cos\theta)(1-\cos\theta)}{1-\cos^2\theta}\\
\frac{1-\cos\theta}{1+\cos\theta} &\stackrel{?}{=} \frac{(1-\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}\\
\frac{1-\cos\theta}{1+\cos\theta} &= \frac{1-\cos\theta}{1+\cos\theta}
\end{aligned}$$

16.
$$\begin{aligned}
\frac{1 - 2\cos^2\theta}{\sin\theta\cos\theta} &\stackrel{?}{=} \tan\theta - \cot\theta\\
\frac{(1 - \cos^2\theta) - \cos^2\theta}{\sin\theta\cos\theta} &\stackrel{?}{=} \tan\theta - \cot\theta\\
\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} &\stackrel{?}{=} \tan\theta - \cot\theta\\
\frac{\sin^2\theta}{\sin\theta\cos\theta} - \frac{\cos^2\theta}{\sin\theta\cos\theta} &\stackrel{?}{=} \tan\theta - \cot\theta\\
\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta} &\stackrel{?}{=} \tan\theta - \cot\theta\\
\tan\theta - \cot\theta &= \tan\theta - \cot\theta
\end{aligned}$$

17.
$$\begin{aligned}
\cot\theta\csc\theta &\stackrel{?}{=} \frac{\cot\theta + \csc\theta}{\sin\theta + \tan\theta}\\
\cot\theta\csc\theta &\stackrel{?}{=} \frac{\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}}{\sin\theta + \frac{\sin\theta}{\cos\theta}}\\
\cot\theta\csc\theta &\stackrel{?}{=} \frac{\frac{\cos\theta + 1}{\sin\theta}}{\frac{\sin\theta\cos\theta + \sin\theta}{\cos\theta}}\\
\cot\theta\csc\theta &\stackrel{?}{=} \frac{\frac{\cos\theta + 1}{\sin\theta}}{\frac{\sin\theta\,(\cos\theta + 1)}{\cos\theta}}\\
\cot\theta\csc\theta &\stackrel{?}{=} \frac{\cos\theta + 1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta\,(\cos\theta + 1)}\\
\cot\theta\csc\theta &\stackrel{?}{=} \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\\
\cot\theta\csc\theta &= \cot\theta\csc\theta
\end{aligned}$$

18.

$$\sin\theta + \cos\theta \overset{?}{=} \frac{1+\tan\theta}{\sec\theta}$$

$$\sin\theta + \cos\theta \overset{?}{=} \frac{1+\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$$

$$\sin\theta + \cos\theta \overset{?}{=} \frac{\frac{\sin\theta+\cos\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$$

$$\sin\theta + \cos\theta \overset{?}{=} \frac{\sin\theta+\cos\theta}{\cos\theta}\cdot\cos\theta$$

$$\sin\theta + \cos\theta = \sin\theta + \cos\theta$$

19.

$$\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} \overset{?}{=} \cot\theta$$

$$\frac{\frac{1}{\cos\theta}}{\sin\theta} - \frac{\sin\theta}{\cos\theta} \overset{?}{=} \cot\theta$$

$$\frac{1}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta} \overset{?}{=} \cot\theta$$

$$\frac{1-\sin^2\theta}{\sin\theta\cos\theta} \overset{?}{=} \cot\theta$$

$$\frac{\cos^2\theta}{\sin\theta\cos\theta} \overset{?}{=} \cot\theta$$

$$\frac{\cos\theta}{\sin\theta} \overset{?}{=} \cot\theta$$

$$\cot\theta = \cot\theta$$

20.

$$\frac{\sin\theta}{1-\cos\theta} + \frac{1-\cos\theta}{\sin\theta} \overset{?}{=} 2\csc\theta$$

$$\frac{\sin\theta}{\sin\theta}\cdot\frac{\sin\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1-\cos\theta}\cdot\frac{1-\cos\theta}{\sin\theta} \overset{?}{=} 2\csc\theta$$

$$\frac{\sin^2\theta}{\sin\theta\,(1-\cos\theta)} + \frac{1-2\cos\theta+\cos^2\theta}{\sin\theta\,(1-\cos\theta)} \overset{?}{=} 2\csc\theta$$

$$\frac{\sin^2\theta+\cos^2\theta+1-2\cos\theta}{\sin\theta\,(1-\cos\theta)} \overset{?}{=} 2\csc\theta$$

$$\frac{2-2\cos\theta}{\sin\theta\,(1-\cos\theta)} \overset{?}{=} 2\csc\theta$$

$$\frac{2(1-\cos\theta)}{\sin\theta\,(1-\cos\theta)} \overset{?}{=} 2\csc\theta$$

$$\frac{2}{\sin\theta} \overset{?}{=} 2\csc\theta$$

$$2\csc\theta = 2\csc\theta$$

21.

$$\frac{1+\sin\theta}{\sin\theta} \overset{?}{=} \frac{\cot^2\theta}{\csc\theta-1}$$

$$\frac{1+\sin\theta}{\sin\theta} \overset{?}{=} \frac{\cot^2\theta}{\csc\theta-1}\cdot\frac{\csc\theta+1}{\csc\theta+1}$$

$$\frac{1+\sin\theta}{\sin\theta} \overset{?}{=} \frac{\cot^2\theta\,(\csc\theta+1)}{\csc^2\theta-1}$$

$$\frac{1+\sin\theta}{\sin\theta} \overset{?}{=} \frac{\cot^2\theta\,(\csc\theta+1)}{\cot^2\theta}$$

$$\frac{1+\sin\theta}{\sin\theta} \overset{?}{=} \csc\theta+1$$

$$\frac{1+\sin\theta}{\sin\theta} \overset{?}{=} \frac{1}{\sin\theta}+\frac{\sin\theta}{\sin\theta}$$

$$\frac{1+\sin\theta}{\sin\theta} = \frac{1+\sin\theta}{\sin\theta}$$

22.

$$\frac{1+\tan\theta}{1+\cot\theta} \overset{?}{=} \frac{\sin\theta}{\cos\theta}$$

$$\frac{1+\frac{\sin\theta}{\cos\theta}}{1+\frac{\cos\theta}{\sin\theta}} \overset{?}{=} \frac{\sin\theta}{\cos\theta}$$

$$\frac{\frac{\sin\theta+\cos\theta}{\cos\theta}}{\frac{\sin\theta+\cos\theta}{\sin\theta}} \overset{?}{=} \frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta+\cos\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta+\cos\theta} \overset{?}{=} \frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$$

23.

$$\frac{1}{\sec^2\theta} + \frac{1}{\csc^2\theta} \overset{?}{=} 1$$

$$\cos^2\theta + \sin^2\theta \overset{?}{=} 1$$

$$1 = 1$$

24.

$$1 + \frac{1}{\cos\theta} \overset{?}{=} \frac{\tan^2\theta}{\sec\theta-1}$$

$$1 + \frac{1}{\cos\theta} \overset{?}{=} \frac{\tan^2\theta}{\sec\theta-1}\cdot\frac{\sec\theta+1}{\sec\theta+1}$$

$$1 + \frac{1}{\cos\theta} \overset{?}{=} \frac{\tan^2\theta\,(\sec\theta+1)}{\sec^2\theta-1}$$

$$1 + \frac{1}{\cos\theta} \overset{?}{=} \frac{\tan^2\theta\,(\sec\theta+1)}{\tan^2\theta}$$

$$1 + \frac{1}{\cos\theta} \overset{?}{=} \sec\theta+1$$

$$1 + \frac{1}{\cos\theta} = 1 + \frac{1}{\cos\theta}$$

25.

$$1-\tan^4\theta \overset{?}{=} 2\sec^2\theta - \sec^4\theta$$

$$(1-\tan^2\theta)(1+\tan^2\theta) \overset{?}{=} \sec^2\theta\,(2-\sec^2\theta)$$

$$[(1-(\sec^2\theta-1)](\sec^2\theta) \overset{?}{=} (2-\sec^2\theta)(\sec^2\theta)$$

$$(2-\sec^2\theta)(\sec^2\theta) = (2-\sec^2\theta)(\sec^2\theta)$$

26.

$$\cos^4\theta - \sin^4\theta \overset{?}{=} \cos^2\theta - \sin^2\theta$$

$$(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta) \overset{?}{=} \cos^2\theta - \sin^2\theta$$

$$(\cos^2\theta-\sin^2\theta)\cdot 1 \overset{?}{=} \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta - \sin^2\theta = \cos^2\theta - \sin^2\theta$$

27.

$$\frac{1-\cos\theta}{\sin\theta} \overset{?}{=} \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{1-\cos\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} \overset{?}{=} \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{1-\cos^2\theta}{\sin\theta\,(1+\cos\theta)} \overset{?}{=} \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{\sin^2\theta}{\sin\theta\,(1+\cos\theta)} \overset{?}{=} \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta}$$

28.

$$\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta} \overset{?}{=} 2\sec\theta$$

$$\frac{\cos\theta}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta} + \frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta} \overset{?}{=} 2\sec\theta$$

$$\frac{\cos\theta\,(1-\sin\theta)+\cos\theta\,(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} \overset{?}{=} 2\sec\theta$$

$$\frac{\cos\theta - \sin\theta\cos\theta + \cos\theta + \sin\theta\cos\theta}{1-\sin^2\theta} \overset{?}{=} 2\sec\theta$$

$$\frac{2\cos\theta}{\cos^2\theta} \overset{?}{=} 2\sec\theta$$

$$\frac{2}{\cos\theta} \overset{?}{=} 2\sec\theta$$

$$2\sec\theta = 2\sec\theta$$

29.
$$\tan\theta\sin\theta\cos\theta\csc^2\theta \stackrel{?}{=} 1$$
$$\frac{\sin\theta}{\cos\theta}\cdot\sin\theta\cdot\cos\theta\cdot\frac{1}{\sin^2\theta} \stackrel{?}{=} 1$$
$$1 = 1$$

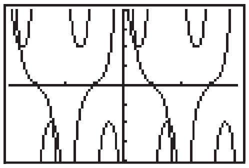
30.
$$\frac{\sin^2\theta}{1-\cos\theta} \stackrel{?}{=} 1+\cos\theta$$
$$\frac{\sin^2\theta}{1-\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} \stackrel{?}{=} 1+\cos\theta$$
$$\frac{\sin^2\theta\,(1+\cos\theta)}{1-\cos^2\theta} \stackrel{?}{=} 1+\cos\theta$$
$$\frac{\sin^2\theta\,(1+\cos\theta)}{\sin^2\theta} \stackrel{?}{=} 1+\cos\theta$$
$$1+\cos\theta = 1+\cos\theta$$

31.
$$\frac{v^2\tan^2\theta}{2g\sec^2\theta} \stackrel{?}{=} \frac{v^2\frac{\sin^2\theta}{\cos^2\theta}}{2g\frac{1}{\cos^2\theta}}$$
$$\stackrel{?}{=} \frac{v^2}{2g}\cdot\frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^2\theta}{1}$$
$$= \frac{v^2\sin^2\theta}{2g}$$

34. Sample answer: Trigonometric identities are verified in a similar manner to proving theorems in geometry before using them. Answers should include the following.

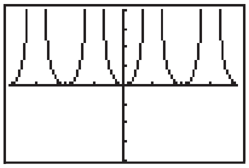
- The expressions have not yet been shown to be equal, so you could not use the properties of equality on them.
- To show two expressions you must transform one, or both independently.
- Graphing two expressions could result in identical graphs for a set interval, that are different elsewhere.

37.



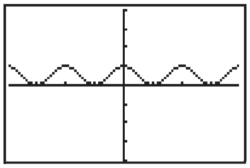
[−360, 360] scl: 90 by [−5, 5] scl: 1

38.



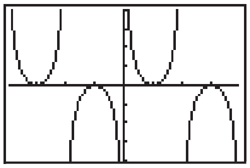
[−360, 360] scl: 90 by [−5, 5] scl: 1

39.



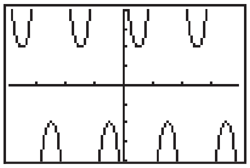
[−360, 360] scl: 90 by [−5, 5] scl: 1

40.



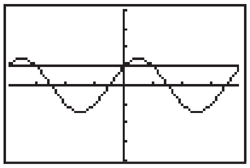
[−360, 360] scl: 90 by [−5, 5] scl: 1

41.



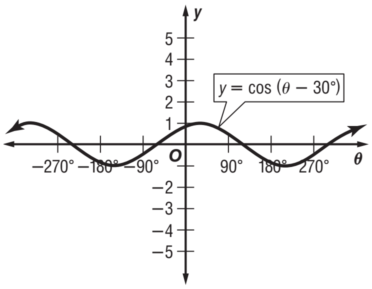
[−360, 360] scl: 90 by [−5, 5] scl: 1

42.

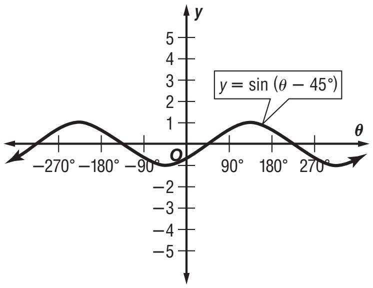


[−360, 360] scl: 90 by [−5, 5] scl: 1

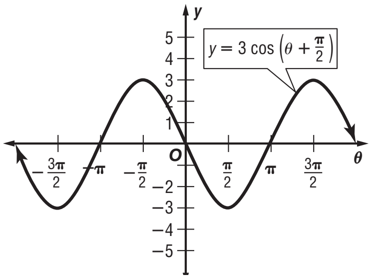
47.



48.



49.



Pages 788–790, Lesson 14-5

10. $\cos(270° - \theta) \stackrel{?}{=} \cos 270°\cos\theta + \sin 270°\sin\theta$
$$\stackrel{?}{=} 0 + (-1\sin\theta)$$
$$= -\sin\theta$$

11.
$$\sin\left(\theta+\frac{\pi}{2}\right) \stackrel{?}{=} \cos\theta$$
$$\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2} \stackrel{?}{=} \cos\theta$$
$$\sin\theta\cdot 0 + \cos\theta\cdot 1 \stackrel{?}{=} \cos\theta$$
$$\cos\theta = \cos\theta$$

12. $\sin(\theta + 30°) + \cos(\theta + 60°)$
$$\stackrel{?}{=} \sin\theta\cos 30° + \cos\theta\sin 30° + \cos\theta\cos 60° - \sin\theta\sin 60°$$
$$\stackrel{?}{=} \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$
$$\stackrel{?}{=} \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta$$
$$= \cos\theta$$

28. $\sin(270° - \theta) \stackrel{?}{=} \sin 270°\cos\theta - \cos 270°\sin\theta$
$$\stackrel{?}{=} -1\cos\theta - 0$$
$$= -\cos\theta$$

29. $\cos(90° + \theta) \stackrel{?}{=} \cos 90°\cos\theta - \sin 90°\sin\theta$
$$\stackrel{?}{=} 0 - 1\sin\theta$$
$$= -\sin\theta$$

30. $\cos(90° - \theta) \stackrel{?}{=} \cos 90°\cos\theta + \sin 90°\sin\theta$
$$\stackrel{?}{=} 0\cdot\cos\theta + 1\cdot\sin\theta$$
$$= \sin\theta$$

31.
$$\begin{aligned}
\sin(90^\circ - \theta) &\stackrel{?}{=} \cos\theta\\
\sin 90^\circ \cos\theta - \cos 90^\circ \sin\theta &\stackrel{?}{=} \cos\theta\\
1\cdot\cos\theta - 0\cdot\sin\theta &\stackrel{?}{=} \cos\theta\\
\cos\theta - 0 &\stackrel{?}{=} \cos\theta\\
\cos\theta &= \cos\theta
\end{aligned}$$

32.
$$\begin{aligned}
\sin\left(\theta + \frac{3\pi}{2}\right) &\stackrel{?}{=} -\cos\theta\\
\sin\theta\cos\frac{3\pi}{2} + \cos\theta\sin\frac{3\pi}{2} &\stackrel{?}{=} -\cos\theta\\
\sin\theta\cdot 0 + \cos\theta\cdot(-1) &\stackrel{?}{=} -\cos\theta\\
0 + (-\cos\theta) &\stackrel{?}{=} -\cos\theta\\
-\cos\theta &= -\cos\theta
\end{aligned}$$

33.
$$\begin{aligned}
\cos(\pi - \theta) &\stackrel{?}{=} -\cos\theta\\
\cos\pi\cos\theta + \sin\pi\sin\theta &\stackrel{?}{=} -\cos\theta\\
-1\cdot\cos\theta + 0\cdot\sin\theta &\stackrel{?}{=} -\cos\theta\\
-\cos\theta &= -\cos\theta
\end{aligned}$$

34.
$$\begin{aligned}
\cos(2\pi + \theta) &\stackrel{?}{=} \cos\theta\\
\cos 2\pi\cos\theta - [\sin 2\pi\sin\theta] &\stackrel{?}{=} \cos\theta\\
1\cdot\cos\theta - [0\cdot\sin\theta] &\stackrel{?}{=} \cos\theta\\
1\cdot\cos\theta - 0 &\stackrel{?}{=} \cos\theta\\
\cos\theta &= \cos\theta
\end{aligned}$$

35.
$$\begin{aligned}
\sin(\pi - \theta) &\stackrel{?}{=} \sin\theta\\
\sin\pi\cos\theta - [\cos\pi\sin\theta] &\stackrel{?}{=} \sin\theta\\
0\cdot\cos\theta - [-1\cdot\sin\theta] &\stackrel{?}{=} \sin\theta\\
0 - [-\sin\theta] &\stackrel{?}{=} \sin\theta\\
\sin\theta &= \sin\theta
\end{aligned}$$

36.
$$\begin{aligned}
&\sin(60^\circ + \theta) + \sin(60^\circ - \theta)\\
&\stackrel{?}{=} \sin 60^\circ\cos\theta + \cos 60^\circ\sin\theta + \sin 60^\circ\cos\theta - \cos 60^\circ\sin\theta\\
&\stackrel{?}{=} \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\\
&= \sqrt{3}\cos\theta
\end{aligned}$$

37.
$$\begin{aligned}
&\sin\left(\theta + \frac{\pi}{3}\right) - \cos\left(\theta + \frac{\pi}{6}\right)\\
&\stackrel{?}{=} \sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3} - \cos\theta\cos\frac{\pi}{6} + \sin\theta\sin\frac{\pi}{6}\\
&\stackrel{?}{=} \frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta - \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\\
&\stackrel{?}{=} \frac{1}{2}\sin\theta + \frac{1}{2}\sin\theta\\
&= \sin\theta
\end{aligned}$$

38.
$$\begin{aligned}
&\sin(\alpha + \beta)\sin(\alpha - \beta) \stackrel{?}{=} \sin^2\alpha - \sin^2\beta\\
&\stackrel{?}{=} (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)\\
&\stackrel{?}{=} \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta\\
&\stackrel{?}{=} \sin^2\alpha\,(1 - \sin^2\beta) - (1 - \sin^2\alpha)\sin^2\beta\\
&\stackrel{?}{=} \sin^2\alpha - \sin^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\alpha\sin^2\beta\\
&= \sin^2\alpha - \sin^2\beta
\end{aligned}$$

39.
$$\begin{aligned}
\cos(\alpha + \beta) &\stackrel{?}{=} \frac{1 - \tan\alpha\tan\beta}{\sec\alpha\sec\beta}\\
\cos(\alpha + \beta) &\stackrel{?}{=} \frac{1 - \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}}{\frac{1}{\cos\alpha}\cdot\frac{1}{\cos\beta}}\\
\cos(\alpha + \beta) &\stackrel{?}{=} \frac{1 - \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}}{\frac{1}{\cos\alpha}\cdot\frac{1}{\cos\beta}}\cdot\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}\\
\cos(\alpha + \beta) &\stackrel{?}{=} \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{1}\\
\cos(\alpha + \beta) &= \cos(\alpha + \beta)
\end{aligned}$$

46.
$$\begin{aligned}
\tan(\alpha + \beta) &\stackrel{?}{=} \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}\\
&\stackrel{?}{=} \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\\
&\stackrel{?}{=} \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\
&= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}
\end{aligned}$$

$$\begin{aligned}
\tan(\alpha - \beta) &\stackrel{?}{=} \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)}\\
&\stackrel{?}{=} \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}\\
&\stackrel{?}{=} \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\
&= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}
\end{aligned}$$

50.
$$\begin{aligned}
\cot\theta + \sec\theta &\stackrel{?}{=} \frac{\cos^2\theta + \sin\theta}{\sin\theta\cos\theta}\\
\cot\theta + \sec\theta &\stackrel{?}{=} \frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin\theta}{\sin\theta\cos\theta}\\
\cot\theta + \sec\theta &\stackrel{?}{=} \frac{\cos\theta}{\sin\theta} + \frac{1}{\cos\theta}\\
\cot\theta + \sec\theta &= \cot\theta + \sec\theta
\end{aligned}$$

51.
$$\begin{aligned}
\sin^2\theta + \tan^2\theta &\stackrel{?}{=} (1 - \cos^2\theta) + \frac{\sec^2\theta}{\csc^2\theta}\\
\sin^2\theta + \tan^2\theta &\stackrel{?}{=} \sin^2\theta + \frac{\sec^2\theta}{\csc^2\theta}\\
\sin^2\theta + \tan^2\theta &\stackrel{?}{=} \sin^2\theta + \frac{1}{\cos^2\theta} \div \frac{1}{\sin^2\theta}\\
\sin^2\theta + \tan^2\theta &\stackrel{?}{=} \sin^2\theta + \frac{\sin^2\theta}{\cos^2\theta}\\
\sin^2\theta + \tan^2\theta &= \sin^2\theta + \tan^2\theta
\end{aligned}$$

52.
$$\begin{aligned}
\sin\theta\,(\sin\theta + \csc\theta) &\stackrel{?}{=} 2 - \cos^2\theta\\
\sin^2\theta + 1 &\stackrel{?}{=} 2 - \cos^2\theta\\
1 - \cos^2\theta + 1 &\stackrel{?}{=} 2 - \cos^2\theta\\
2 - \cos^2\theta &= 2 - \cos^2\theta
\end{aligned}$$

53.
$$\begin{aligned}
\frac{\sec\theta}{\tan\theta} &\stackrel{?}{=} \csc\theta\\
\frac{1}{\cos\theta} \div \frac{\sin\theta}{\cos\theta} &\stackrel{?}{=} \csc\theta\\
\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} &\stackrel{?}{=} \csc\theta\\
\frac{1}{\sin\theta} &\stackrel{?}{=} \csc\theta\\
\csc\theta &= \csc\theta
\end{aligned}$$

Pages 795–797, Lesson 14-6

31.
$$\begin{aligned}
\sin 2x &\stackrel{?}{=} 2\cot x\sin^2 x\\
2\sin x\cos x &\stackrel{?}{=} 2\cdot\frac{\cos x}{\sin x}\cdot\sin^2 x\\
2\sin x\cos x &= 2\sin x\cos x
\end{aligned}$$

32.
$$\begin{aligned} 2\cos^2\frac{x}{2} &\stackrel{?}{=} 1+\cos x \\ 2\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2 &\stackrel{?}{=} 1+\cos x \\ 2\left(\frac{1+\cos x}{2}\right) &\stackrel{?}{=} 1+\cos x \\ 1+\cos x &= 1+\cos x \end{aligned}$$

33.
$$\begin{aligned} \sin^4 x-\cos^4 x &\stackrel{?}{=} 2\sin^2 x-1 \\ (\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x) &\stackrel{?}{=} 2\sin^2 x-1 \\ (\sin^2 x-\cos^2 x)\cdot 1 &\stackrel{?}{=} 2\sin^2 x-1 \\ [\sin^2 x-(1-\sin^2 x)]\cdot 1 &\stackrel{?}{=} 2\sin^2 x-1 \\ \sin^2 x-1+\sin^2 x &\stackrel{?}{=} 2\sin^2 x-1 \\ 2\sin^2 x-1 &= 2\sin^2 x-1 \end{aligned}$$

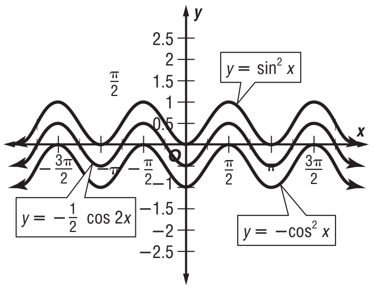
34.
$$\begin{aligned} \sin^2 x &\stackrel{?}{=} \frac{1}{2}(1-\cos 2x) \\ \sin^2 x &\stackrel{?}{=} \frac{1}{2}[1-(1-2\sin^2 x)] \\ \sin^2 x &\stackrel{?}{=} \frac{1}{2}(2\sin^2 x) \\ \sin^2 x &= \sin^2 x \end{aligned}$$

35.
$$\begin{aligned} \tan^2\frac{x}{2} &\stackrel{?}{=} \frac{1-\cos x}{1+\cos x} \\ \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} &\stackrel{?}{=} \frac{1-\cos x}{1+\cos x} \\ \frac{\left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2}{\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2} &\stackrel{?}{=} \frac{1-\cos x}{1+\cos x} \\ \frac{1-\cos x}{1+\cos x} &= \frac{1-\cos x}{1+\cos x} \end{aligned}$$

36.
$$\begin{aligned} \frac{1}{\sin x\cos x}-\frac{\cos x}{\sin x} &\stackrel{?}{=} \tan x \\ \frac{1-\cos^2 x}{\sin x\cos x} &\stackrel{?}{=} \tan x \\ \frac{\sin^2 x}{\sin x\cos x} &\stackrel{?}{=} \tan x \\ \frac{\sin x}{\cos x} &\stackrel{?}{=} \tan x \\ \tan x &= \tan x \end{aligned}$$

40.
$$\begin{aligned} \frac{2}{g}v^2(\tan\theta-\tan\theta\sin^2\theta) &\stackrel{?}{=} \frac{2}{g}v^2\tan\theta\,(1-\sin^2\theta) \\ &\stackrel{?}{=} \frac{2}{g}v^2\tan\theta\cos^2\theta \\ &\stackrel{?}{=} \frac{2}{g}v^2\sin\theta\cos\theta \\ &= \frac{v^2\sin 2\theta}{g} \end{aligned}$$

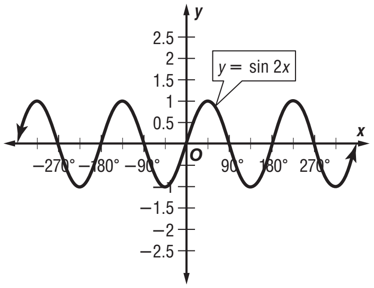
42.



Sample answer: They all have the same shape and are vertical translations of each other.

43. The maxima occur at $\pm\frac{\pi}{2}$ and $\pm\frac{3\pi}{2}$. The minima occur at $x = 0$, $\pm\pi$, and $\pm 2\pi$.

44.



45. The graph of $f(x)$ crosses the x-axis at the points specified in Exercise 43.

46. $c = 1$ and $d = 0.5$

56.
$$\begin{aligned} \cot^2\theta-\sin^2\theta &\stackrel{?}{=} \frac{\cos^2\theta\csc^2\theta-\sin^2\theta}{\sin^2\theta\csc^2\theta} \\ \cot^2\theta-\sin^2\theta &\stackrel{?}{=} \frac{\cos^2\theta\,\frac{1}{\sin^2\theta}-\sin^2\theta}{\sin^2\theta\,\frac{1}{\sin^2\theta}} \\ \cot^2\theta-\sin^2\theta &\stackrel{?}{=} \frac{\cot^2\theta-\sin^2\theta}{1} \\ \cot^2\theta-\sin^2\theta &= \cot^2\theta-\sin^2\theta \end{aligned}$$

57.
$$\begin{aligned} \cos\theta\,(\cos\theta+\cot\theta) &\stackrel{?}{=} \cot\theta\cos\theta\,(\sin\theta+1) \\ \cos\theta\,(\cos\theta+\cot\theta) &\stackrel{?}{=} \frac{\cos\theta}{\sin\theta}\cos\theta\sin\theta+\cot\theta\cos\theta \\ \cos\theta\,(\cos\theta+\cot\theta) &\stackrel{?}{=} \cos^2\theta+\cot\theta\cos\theta \\ \cos\theta\,(\cos\theta+\cot\theta) &= \cos\theta\,(\cos\theta+\cot\theta) \end{aligned}$$

Page 797, Practice Quiz 2

1.
$$\begin{aligned} \sin\theta\sec\theta &\stackrel{?}{=} \tan\theta \\ \sin\theta\cdot\frac{1}{\cos\theta} &\stackrel{?}{=} \tan\theta \\ \frac{\sin\theta}{\cos\theta} &\stackrel{?}{=} \tan\theta \\ \tan\theta &= \tan\theta \end{aligned}$$

2.
$$\begin{aligned} \sec\theta-\cos\theta &\stackrel{?}{=} \sin\theta\tan\theta \\ \frac{1}{\cos\theta}-\cos\theta\cdot\frac{\cos\theta}{\cos\theta} &\stackrel{?}{=} \sin\theta\tan\theta \\ \frac{1}{\cos\theta}-\frac{\cos^2\theta}{\cos\theta} &\stackrel{?}{=} \sin\theta\tan\theta \\ \frac{1-\cos^2\theta}{\cos\theta} &\stackrel{?}{=} \sin\theta\tan\theta \\ \frac{\sin^2\theta}{\cos\theta} &\stackrel{?}{=} \sin\theta\tan\theta \\ \sin\theta\,\frac{\sin\theta}{\cos\theta} &\stackrel{?}{=} \sin\theta\tan\theta \\ \sin\theta\tan\theta &= \sin\theta\tan\theta \end{aligned}$$

3.
$$\begin{aligned} \sin\theta+\tan\theta &\stackrel{?}{=} \frac{\sin\theta\,(\cos\theta+1)}{\cos\theta} \\ \sin\theta+\tan\theta &\stackrel{?}{=} \frac{\sin\theta\cos\theta+\sin\theta}{\cos\theta} \\ \sin\theta+\tan\theta &\stackrel{?}{=} \frac{\sin\theta\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta} \\ \sin\theta+\tan\theta &= \sin\theta+\tan\theta \end{aligned}$$

4.
$$\begin{aligned} \sin(90^\circ+\theta) &\stackrel{?}{=} \cos\theta \\ \sin 90^\circ\cos\theta+\cos 90^\circ\sin\theta &\stackrel{?}{=} \cos\theta \\ \cos\theta+0 &\stackrel{?}{=} \cos\theta \\ \cos\theta &= \cos\theta \end{aligned}$$

5. $\cos\left(\frac{3\pi}{2} - \theta\right) \stackrel{?}{=} -\sin\theta$

$\cos\frac{3\pi}{2}\cos\theta + \sin\frac{3\pi}{2}\sin\theta \stackrel{?}{=} -\sin\theta$

$0 + (-1 \cdot \sin\theta) \stackrel{?}{=} -\sin\theta$

$-\sin\theta = -\sin\theta$

6. $\sin(\theta + 30°) + \cos(\theta + 60°)$

$\stackrel{?}{=} (\sin\theta\cos 30° + \cos\theta\sin 30°) + (\cos\theta\cos 60° - \sin\theta\sin 60°)$

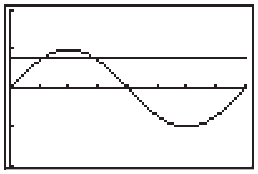
$\stackrel{?}{=} \left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right) + \left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right)$

$\stackrel{?}{=} \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta$

$= \cos\theta$

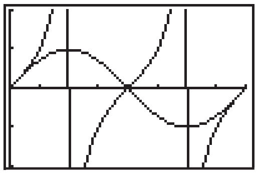
Page 798, Preview of Lesson 14-7 Graphing Calculator Investigation

1.



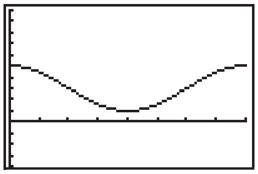
[0, 360] scl: 45 by [−2, 2] scl: 1

2.



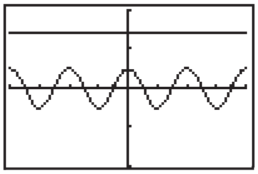
[0, 360] scl: 45 by [−2, 2] scl: 1

3.



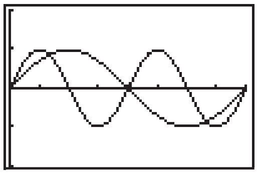
[0, 360] scl: 45 by [−5, 10] scl: 1

4.



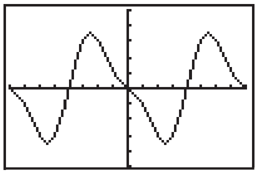
[−720, 720] scl: 90 by [−2, 2] scl: 1

5.



[0, 360] scl: 45 by [−2, 2] scl: 1

6.



[−360, 360] scl: 45 by [−5, 5] scl: 1

Page 803, Lesson 14-7

35. $0 + 2k\pi, \frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi$ or $0° + k \cdot 360°$, $90° + k \cdot 360°$, $270° + k \cdot 360°$

36. $\frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi$ or $210° + k \cdot 360°$, $330° + k \cdot 360°$

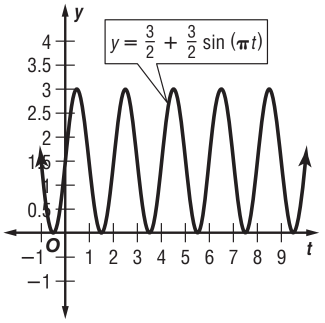
37. $0 + k\pi$ or $0° + k \cdot 180°$

38. $\frac{\pi}{2} + k\pi, \frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi$ or $90° + k \cdot 180°$, $120° + k \cdot 360°$, $240° + k \cdot 360°$

39. $0 + 2k\pi, \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi$ or $0° + k \cdot 360°$, $60° + k \cdot 360°$, $300° + k \cdot 360°$

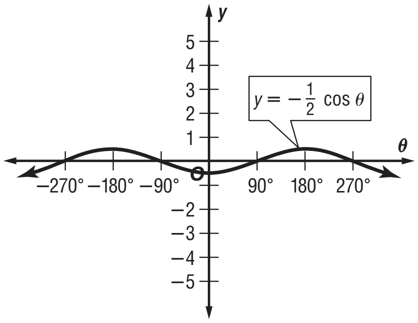
40. $\frac{\pi}{2} + 4k\pi$ or $90° + k \cdot 720°$

43. $y = \frac{3}{2} + \frac{3}{2}\sin(\pi t)$

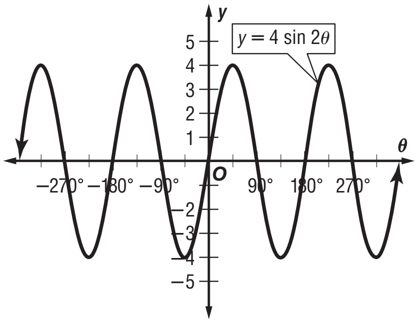


Pages 805–807, Chapter 14 Study Guide and Review

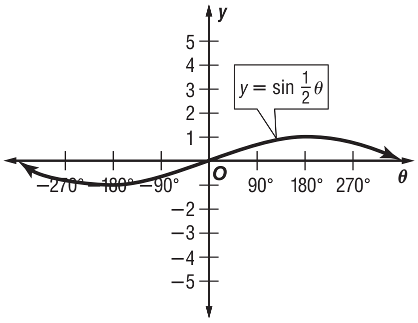
9. amplitude: $\frac{1}{2}$; period: 360° or 2π



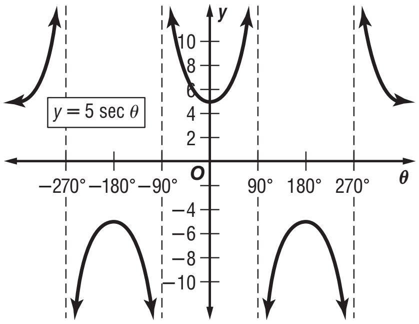
10. amplitude: 4; period: 180° or π



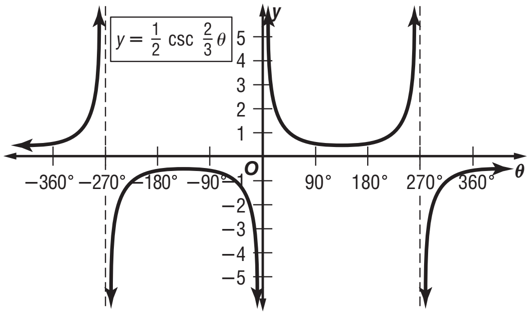
11. amplitude: 1; period: 720° or 4π



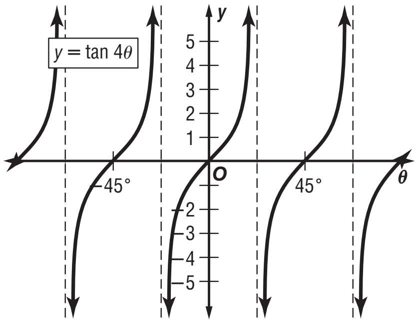
12. amplitude: does not exist; period: 360° or 2π



13. amplitude: does not exist; period: 540° or 3π



14. amplitude: does not exist; period: 45° or $\frac{\pi}{4}$



34.
$$\cos(90° + \theta) \stackrel{?}{=} -\sin\theta$$
$$\cos 90° \cos\theta - \sin 90° \sin\theta \stackrel{?}{=} -\sin\theta$$
$$0 \cdot \cos\theta - 1 \cdot \sin\theta \stackrel{?}{=} -\sin\theta$$
$$-\sin\theta = -\sin\theta$$

35.
$$\sin(30° - \theta) \stackrel{?}{=} \cos(60° + \theta)$$
$$\sin 30° \cos\theta - \cos 30° \sin\theta \stackrel{?}{=} \cos 60° \cos\theta - \sin 60° \sin\theta$$
$$\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$

36.
$$\sin(\theta + \pi) \stackrel{?}{=} -\sin\theta$$
$$\sin\theta\cos\pi + \cos\theta\sin\pi \stackrel{?}{=} -\sin\theta$$
$$(\sin\theta)(-1) + (\cos\theta)(0) \stackrel{?}{=} -\sin\theta$$
$$-\sin\theta = -\sin\theta$$

37.
$$-\cos\theta \stackrel{?}{=} \cos(\pi + \theta)$$
$$-\cos\theta \stackrel{?}{=} \cos\pi\cos\theta - \sin\pi\sin\theta$$
$$-\cos\theta \stackrel{?}{=} -1 \cdot \cos\theta - 0 \cdot \sin\theta$$
$$-\cos\theta = -\cos\theta$$

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9.
$$(\sin\theta - \cos\theta)^2 \stackrel{?}{=} 1 - \sin 2\theta$$
$$\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta \stackrel{?}{=} 1 - \sin 2\theta$$
$$(\sin^2\theta + \cos^2\theta) - 2\sin\theta\cos\theta \stackrel{?}{=} 1 - \sin 2\theta$$
$$1 - \sin 2\theta = 1 - \sin 2\theta$$

10.
$$\frac{\cos\theta}{1 - \sin^2\theta} \stackrel{?}{=} \sec\theta$$
$$\frac{\cos\theta}{\cos^2\theta} \stackrel{?}{=} \sec\theta$$
$$\frac{1}{\cos\theta} \stackrel{?}{=} \sec\theta$$
$$\sec\theta = \sec\theta$$

11.
$$\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} \stackrel{?}{=} \cot\theta$$
$$\frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta} \stackrel{?}{=} \cot\theta$$
$$\frac{1}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta} \stackrel{?}{=} \cot\theta$$
$$\frac{\cos^2\theta}{\sin\theta\cos\theta} \stackrel{?}{=} \cot\theta$$
$$\frac{\cos\theta}{\sin\theta} \stackrel{?}{=} \cot\theta$$
$$\cot\theta = \cot\theta$$

12.
$$\frac{1 + \tan^2\theta}{\cos^2\theta} \stackrel{?}{=} \sec^4\theta$$
$$\frac{\sec^2\theta}{\cos^2\theta} \stackrel{?}{=} \sec^4\theta$$
$$\sec^2\theta\sec^2\theta \stackrel{?}{=} \sec^4\theta$$
$$\sec^4\theta = \sec^4\theta$$