Chapter 7

Polynomial Functions

Chapter Overview and Pacing

LESSON OBJECTIVES	PACING (days) Regular Basic/Average	Regular Advanced	Block Basic/Average	Block Advanced
7-1 Polynomial Functions *(pp. 346–352)* • Evaluate polynomial functions. • Identify general shapes of graphs of polynomial functions.	1	1	0.5	0.5
7-2 Graphing Polynomial Functions *(pp. 353–359)* • Graph polynomial functions and locate their real zeros. • Find the maxima and minima of polynomial functions. ***Follow-Up:*** Modeling Real-World Data	1	2 (with 7-2 Follow-Up)	0.5	1.5 (with 7-2 Follow-Up)
7-3 Solving Equations Using Quadratic Techniques *(pp. 360–364)* • Write expressions in quadratic form. • Use quadratic techniques to solve equations.	2	2	1	1
7-4 The Remainder and Factor Theorems *(pp. 365–370)* • Evaluate functions using synthetic substitution. • Determine whether a binomial is a factor of a polynomial by using synthetic substitution.	2	2	1	1
7-5 Roots and Zeros *(pp. 371–377)* • Determine the number and type of roots for a polynomial equation. • Find the zeros of a polynomial function.	1	1	0.5	0.5
7-6 Rational Zero Theorem *(pp. 378–382)* • Identify the possible rational zeros of a polynomial function. • Find all the rational zeros of a polynomial function.	2	2	1	1
7-7 Operations on Functions *(pp. 383–389)* • Find the sum, difference, product, and quotient of functions. • Find the composition of functions.	1	1	0.5	0.5
7-8 Inverse Functions and Relations *(pp. 390–394)* • Find the inverse of a function or relation. • Determine whether two functions or relations are inverses.	1	1	0.5	0.5
7-9 Square Root Functions and Inequalities *(pp. 395–399)* • Graph and analyze square root functions. • Graph square root inequalities.	1	1	0.5	0.5
Study Guide and **Practice Test** *(pp. 400–405)* **Standardized Test Practice** *(pp. 406–407)*	1	1	0.5	0.5
Chapter Assessment	1	1	0.5	0.5
TOTAL	**14**	**15**	**7**	**8**

Pacing suggestions for the entire year can be found on pages T20–T21.

Timesaving Tools
TeacherWorks™
All-In-One Planner and Resource Center
See pages T12–T13.

Chapter Resource Manager

Chapter 7 Resource Masters

Study Guide and Intervention	Practice (Skills and Average)	Reading to Learn Mathematics	Enrichment	Assessment	Applications*	5-Minute Check Transparencies	Interactive Chalkboard	Alge2PASS: Tutorial Plus (lessons)	Materials
375–376	377–378	379	380			7-1	7-1		graphing calculator, grid paper, string
381–382	383–384	385	386		SM 71–76	7-2	7-2		graphing calculator (*Follow-Up:* graphing calculator)
387–388	389–390	391	392	443		7-3	7-3	13	colored pencils
393–394	395–396	397	398			7-4	7-4		
399–400	401–402	403	404	443, 445		7-5	7-5	14	slips of paper
405–406	407–408	409	410		GCS 39	7-6	7-6		
411–412	413–414	415	416	444	GCS 40, SC 13	7-7	7-7		
417–418	419–420	421	422		SC 14	7-8	7-8		grid paper, string, spaghetti
423–424	425–426	427	428	444		7-9	7-9		graphing calculator
				429–442, 446–448					

**Key to Abbreviations:* GCS = Graphing Calculator and Speadsheet Masters,
SC = School-to-Career Masters,
SM = Science and Mathematics Lab Manual

Chapter 7 Mathematical Connections and Background

Continuity of Instruction

Prior Knowledge

Students have evaluated polynomials and terms of polynomials, graphed quadratic functions, and solved quadratic equations and inequalities. They explored synthetic division and the equivalence between zeros of functions and roots of equations. Also, they worked with function notation and manipulated and evaluated functions.

This Chapter

Students look at algebraic relationships among algebraic entities. They use the leading term of a polynomial function to describe and identify the end behavior of the function. They see how synthetic division of polynomials helps them explore the number of roots of a polynomial function and then find those roots. They perform operations on functions, and examine restrictions on domains and ranges necessary for combining functions.

Future Connections

Students will return to polynomials in later mathematics and other courses. They will continue their study of complex numbers, and extend it to study the complex plane. They will frequently use functions, combinations of functions, and inverses of functions in future mathematics topics.

7-1 Polynomial Functions

While this chapter uses some graphs to illustrate ideas, the primary focus of the chapter is on algebraic relationships between algebraic entities. In this lesson, the algebraic entities are the terms of a polynomial function. The main algebraic relationship in the lesson is how the degree and coefficient of one of those entities, the leading term of a polynomial function, determines the end behavior of the function.

The lesson begins by evaluating polynomial functions for various values of the variable, and uses the resulting sets of ordered pairs to illustrate the functions. Students use those illustrations to discuss when a graph may or must cross the x-axis. The illustrations also lead to the lesson's primarily idea of end behavior: for even-degree polynomial functions the left and right extremes of the graph are both positive or both negative, while for odd-degree polynomial functions one extreme is positive and the other extreme is negative.

7-2 Graphing Polynomial Functions

In this lesson, the primary algebraic entities are sets of x values, each set representing an interval on the x-axis. These entities are used to explore two types of changes in the values of a polynomial function. One type of change, a change between positive and negative function values, indicates that the graph of the function is crossing the x-axis, which is where the function has a zero. The other type of change, a change between increasing values and decreasing values for that x-axis interval, is called a turning point of the function. A turning point is a relative maximum or a relative minimum of the function depending on whether the switch is from increasing values to decreasing values, or vice versa. The lesson also uses the end-behavior analysis of the previous lesson and the two types of changes to help sketch graphs of polynomial functions.

7-3 Solving Equations Using Quadratic Techniques

In this lesson, the algebraic entities are polynomial expressions that can be rewritten as quadratic trinomials or binomials; for example, $a^4 - 3a^2 + 2$ can be rewritten as $(a^2)^2 - 3(a^2) + 2$. Students rewrite polynomial expressions whose exponents are integers or rational numbers, and then factor or use the Quadratic Formula to solve related quadratic equations.

7-4 The Remainder and Factor Theorems

In this lesson, the algebraic entity is a statement of polynomial division, $\frac{f(x)}{x-a} = q(x) + \frac{k}{x-a}$, where the right side of the equation is a polynomial plus a fraction whose numerator is a constant and whose denominator is the divisor. The lesson observes two properties of polynomials. One property focuses on the remainder. Multiplying both sides by the divisor $(x - a)$ and then finding the value of the polynomial at a gives the result $f(a) = q(a)(a - a) + k$. This simplifies to $f(a) = k$. In words, if a polynomial is divided by $(x - a)$, then the remainder k is equal to the value of the polynomial at a, or $f(a) = k$. This property is called the Remainder Theorem. The other property focuses on $(x - a)$, a binomial that divides a polynomial $f(x)$ evenly. In such a case, $(x - a)$ is a factor of $f(x)$ and the remainder must be zero; since $f(a)$ is the value of the remainder, $f(a) = 0$. This property is called the Factor Theorem. Students use these properties, along with the result that the degree of $q(x)$ is one less than the degree of $f(x)$, to find all the factors of third-degree polynomials.

7-5 Roots and Zeros

This lesson focuses on the equivalent algebraic entities of roots of a polynomial equation, zeros of functions, factors of polynomials, and x-intercepts of graphs. The lesson presents two properties. One property, called the Fundamental Theorem of Algebra, states the existence of at least one complex root for a polynomial equation. Based on this property, students learn that the degree of a polynomial equation is the same as the number of its complex roots. The other property, called Descartes' Rule of Signs, lets students calculate the number of positive and negative zeros of a polynomial equation. Then, given a polynomial equation, students use the two properties to find out how many roots they are looking for, how many will be real, and how many (as complex conjugates) will be complex. With that information, they use methods from the previous lessons to find the roots of the polynomial equation.

7-6 Rational Zero Theorem

In this lesson, the algebraic entities are the leading term a_0x_n and constant term a_n of a polynomial function with integer coefficients. The algebraic property is that if a reduced fraction is a zero of a function, then the fraction's numerator and denominator must be factors of a_0 and a_n, respectively. Students use this property in a two-step process for finding all the rational zeros of a polynomial function. First, they find all possible factors p of a_0, all possible factors q of a_n, and list (and reduce) all possible fractions $\frac{p}{q}$. Second, they test whether particular values are zeros. Once they find zeros, they can also use the Remainder Theorem to find the depressed polynomial for that zero.

7-7 Operations on Functions

In this lesson (and the next), the algebraic entities are functions themselves. The algebraic relationship is to look at the result of combining functions. Four combinations interpret the arithmetic operations addition, subtraction, multiplication, and division for functions. Another type of combination of functions is the composition $f \circ g$, where $[f \circ g](x) = f[g(x)]$. Students also identify relationships between and restrictions on ranges and domains for combining functions.

7-8 Inverse Functions and Relations

In this lesson, the algebraic entities are functions and the relationship explored is that of inverse functions. Given a function $f(x)$, students find another function $g(x)$ by switching the variables x and y and solving the resulting equation for y. Then they test whether $f(x)$ and $g(x)$ are inverses by checking that each of the two compositions $[f \circ g](x)$ and $[g \circ f](x)$ has the value x. The lesson uses the notation $f^{-1}(x)$ for the inverse function of $f(x)$ and the notation $I(x)$ for the identify function, and introduces the term *one-to-one* for a function that passes the horizontal line test.

7-9 Square Root Functions and Inequalities

In this lesson, the algebraic entity is a square root function, so called because the function contains a variable inside a square root symbol. The lesson explores two main ideas. One idea is that while the inverse of a quadratic function is not a function, you can restrict the domain of the inverse so that the result is a function. The other idea is that to graph a square root inequality, first you graph the related square root equation, forming two regions. Then you check points to see which region is represented by the inequality, and you use the inequality symbol to decide whether or not the boundary line is part of the solution region.

Chapter 7

DAILY INTERVENTION and Assessment

INTERVENTION / ASSESSMENT

Type	Student Edition	Teacher Resources	Technology/Internet
Ongoing	Prerequisite Skills, pp. 345, 352, 358, 364, 370, 377, 382, 389, 394 Practice Quiz 1, p. 364 Practice Quiz 2, p. 382	5-Minute Check Transparencies Quizzes, *CRM* pp. 443–444 Mid-Chapter Test, *CRM* p. 445 Study Guide and Intervention, *CRM* pp. 375–376, 381–382, 387–388, 393–394, 399–400, 405–406, 411–412, 417–418, 423–424	Alge2PASS: Tutorial Plus www.algebra2.com/self_check_quiz www.algebra2.com/extra_examples
Mixed Review	pp. 352, 358, 364, 370, 377, 382, 389, 394, 399	Cumulative Review, *CRM* p. 446	
Error Analysis	Find the Error, pp. 380, 386	Find the Error, *TWE* pp. 380, 386 Unlocking Misconceptions, *TWE* pp. 354, 361, 375 Tips for New Teachers, *TWE* p. 384	
Standardized Test Practice	pp. 352, 358, 364, 370, 374, 375, 377, 382, 389, 394, 399, 405, 406–407	*TWE* p. 374 Standardized Test Practice, *CRM* pp. 447–448	Standardized Test Practice CD-ROM www.algebra2.com/standardized_test
Open-Ended Assessment	Writing in Math, pp. 352, 357, 364, 370, 377, 382, 389, 394, 399 Open Ended, pp. 350, 356, 362, 368, 375, 380, 382, 386, 393, 397	Modeling: *TWE* pp. 352, 389, 394 Speaking: *TWE* pp. 364, 370, 382 Writing: *TWE* pp. 358, 377, 399 Open-Ended Assessment, *CRM* p. 441	
Chapter Assessment	Study Guide, pp. 400–404 Practice Test, p. 405	Multiple-Choice Tests (Forms 1, 2A, 2B), *CRM* pp. 429–434 Free-Response Tests (Forms 2C, 2D, 3), *CRM* pp. 435–440 Vocabulary Test/Review, *CRM* p. 442	TestCheck and Worksheet Builder (see below) MindJogger Videoquizzes www.algebra2.com/vocabulary_review www.algebra2.com/chapter_test

Key to Abbreviations: TWE = Teacher Wraparound Edition; CRM = Chapter Resource Masters

Additional Intervention Resources

The Princeton Review's *Cracking the SAT & PSAT*
The Princeton Review's *Cracking the ACT*
ALEKS

TestCheck and Worksheet Builder

This **networkable** software has three modules for intervention and assessment flexibility:

- **Worksheet Builder** to make worksheet and tests
- **Student Module** to take tests on screen (optional)
- **Management System** to keep student records (optional)

Special banks are included for SAT, ACT, TIMSS, NAEP, and End-of-Course tests.

Intervention Technology

Alge2PASS: Tutorial Plus CD-ROM offers a complete, self-paced algebra curriculum.

Algebra 2 Lesson	Alge2PASS Lesson
7-3	**13** *Graphing Polynomial Functions*
7-5	**14** *Finding Roots and Zeros*

ALEKS is an online mathematics learning system that adapts assessment and tutoring to the student's needs. Subscribe at www.k12aleks.com.

Intervention at Home

Log on for student study help.

- For each lesson in the Student Edition, there are Extra Examples and Self-Check Quizzes.
  www.algebra2.com/extra_examples
  www.algebra2.com/self_check_quiz
- For chapter review, there is vocabulary review, test practice, and standardized test practice.
  www.algebra2.com/vocabulary_review
  www.algebra2.com/chapter_test
  www.algebra2.com/standardized_test

For more information on Intervention and Assessment, see pp. T8–T11.

Reading and Writing in Mathematics

Glencoe Algebra 2 provides numerous opportunities to incorporate reading and writing into the mathematics classroom.

Student Edition

- Foldables Study Organizer, p. 345
- Concept Check questions require students to verbalize and write about what they have learned in the lesson. (pp. 350, 356, 362, 368, 375, 380, 386, 393, 397, 400)
- Writing in Math questions in every lesson, pp. 352, 357, 364, 370, 377, 382, 389, 394, 399
- Reading Study Tip, pp. 354, 372, 384, 391
- WebQuest, p. 399

Teacher Wraparound Edition

- Foldables Study Organizer, pp. 345, 400
- Study Notebook suggestions, pp. 350, 356, 362, 368, 375, 380, 386, 392, 397
- Modeling activities, pp. 352, 389, 394
- Speaking activities, pp. 364, 370, 382
- Writing activities, pp. 358, 377, 399
- Differentiated Instruction, (Verbal/Linguistic), p. 356
- ELL Resources, pp. 344, 351, 356, 357, 363, 369, 376, 381, 388, 393, 398, 400

Additional Resources

- Vocabulary Builder worksheets require students to define and give examples for key vocabulary terms as they progress through the chapter. (*Chapter 7 Resource Masters*, pp. vii-viii)
- Reading to Learn Mathematics master for each lesson (*Chapter 7 Resource Masters*, pp. 379, 385, 391, 397, 403, 409, 415, 421, 427)
- *Vocabulary PuzzleMaker* software creates crossword, jumble, and word search puzzles using vocabulary lists that you can customize.
- *Teaching Mathematics with Foldables* provides suggestions for promoting cognition and language.
- *Reading and Writing in the Mathematics Classroom*
- *WebQuest and Project Resources*

For more information on Reading and Writing in Mathematics, see pp. T6–T7.

Chapter 7 Notes

What You'll Learn

Have students read over the list of objectives and make a list of any words with which they are not familiar.

Why It's Important

Point out to students that this is only one of many reasons why each objective is important. Others are provided in the introduction to each lesson.

Lesson	NCTM Standards	Local Objectives
7-1	1, 2, 6, 7, 8, 9, 10	
7-2	1, 2, 6, 8, 9, 10	
7-2 Follow-Up	2, 5, 6, 9, 10	
7-3	1, 2, 3, 4, 6, 8, 9, 10	
7-4	1, 2, 3, 6, 8, 9, 10	
7-5	1, 2, 3, 4, 6, 7, 8, 9, 10	
7-6	1, 2, 3, 4, 6, 7, 8, 9	
7-7	1, 2, 6, 7, 8, 9, 10	
7-8	1, 2, 3, 4, 6, 7, 8, 9, 10	
7-9	1, 2, 6, 7, 8, 9, 10	

Key to NCTM Standards:

1=Number & Operations, 2=Algebra, 3=Geometry, 4=Measurement, 5=Data Analysis & Probability, 6=Problem Solving, 7=Reasoning & Proof, 8=Communication, 9=Connections, 10=Representation

Chapter 7 Polynomial Functions

What You'll Learn

- **Lessons 7-1 and 7-3** Evaluate polynomial functions and solve polynomial equations.
- **Lessons 7-2 and 7-9** Graph polynomial and square root functions.
- **Lessons 7-4, 7-5, and 7-6** Find factors and zeros of polynomial functions.
- **Lesson 7-7** Find the composition of functions.
- **Lesson 7-8** Determine the inverses of functions or relations.

Key Vocabulary

- polynomial function (p. 347)
- synthetic substitution (p. 365)
- Fundamental Theorem of Algebra (p. 371)
- composition of functions (p. 384)
- inverse function (p. 391)

Why It's Important

According to the Fundamental Theorem of Algebra, every polynomial equation has at least one root. Sometimes the roots have real-world meaning. Many real-world situations that cannot be modeled using a linear function can be approximated using a polynomial function. *You will learn how the power generated by a windmill can be modeled by a polynomial function in Lesson 7-1.*

344 Chapter 7 Polynomial Functions

Vocabulary Builder

ELL

The Key Vocabulary list introduces students to some of the main vocabulary terms included in this chapter. For a more thorough vocabulary list with pronunciations of new words, give students the Vocabulary Builder worksheets found on pages vii and viii of the *Chapter 7 Resource Masters*. Encourage them to complete the definition of each term as they progress through the chapter. You may suggest that they add these sheets to their study notebooks for future reference when studying for the Chapter 7 test.

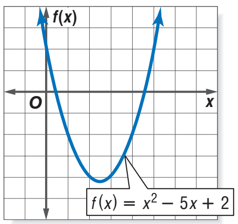
Getting Started

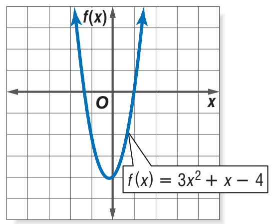
Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 7.

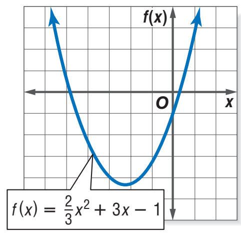
For Lesson 7-2 **3. between −5 and −4, between 0 and 1** **Solve Equations by Graphing**

Use the related graph of each equation to determine its roots. If exact roots cannot be found, state the consecutive integers between which the roots are located.
(For review, see Lesson 6-2.) **1. between 0 and 1, between 4 and 5** **2. between −2 and −1, 1**

1. $x^2 - 5x + 2 = 0$ **2.** $3x^2 + x - 4 = 0$ **3.** $\frac{2}{3}x^2 + 3x - 1 = 0$







For Lesson 7-3 **Quadratic Formula**

Solve each equation. *(For review, see Lesson 6-5.)*

4. $x^2 - 17x + 60 = 0$ **5, 12** **5.** $14x^2 + 23x + 3 = 0$ $-\frac{3}{2}, -\frac{1}{7}$ **6.** $2x^2 + 5x + 1 = 0$ $\frac{-5 \pm \sqrt{17}}{4}$

For Lessons 7-4 through 7-6 **Synthetic Division**

Simplify each expression using synthetic division. *(For review, see Lesson 5-3.)*

7. $(3x^2 - 14x - 24) \div (x - 6)$ $3x + 4$ **8.** $(a^2 - 2a - 30) \div (a + 7)$ $a - 9 + \frac{33}{a + 7}$

For Lessons 7-1 and 7-7 **Evaluating Functions**

Find each value if $f(x) = 4x - 7$ and $g(x) = 2x^2 - 3x + 1$. *(For review, see Lesson 2-1.)*

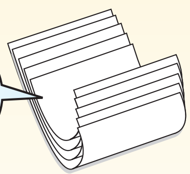
9. $f(-3)$ -19 **10.** $g(2a)$ $8a^2 - 6a + 1$ **11.** $f(4b^2) + g(b)$ $18b^2 - 3b - 6$

FOLDABLES™ Study Organizer

Make this Foldable to help you organize information about polynomial functions. Begin with five sheets of plain $8\frac{1}{2}$" by 11" paper.

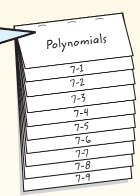
Step 1 Stack and Fold

Stack sheets of paper with edges $\frac{3}{4}$-inch apart. Fold up bottom edges to create equal tabs.



Step 2 Staple and Label

Staple along fold. Label the tabs with lesson numbers.



Reading and Writing As you read and study the chapter, use each page to write notes and examples.

Getting Started

This section provides a review of the basic concepts needed before beginning Chapter 7. Page references are included for additional student help.

Prerequisite Skills in the Getting Ready for the Next Lesson section at the end of each exercise set review a skill needed in the next lesson.

For Lesson	Prerequisite Skill
7-2	Graphing Quadratic Functions (p. 352)
7-3	Factoring Polynomials (p. 358)
7-4	Dividing Polynomials (p. 364)
7-5	Quadratic Formula (p. 370)
7-7	Operations with Polynomials (p. 382)
7-8	Solving Equations for a Variable (p. 389)
7-9	Solving Radical Equations (p. 394)

FOLDABLES™ Study Organizer

For more information about Foldables, see *Teaching Mathematics with Foldables.*

Making Generalizations while Previewing and Reviewing Data Have students label each tab of their Foldable to correspond to a lesson in Chapter 7. Before reading each lesson, have students preview it and write generalizations about what they think they will learn. As they read and study each lesson, students use their Foldable to take notes, define terms, record concepts, and write examples. After each lesson, ask them to compare what they thought they would learn with what they did learn as they review their notes.

7-1 Lesson Notes

1 Focus

5-Minute Check Transparency 7-1 Use as a quiz or review of Chapter 6.

Mathematical Background notes are available for this lesson on p. 344C.

Where are polynomial functions found in nature?

Ask students:

- How could you confirm that this model works for $r = 1$ and $r = 2$? **Substitute 1 and 2 for *r* in the formula and compare the results with the number of hexagons you count in the photograph.**
- Using the formula, what is the value of $f(8)$? **169**
- Why might it be useful for a beekeeper to know the approximate number of hexagons in a honeycomb? **Sample answer: The amount of honey stored in a honeycomb could be predicted by the number of hexagons.**

7-1 Polynomial Functions

What You'll Learn

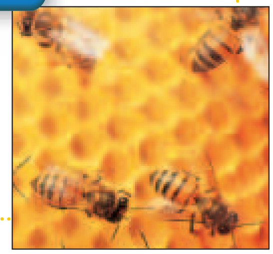
- Evaluate polynomial functions.
- Identify general shapes of graphs of polynomial functions.

Vocabulary

- polynomial in one variable
- degree of a polynomial
- leading coefficients
- polynomial function
- end behavior

Where are polynomial functions found in nature?

If you look at a cross section of a honeycomb, you see a pattern of hexagons. This pattern has one hexagon surrounded by six more hexagons. Surrounding these is a third ring of 12 hexagons, and so on. The total number of hexagons in a honeycomb can be modeled by the function $f(r) = 3r^2 - 3r + 1$, where r is the number of rings and $f(r)$ is the number of hexagons.



Study Tip

Look Back
To review **polynomials**, see Lesson 5-2.

POLYNOMIAL FUNCTIONS

Recall that a polynomial is a monomial or a sum of monomials. The expression $3r^2 - 3r + 1$ is a **polynomial in one variable** since it only contains one variable, r.

Key Concept — *Polynomial in One Variable*

- **Words** A polynomial of degree n in one variable x is an expression of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$, where the coefficients $a_0, a_1, a_2, \ldots, a_n$ represent real numbers, a_0 is not zero, and n represents a nonnegative integer.
- **Examples** $3x^5 + 2x^4 - 5x^3 + x^2 + 1$
  $n = 5$, $a_0 = 3$, $a_1 = 2$, $a_2 = -5$, $a_3 = 1$, $a_4 = 0$, and $a_5 = 1$

The **degree of a polynomial** in one variable is the greatest exponent of its variable. The **leading coefficient** is the coefficient of the term with the highest degree.

Polynomial	Expression	Degree	Leading Coefficient
Constant	9	0	9
Linear	$x - 2$	1	1
Quadratic	$3x^2 + 4x - 5$	2	3
Cubic	$4x^3 - 6$	3	4
General	$a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$	n	a_0

Example 1 Find Degree and Leading Coefficients

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

a. $7x^4 + 5x^2 + x - 9$

This is a polynomial in one variable.
The degree is 4, and the leading coefficient is 7.

Resource Manager

Workbook and Reproducible Masters

Chapter 7 Resource Masters

- Study Guide and Intervention, pp. 375–376
- Skills Practice, p. 377
- Practice, p. 378
- Reading to Learn Mathematics, p. 379
- Enrichment, p. 380

Transparencies

5-Minute Check Transparency 7-1
Answer Key Transparencies

Technology

Interactive Chalkboard

b. $8x^2 + 3xy - 2y^2$

This is not a polynomial in one variable. It contains two variables, x and y.

c. $7x^6 - 4x^3 + \frac{1}{x}$

This is not a polynomial. The term $\frac{1}{x}$ cannot be written in the form x^n, where n is a nonnegative integer.

d. $\frac{1}{2}x^2 + 2x^3 - x^5$

Rewrite the expression so the powers of x are in decreasing order.

$$-x^5 + 2x^3 + \frac{1}{2}x^2$$

This is a polynomial in one variable with degree of 5 and leading coefficient of -1.

Study Tip

Power Function

A common type of function is a **power function**, which has an equation in the form $f(x) = ax^b$, where a and b are real numbers. When b is a positive integer, $f(x) = ax^b$ is a polynomial function.

A polynomial equation used to represent a function is called a **polynomial function**. For example, the equation $f(x) = 4x^2 - 5x + 2$ is a quadratic polynomial function, and the equation $p(x) = 2x^3 + 4x^2 - 5x + 7$ is a cubic polynomial function. Other polynomial functions can be defined by the following general rule.

Key Concept — Definition of a Polynomial Function

- **Words** A polynomial function of degree n can be described by an equation of the form $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$, where the coefficients $a_0, a_1, a_2, \ldots, a_n$, represent real numbers, a_0 is not zero, and n represents a nonnegative integer.
- **Examples** $f(x) = 4x^2 - 3x + 2$
  $n = 2, a_0 = 4, a_1 = -3, a_2 = 2$

If you know an element in the domain of any polynomial function, you can find the corresponding value in the range. Recall that $f(3)$ can be found by evaluating the function for $x = 3$.

Example 2 Evaluate a Polynomial Function

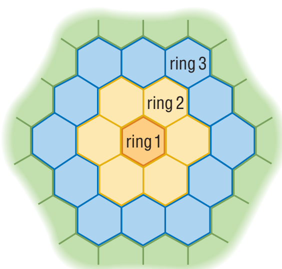
NATURE Refer to the application at the beginning of the lesson.

a. Show that the polynomial function $f(r) = 3r^2 - 3r + 1$ gives the total number of hexagons when $r = 1, 2,$ and 3.

Find the values of $f(1)$, $f(2)$, and $f(3)$.

$f(r) = 3r^2 - 3r + 1$	$f(r) = 3r^2 - 3r + 1$	$f(r) = 3r^2 - 3r + 1$
$f(1) = 3(1)^2 - 3(1) + 1$	$f(2) = 3(2)^2 - 3(2) + 1$	$f(3) = 3(3)^2 - 3(3) + 1$
$= 3 - 3 + 1$ or 1	$= 12 - 6 + 1$ or 7	$= 27 - 9 + 1$ or 19

From the information given, you know the number of hexagons in the first ring is 1, the number of hexagons in the second ring is 6, and the number of hexagons in the third ring is 12. So, the total number of hexagons with one ring is 1, two rings is 6 + 1 or 7, and three rings is 12 + 6 + 1 or 19. These match the functional values for $r = 1, 2,$ and 3, respectively.



Rings of a Honeycomb

b. Find the total number of hexagons in a honeycomb with 12 rings.

$f(r) = 3r^2 - 3r + 1$	Original function
$f(12) = 3(12)^2 - 3(12) + 1$	Replace r with 12.
$= 432 - 36 + 1$ or 397	Simplify.

www.algebra2.com/extra_examples

2 Teach

POLYNOMIAL FUNCTIONS

In-Class Examples Power Point®

Teaching Tip Stress that the leading coefficient is not always the coefficient of the first term of a polynomial.

1 State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

a. $7z^3 - 4z^2 + z$ **degree 3, leading coefficient 7**

b. $6a^3 - 4a^2 + ab^2$ **This is not a polynomial in one variable. It contains two variables, a and b.**

c. $3c^2 + 4c - 2c^{-1}$ **This is not a polynomial. The term $-2c^{-1}$ is not of the form a_nc^n, where n is a nonnegative integer.**

d. $9y - 3y^2 + y^4$ **degree 4, leading coefficient 1**

2 **NATURE** Refer to the application at the beginning of the lesson. A sketch of the arrangement of hexagons shows a fourth ring of 18 hexagons, a fifth ring of 24 hexagons, and a sixth ring of 30 hexagons.

a. Show that the polynomial function $f(r) = 3r^2 - 3r + 1$ gives the total number of hexagons when $r = 4, 5,$ and 6. **$f(4) = 48 - 12 + 1$, or 37; $f(5) = 75 - 15 + 1$, or 61; $f(6) = 108 - 18 + 1$, or 91; The total number of hexagons for four rings is 19 + 18 or 37, five rings is 37 + 24 or 61, and six rings is 61 + 30 or 91. These match the functional values for $r = 4, 5,$ and 6, respectively.**

b. Find the total number of hexagons in a honeycomb with 20 rings. **1141**

In-Class Example

3 **a.** Find $p(y^3)$ if $p(x) = 2x^4 - x^3 + 3x$. $2y^{12} - y^9 + 3y^3$

b. Find $b(2x - 1) - 3b(x)$ if $b(m) = 2m^2 + m - 1$. $2x^2 - 9x + 3$

GRAPHS OF POLYNOMIAL FUNCTIONS

Teaching Tip Since the graphs on this page show the *maximum* number of times each type of graph may intersect the *x*-axis, some students may ask about the minimum number of times each graph type may intersect the *x*-axis. Have students work in pairs using the given graphs to discuss this issue. Lead students to see that for functions of degree 1 the minimum is 1 (the same as the maximum), for functions of degree 2 the minimum is 0, for functions of degree 3 the minimum is 1, for functions of degree 4 the minimum is 0, and for functions of degree 5 the minimum is 1. Some students may notice the pattern for functions with odd and even degrees.

Interactive Chalkboard
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- Step-by-step, dynamic solutions of each In-Class Example from the Teacher Wraparound Edition
- Additional, Your Turn exercises for each example
- The 5-Minute Check Transparencies
- Hot links to Glencoe Online Study Tools

You can also evaluate functions for variables and algebraic expressions.

Example 3 Functional Values of Variables

a. Find $p(a^2)$ if $p(x) = x^3 + 4x^2 - 5x$.

$p(x) = x^3 + 4x^2 - 5x$ Original function

$p(a^2) = (a^2)^3 + 4(a^2)^2 - 5(a^2)$ Replace x with a^2.

$= a^6 + 4a^4 - 5a^2$ Property of powers

b. Find $q(a + 1) - 2q(a)$ if $q(x) = x^2 + 3x + 4$.

To evaluate $q(a + 1)$, replace x in $q(x)$ with $a + 1$.

$q(x) = x^2 + 3x + 4$ Original function

$q(a + 1) = (a + 1)^2 + 3(a + 1) + 4$ Replace x with $a + 1$.

$= a^2 + 2a + 1 + 3a + 3 + 4$ Evaluate $(a + 1)^2$ and $3(a + 1)$.

$= a^2 + 5a + 8$ Simplify.

To evaluate $2q(a)$, replace x with a in $q(x)$, then multiply the expression by 2.

$q(x) = x^2 + 3x + 4$ Original function

$2q(a) = 2(a^2 + 3a + 4)$ Replace x with a.

$= 2a^2 + 6a + 8$ Distributive Property

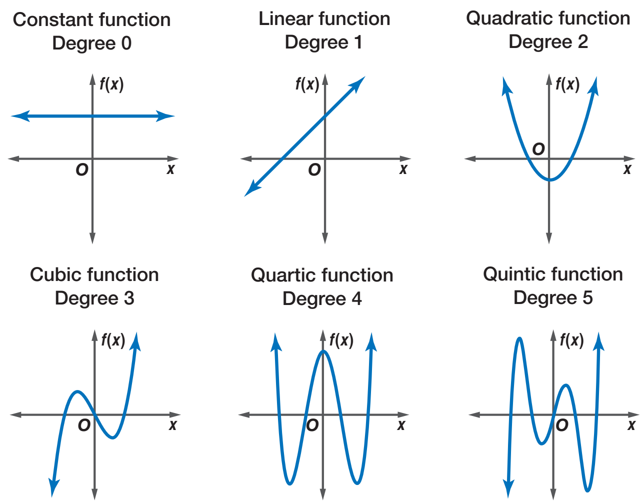
Now evaluate $q(a + 1) - 2q(a)$.

$q(a + 1) - 2q(a) = a^2 + 5a + 8 - (2a^2 + 6a + 8)$ Replace $q(a + 1)$ and $2q(a)$ with evaluated expressions.

$= a^2 + 5a + 8 - 2a^2 - 6a - 8$

$= -a^2 - a$ Simplify.

GRAPHS OF POLYNOMIAL FUNCTIONS The general shapes of the graphs of several polynomial functions are shown below. These graphs show the *maximum* number of times the graph of each type of polynomial may intersect the *x*-axis. Recall that the *x*-coordinate of the point at which the graph intersects the *x*-axis is called a *zero* of a function. How does the degree compare to the maximum number of real zeros?



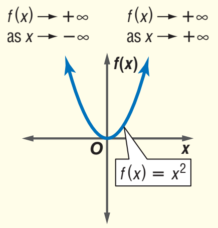
Notice the shapes of the graphs for even-degree polynomial functions and odd-degree polynomial functions. The degree and leading coefficient of a polynomial function determine the graph's end behavior.

348 Chapter 7 Polynomial Functions

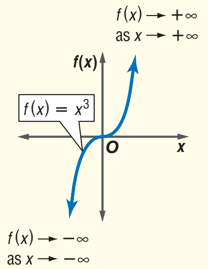
The **end behavior** is the behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). This is represented as $x \to +\infty$ and $x \to -\infty$, respectively. *$x \to +\infty$ is read x approaches positive infinity.*

Concept Summary — *End Behavior of a Polynomial Function*

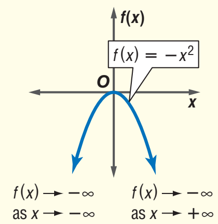
Degree: even
Leading Coefficient: positive
End Behavior:



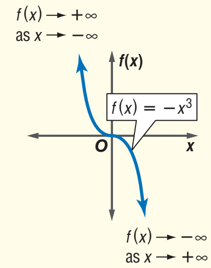
Degree: odd
Leading Coefficient: positive
End Behavior:



Degree: even
Leading Coefficient: negative
End Behavior:



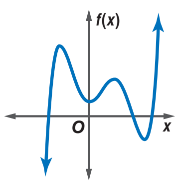
Degree: odd
Leading Coefficient: negative
End Behavior:



The graph of an even-degree function may or may not intersect the x-axis, depending on its location in the coordinate plane. If it intersects the x-axis in two places, the function has two real zeros. If it does not intersect the x-axis, the roots of the related equation are imaginary and cannot be determined from the graph. If the graph is tangent to the x-axis, as shown above, there are two zeros that are the same number. The graph of an odd-degree function always crosses the x-axis at least once, and thus the function always has at least one real zero.

Study Tip

Number of Zeros The number of zeros of an odd-degree function may be less than the maximum by a multiple of 2. For example, the graph of a quintic function may only cross the x-axis 3 times.



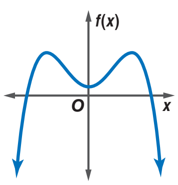
The same is true for an even-degree function. One exception is when the graph of $f(x)$ touches the x-axis.

Example 4 *Graphs of Polynomial Functions*

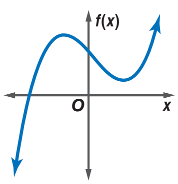
For each graph,

- **describe the end behavior,**
- **determine whether it represents an odd-degree or an even-degree polynomial function, and**
- **state the number of real zeros.**

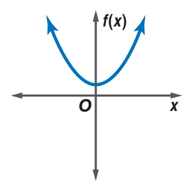
a.



b.



c.



a.
- $f(x) \to -\infty$ as $x \to +\infty$. $f(x) \to -\infty$ as $x \to -\infty$.
- It is an even-degree polynomial function.
- The graph intersects the x-axis at two points, so the function has two real zeros.

b.
- $f(x) \to +\infty$ as $x \to +\infty$. $f(x) \to -\infty$ as $x \to -\infty$.
- It is an odd-degree polynomial function.
- The graph has one real zero.

c.
- $f(x) \to +\infty$ as $x \to +\infty$. $f(x) \to +\infty$ as $x \to -\infty$.
- It is an even-degree polynomial function.
- This graph does not intersect the x-axis, so the function has no real zeros.

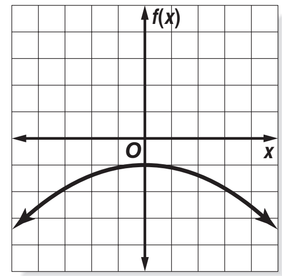
DAILY INTERVENTION — Differentiated Instruction

Interpersonal Arrange students in groups of 3 or 4, providing each group with a graphing calculator. Have each student write a polynomial function. As a group, have students state whether each function is an odd-degree or an even-degree polynomial function before predicting the end behavior and the number of zeros of the function. Then have students check their predictions by graphing each function. Challenge students to find at least one polynomial function that crosses the x-axis 3 or 4 times.

In-Class Example — Power Point®

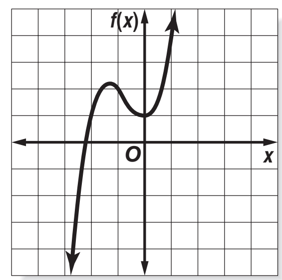
4 For each graph,
- describe the end behavior,
- determine whether it represents an odd-degree or an even-degree polynomial function, and
- state the number of real zeros.

a.



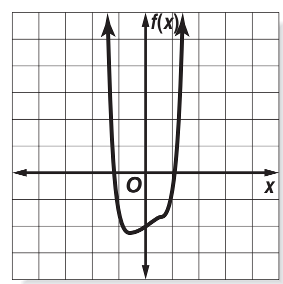
- $f(x) \to -\infty$ as $x \to +\infty$. $f(x) \to -\infty$ as $x \to -\infty$.
- It is an even-degree polynomial function.
- The graph does not intersect the x-axis, so the function has no real zeros.

b.



- $f(x) \to +\infty$ as $x \to +\infty$. $f(x) \to -\infty$ as $x \to -\infty$.
- It is an odd-degree polynomial function.
- The graph intersects the x-axis at one point, so the function has one real zero.

c.



- $f(x) \to +\infty$ as $x \to +\infty$. $f(x) \to +\infty$ as $x \to -\infty$.
- It is an even-degree polynomial function.
- The function has two real zeros.

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 7.
- copy the information in the Concept Summary about end behavior shown on p. 349.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

Check for Understanding

Concept Check

1. **Explain** why a constant polynomial such as $f(x) = 4$ has degree 0 and a linear polynomial such as $f(x) = x + 5$ has degree 1. **$4 = 4x^0$; $x = x^1$**
2. **Describe** the characteristics of the graphs of odd-degree and even-degree polynomial functions whose leading coefficients are positive. **See margin.**
3. **OPEN ENDED** Sketch the graph of an odd-degree polynomial function with a negative leading coefficient and three real roots. **See margin.**
4. **Tell** whether the following statement is *always*, *sometimes* or *never* true. Explain.
   A polynomial function that has four real roots is a fourth-degree polynomial.

4. Sometimes; a polynomial function with 4 real roots may be a sixth-degree polynomial function with 2 imaginary roots. A polynomial function that has 4 real roots is *at least* a fourth-degree polynomial.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
5, 6	1
7, 8, 15	2
9–11	3
12–14	4

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

5. $5x^6 - 8x^2$ **6; 5**
6. $2b + 4b^3 - 3b^5 - 7$ **5; −3**

Find $p(3)$ and $p(-1)$ for each function.

7. $p(x) = -x^3 + x^2 - x$ **−21; 3**
8. $p(x) = x^4 - 3x^3 + 2x^2 - 5x + 1$ **4; 12**

If $p(x) = 2x^3 + 6x - 12$ and $q(x) = 5x^2 + 4$, find each value. **11. $6a^3 - 5a^2 + 8a - 45$**

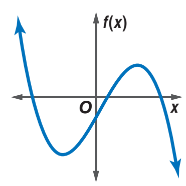
9. $p(a^3)$ **$2a^9 + 6a^3 - 12$**
10. $5[q(2a)]$ **$100a^2 + 20$**
11. $3p(a) - q(a + 1)$

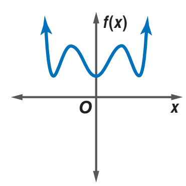
For each graph,

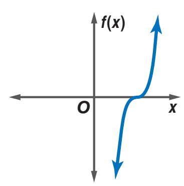
a. describe the end behavior,

b. determine whether it represents an odd-degree or an even-degree polynomial function, and

c. state the number of real zeros.

12.


13.


14.


12. a. $f(x) \to -\infty$ as $x \to +\infty$, $f(x) \to +\infty$ as $x \to -\infty$; b. odd; c. 3

13. a. $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to +\infty$ as $x \to -\infty$; b. even; c. 0

14. a. $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$; b. odd; c. 1

Application

15. **BIOLOGY** The intensity of light emitted by a firefly can be determined by $L(t) = 10 + 0.3t + 0.4t^2 - 0.01t^3$, where t is temperature in degrees Celsius and $L(t)$ is light intensity in lumens. If the temperature is 30°C, find the light intensity. **109 lumens**

★ indicates increased difficulty

Practice and Apply

Homework Help

For Exercises	See Examples
16–21	1
22–29, 45	2
30–38	3
39–44, 46–48	4

Extra Practice

See page 842.

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

16. $7 - x$ **1; −1**
17. $(a + 1)(a^2 - 4)$ **3; 1**
18. $a^2 + 2ab + b^2$ **See margin.**
19. $6x^4 + 3x^2 + 4x - 8$ **4; 6**
20. $7 + 3x^2 - 5x^3 + 6x^2 - 2x$ **3; −5**
21. $c^2 + c - \frac{1}{c}$ **See margin.**

Find $p(4)$ and $p(-2)$ for each function.

22. $p(x) = 2 - x$ **−2; 4**
23. $p(x) = x^2 - 3x + 8$ **12; 18**
24. $p(x) = 2x^3 - x^2 + 5x - 7$ **125; −37**
25. $p(x) = x^5 - x^2$ **1008; −36**
26. $p(x) = x^4 - 7x^3 + 8x - 6$ **−166; 50**
27. $p(x) = 7x^2 - 9x + 10$ **86; 56**
28. $p(x) = \frac{1}{2}x^4 - 2x^2 + 4$ **100; 4**
29. $p(x) = \frac{1}{8}x^3 - \frac{1}{4}x^2 - \frac{1}{2}x + 5$ **7; 4**

About the Exercises…

Organization by Objective
- **Polynomial Functions:** 16–38
- **Graphs of Polynomial Functions:** 39–44

Odd/Even Assignments

Exercises 16–44 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 17–45 odd, 49–52, 56–70

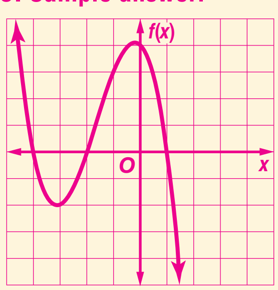
Average: 17–45 odd, 46–52, 56–70

Advanced: 16–44 even, 46–67 (optional: 68–70)

Answers

2. Sample answer: Even-degree polynomial functions with positive leading coefficients have graphs in which $f(x) \to +\infty$ as $x \to +\infty$ and as $x \to -\infty$. Odd-degree polynomial functions with positive leading coefficients have graphs in which $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

3. Sample answer:



18. No, the polynomial contains two variables, *a* and *b*.

21. No, this is not a polynomial because the term $\frac{1}{c}$ cannot be written in the form x^n, where n is a nonnegative integer.

39a. $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$; b. odd; c. 3

40a. $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to +\infty$ as $x \to -\infty$; b. even; c. 4

41a. $f(x) \to -\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$; b. even; c. 0

42a. $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$; b. odd; c. 5

43a. $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$; b. odd; c. 1

44a. $f(x) \to -\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$; b. even; c. 2

If $p(x) = 3x^2 - 2x + 5$ and $r(x) = x^3 + x + 1$, find each value.

30. $r(3a)$ $27a^3 + 3a + 1$
31. $4p(a)$ $12a^2 - 8a + 20$
32. $p(a^2)$
33. $p(2a^3)$ $12a^6 - 4a^3 + 5$
34. $r(x + 1)$
35. $p(x^2 + 3)$
36. $2[p(x + 4)]$ $6x^2 + 44x + 90$
37. $r(x + 1) - r(x^2)$ $-x^6 + x^3 + 2x^2 + 4x + 2$
38. $3[p(x^2 - 1)] + 4p(x)$ $9x^4 - 12x^2 - 8x + 50$

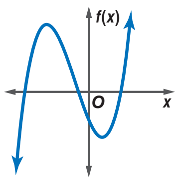
32. $3a^4 - 2a^2 + 5$
34. $x^3 + 3x^2 + 4x + 3$
35. $3x^4 + 16x^2 + 26$

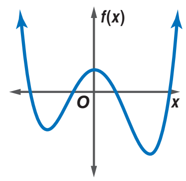
For each graph,

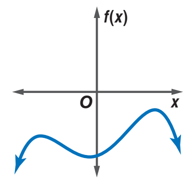
a. describe the end behavior,

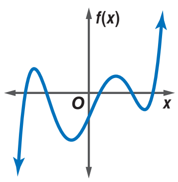
b. determine whether it represents an odd-degree or an even-degree polynomial function, and

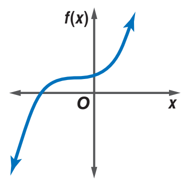
c. state the number of real zeros. 39–44. See margin.

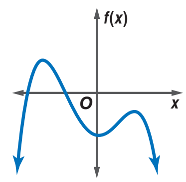
39.


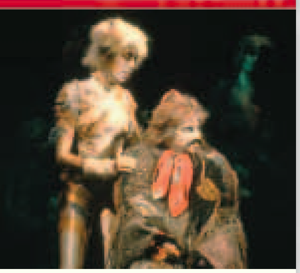
40.


41.


42.


43.


44.


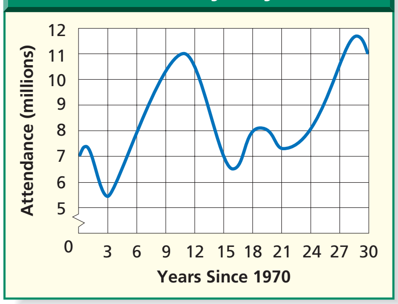


Theater In 1997, *Cats* surpassed *A Chorus Line* as the longest-running Broadway show.

Source: www.newsherald.com

45. **ENERGY** The power generated by a windmill is a function of the speed of the wind. The approximate power is given by the function $P(s) = \frac{s^3}{1000}$, where s represents the speed of the wind in kilometers per hour. Find the units of power $P(s)$ generated by a windmill when the wind speed is 18 kilometers per hour. 5.832 units

THEATER For Exercises 46–48, use the graph that models the attendance to Broadway plays (in millions) from 1970–2000.



46. Is the graph an odd-degree or even-degree function? even

47. Discuss the end behavior of the graph.

48. Do you think attendance at Broadway plays will increase or decrease after 2000? Explain your reasoning.

47. $f(x) \to -\infty$ as $x \to +\infty$; $f(x) \to -\infty$ as $x \to -\infty$

48. Sample answer: Decrease; the graph appears to be turning at $x = 30$ indicating a relative maximum at that point. So attendance will decrease after 2000.

CRITICAL THINKING For Exercises 49–52, use the following information.

The graph of the polynomial function $f(x) = ax(x - 4)(x + 1)$ goes through the point at (5, 15).

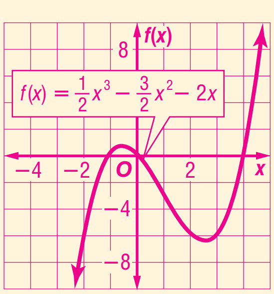
49. Find the value of a. $\frac{1}{2}$

50. For what value(s) of x will $f(x) = 0$? $-1, 0, 4$

51. Rewrite the function as a cubic function. $f(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 - 2x$

52. Sketch the graph of the function. See margin.

www.algebra2.com/self_check_quiz

52.


Study Guide and Intervention, p. 375 (shown) and p. 376

Polynomial Functions

Polynomial in One Variable	A polynomial of degree n in one variable x is an expression of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$, where the coefficients $a_0, a_1, a_2, \ldots, a_n$ represent real numbers, a_0 is not zero, and n represents a nonnegative integer.

The **degree of a polynomial** in one variable is the greatest exponent of its variable. The **leading coefficient** is the coefficient of the term with the highest degree.

Polynomial Function	A polynomial function of degree n can be described by an equation of the form $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$, where the coefficients $a_0, a_1, a_2, \ldots, a_n$ represent real numbers, a_0 is not zero, and n represents a nonnegative integer.

Example 1 **What are the degree and leading coefficient of $3x^2 - 2x^4 - 7 + x^3$?**

Rewrite the expression so the powers of x are in decreasing order.

$-2x^4 + x^3 + 3x^2 - 7$

This is a polynomial in one variable. The degree is 4, and the leading coefficient is −2.

Example 2 **Find $f(-5)$ if $f(x) = x^3 + 2x^2 - 10x + 20$.**

$f(x) = x^3 + 2x^2 - 10x + 20$ Original function
$f(-5) = (-5)^3 + 2(-5)^2 - 10(-5) + 20$ Replace x with −5.
$= -125 + 50 + 50 + 20$ Evaluate.
$= -5$ Simplify.

Example 3 **Find $g(a^2 - 1)$ if $g(x) = x^2 + 3x - 4$.**

$g(x) = x^2 + 3x - 4$ Original function
$g(a^2 - 1) = (a^2 - 1)^2 + 3(a^2 - 1) - 4$ Replace x with $a^2 - 1$.
$= a^4 - 2a^2 + 1 + 3a^2 - 3 - 4$ Evaluate.
$= a^4 + a^2 - 6$ Simplify.

Exercises

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

1. $3x^4 + 6x^3 - x^2 + 12$ 4; 3
2. $100 - 5x^3 + 10x^7$ 7; 10
3. $4x^6 + 6x^4 + 8x^8 - 10x^2 + 20$ 8; 8
4. $4x^2 - 3xy + 16y^2$ not a polynomial in one variable; contains two variables
5. $8x^3 - 9x^5 + 4x^2 - 36$ 5; −9
6. $\frac{x^2}{18} - \frac{x^6}{25} + \frac{x^3}{36} - \frac{1}{72}$ 6; $-\frac{1}{25}$

Find $f(2)$ and $f(-5)$ for each function.

7. $f(x) = x^2 - 9$ −5; 16
8. $f(x) = 4x^3 - 3x^2 + 2x - 1$ 23; −586
9. $f(x) = 9x^3 - 4x^2 + 5x + 7$ 73; −1243

Skills Practice, p. 377 and Practice, p. 378 (shown)

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

1. $(3x^2 + 1)(2x^2 - 9)$ 4; 6
2. $\frac{1}{5}a^3 - \frac{3}{5}a^2 + \frac{4}{5}a$ 3; $\frac{1}{5}$
3. $\frac{2}{m^2} + 3m - 12$ Not a polynomial; $\frac{2}{m^2}$ cannot be written in the form m^n for a nonnegative integer n.
4. $27 + 3xy^3 - 12x^2y^2 - 10y$ No, this polynomial contains two variables, x and y.

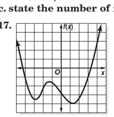
Find $p(-2)$ and $p(3)$ for each function.

5. $p(x) = x^3 - x^5$ 24; −216
6. $p(x) = -7x^2 + 5x + 9$ −29; −39
7. $p(x) = -x^5 + 4x^3$ 0; −135
8. $p(x) = 3x^3 - x^2 + 2x - 5$ −37; 73
9. $p(x) = x^4 + \frac{1}{2}x^3 - \frac{1}{2}x$ 13; 93
10. $p(x) = \frac{1}{3}x^3 + \frac{2}{3}x^2 + 3x$ −6; 24

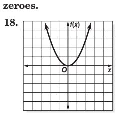
If $p(x) = 3x^2 - 4$ and $r(x) = 2x^2 - 5x + 1$, find each value.

11. $p(8a)$ $192a^2 - 4$
12. $r(a^2)$ $2a^4 - 5a^2 + 1$
13. $-5r(2a)$ $-40a^2 + 50a - 5$
14. $r(x + 2)$ $2x^2 + 3x - 1$
15. $p(x^2 - 1)$ $3x^4 - 6x^2 - 1$
16. $5[p(x + 2)]$ $15x^2 + 60x + 40$

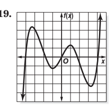
For each graph,
a. describe the end behavior,
b. determine whether it represents an odd-degree or an even-degree polynomial function, and
c. state the number of real zeroes.

17.


$f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to +\infty$ as $x \to -\infty$; even; 2

18.


$f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to +\infty$ as $x \to -\infty$; even; 1

19.


$f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$; odd; 5

20. **WIND CHILL** The function $C(s) = 0.013s^2 - s - 7$ estimates the wind chill temperature $C(s)$ at 0°F for wind speeds s from 5 to 30 miles per hour. Estimate the wind chill temperature at 0°F if the wind speed is 20 miles per hour. about −22°F

Reading to Learn Mathematics, p. 379 ELL

Pre-Activity **Where are polynomial functions found in nature?**

Read the introduction to Lesson 7-1 at the top of page 346 in your textbook.

- In the honeycomb cross section shown in your textbook, there is 1 hexagon in the center, 6 hexagons in the second ring, and 12 hexagons in the third ring. How many hexagons will there be in the fourth, fifth, and sixth rings? 18; 24; 30
- There is 1 hexagon in a honeycomb with 1 ring. There are 7 hexagons in a honeycomb with 2 rings. How many hexagons are there in honeycombs with 3 rings, 4 rings, 5 rings, and 6 rings? 19; 37; 61; 91

Reading the Lesson

1. Give the degree and leading coefficient of each polynomial in one variable.

	degree	leading coefficient
a. $10x^3 + 3x^2 - x + 7$	3	10
b. $7y^2 - 2y^5 + y - 4y^3$	5	−2
c. 100	0	100

2. Match each description of a polynomial function from the list on the left with the corresponding end behavior from the list on the right.

a. even degree, negative leading coefficient iii
b. odd degree, positive leading coefficient iv
c. odd degree, negative leading coefficient ii
d. even degree, positive leading coefficient i

i. $f(x) \to +\infty$ as $x \to +\infty$; $f(x) \to +\infty$ as $x \to -\infty$
ii. $f(x) \to -\infty$ as $x \to +\infty$; $f(x) \to +\infty$ as $x \to -\infty$
iii. $f(x) \to -\infty$ as $x \to +\infty$; $f(x) \to -\infty$ as $x \to -\infty$
iv. $f(x) \to +\infty$ as $x \to +\infty$; $f(x) \to -\infty$ as $x \to -\infty$

Helping You Remember

3. What is an easy way to remember the difference between the end behavior of the graphs of even-degree and odd-degree polynomial functions?

Sample answer: Both ends of the graph of an even-degree function eventually keep going in the same direction. For odd-degree functions, the two ends eventually head in opposite directions, one upward, the other downward.

Enrichment, p. 380

Approximation by Means of Polynomials

Many scientific experiments produce pairs of numbers $[x, f(x)]$ that can be related by a formula. If the pairs form a function, you can fit a polynomial to the pairs in exactly one way. Consider the pairs given by the following table.

x	1	2	4	7
$f(x)$	6	11	39	−54

We will assume the polynomial is of degree three. Substitute the given values into this expression.

$$f(x) = A + B(x - x_0) + C(x - x_0)(x - x_1) + D(x - x_0)(x - x_1)(x - x_2)$$

You will get the system of equations shown below. You can solve this system and use the values for A, B, C, and D to find the desired polynomial.

$6 = A$
$11 = A + B(2 - 1) = A + B$

4 Assess

Open-Ended Assessment

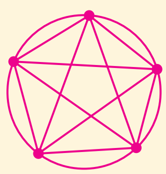
Modeling Provide students with grid paper and a length of string. Describe the end behavior and number of real zeros of the graph of a function and have students use their string to model a possible graph that exhibits these characteristics.

Getting Ready for Lesson 7-2

PREREQUISITE SKILL In Lesson 7-2, students will graph polynomial functions by making a table of values. It is important that students know how to make tables of values and how to use them to graph equations. Use Exercises 68–70 to determine your students' familiarity with graphing quadratic functions by making a table of values.

Answers

54.

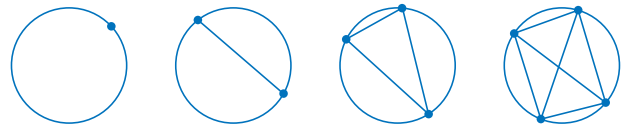


56. Many relationships in nature can be modeled by polynomial functions; for example, the pattern in a honeycomb or the rings in a tree trunk. Answers should include the following.

- You can use the equation to find the number of hexagons in a honeycomb with 10 rings and the number of hexagons in a honeycomb with 9 rings. The difference is the number of hexagons in the tenth ring.
- Other examples of patterns found in nature include pinecones, pineapples, and flower petals.

PATTERNS For Exercises 53–55, use the diagrams below that show the maximum number of regions formed by connecting points on a circle.

1 point, 1 region | 2 points, 2 regions | 3 points, 4 regions | 4 points, 8 regions



53. The maximum number of regions formed by connecting n points of a circle can be described by the function $f(n) = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$. What is the degree of this polynomial function? **4**

★ 54. Find the maximum number of regions formed by connecting 5 points of a circle. Draw a diagram to verify your solution. **16 regions; See margin for diagram.**

★ 55. How many points would you have to connect to form 99 regions? **8 points**

56. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

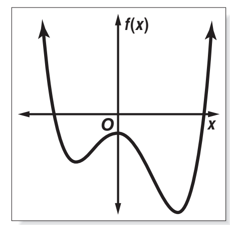
Where are polynomial functions found in nature?

Include the following in your answer:
- an explanation of how you could use the equation to find the number of hexagons in the tenth ring, and
- any other examples of patterns found in nature that might be modeled by a polynomial equation.

Standardized Test Practice

57. The figure at the right shows the graph of the polynomial function $f(x)$. Which of the following could be the degree of $f(x)$? **C**

(A) 2 (B) 3 (C) 4 (D) 5



58. If $\frac{1}{2}x^2 - 6x + 2 = 0$, then x could equal which of the following? **C**

(A) −1.84 (B) −0.81 (C) 0.34 (D) 2.37

Maintain Your Skills

Mixed Review

Solve each inequality algebraically. *(Lesson 6-7)*

59. $x^2 - 8x + 12 < 0$ **59.** $\{x \mid 2 < x < 6\}$

60. $x^2 + 2x - 86 \geq -23$ **60.** $\{x \mid x \leq -9$ or $x \geq 7\}$

61. $15x^2 + 3x - 12 \leq 0$ $\left\{x \mid -1 \leq x \leq \frac{4}{5}\right\}$

Graph each function. *(Lesson 6-6)* **62–64. See margin.**

62. $y = -2(x - 2)^2 + 3$

63. $y = \frac{1}{3}(x + 5)^2 - 1$

64. $y = \frac{1}{2}x^2 + x + \frac{3}{2}$

Solve each equation by completing the square. *(Lesson 6-4)*

65. $x^2 - 8x - 2 = 0$ $\{4 \pm 3\sqrt{2}\}$

66. $x^2 + \frac{1}{3}x - \frac{35}{36} = 0$ $\left\{-\frac{7}{6}, \frac{5}{6}\right\}$

67. **BUSINESS** Becca is writing a computer program to find the salaries of her employees after their annual raise. The percent of increase is represented by p. Marty's salary is \$23,450 now. Write a polynomial to represent Marty's salary after one year and another to represent Marty's salary after three years. Assume that the rate of increase will be the same for each of the three years. *(Lesson 5-2)* **67.** $23{,}450(1 + p)$; $23{,}450(1 + p)^3$

Getting Ready for the Next Lesson

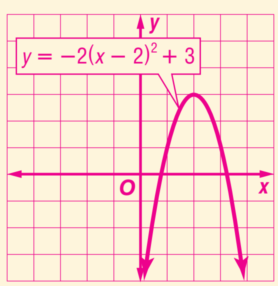
PREREQUISITE SKILL Graph each equation by making a table of values.
*(To review **graphing quadratic functions**, see Lesson 6-1.)* **68–70. See pp. 407A–407H.**

68. $y = x^2 + 4$

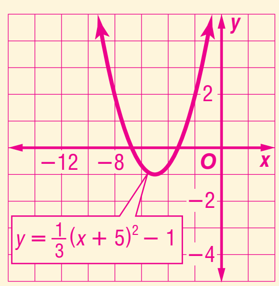
69. $y = -x^2 + 6x - 5$

70. $y = \frac{1}{2}x^2 + 2x - 6$

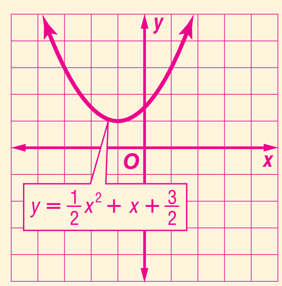
62.



63.



64.



7-2 Graphing Polynomial Functions

What You'll Learn

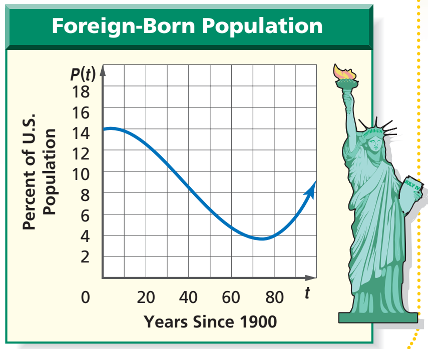
- Graph polynomial functions and locate their real zeros.
- Find the maxima and minima of polynomial functions.

Vocabulary

- Location Principle
- relative maximum
- relative minimum

How can graphs of polynomial functions show trends in data?

The percent of the United States population that was foreign-born since 1900 can be modeled by $P(t) = 0.00006t^3 - 0.007t^2 + 0.05t + 14$, where $t = 0$ in 1900. Notice that the graph is decreasing from $t = 5$ to $t = 75$ and then it begins to increase. The points at $t = 5$ and $t = 75$ are turning points in the graph.



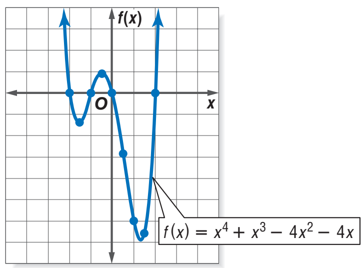
GRAPH POLYNOMIAL FUNCTIONS To graph a polynomial function, make a table of values to find several points and then connect them to make a smooth curve. Knowing the end behavior of the graph will assist you in completing the sketch of the graph.

Example 1 *Graph a Polynomial Function*

Graph $f(x) = x^4 + x^3 - 4x^2 - 4x$ by making a table of values.

x	$f(x)$
−2.5	≈ 8.4
−2.0	0.0
−1.5	≈ −1.3
−1.0	0.0
−0.5	≈ 0.9

x	$f(x)$
0.0	0.0
0.5	≈ −2.8
1.0	−6.0
1.5	≈ −6.6
2.0	0.0



This is an even-degree polynomial with a positive leading coefficient, so $f(x) \to +\infty$ as $x \to +\infty$, and $f(x) \to +\infty$ as $x \to -\infty$. Notice that the graph intersects the x-axis at four points, indicating there are four real zeros of this function.

Study Tip

Graphing Polynomial Functions

To graph polynomial functions it will often be necessary to include x values that are not integers.

In Example 1, the zeros occur at integral values that can be seen in the table used to plot the function. Notice that the values of the function before and after each zero are different in sign. In general, the graph of a polynomial function will cross the x-axis somewhere between pairs of x values at which the corresponding $f(x)$ values change signs. Since zeros of the function are located at x-intercepts, there is a zero between each pair of these x values. This property for locating zeros is called the **Location Principle**.

7-2 Lesson Notes

1 Focus

5-Minute Check Transparency 7-2 Use as a quiz or review of Lesson 7-1.

Mathematical Background notes are available for this lesson on p. 344C.

Building on Prior Knowledge

In Chapter 6, students learned to graph quadratic functions. Those same skills will be used in this lesson to graph polynomial functions.

How can graphs of polynomial functions show trends in data?

Ask students:

- When the graph is sloping downward to the right, what does that tell you about the population it represents? **The percent of the U.S. population that is foreign-born is decreasing during that span of time.**
- If the United States government banned any further immigration, what would happen to the graph? **It would gradually approach the horizontal axis.**
- Why would the graph not immediately reach the horizontal axis, where $P(t) = 0$? **All of the current foreign-born residents of the U.S. may still be part of the population.**

Resource Manager

Workbook and Reproducible Masters

Chapter 7 Resource Masters

- Study Guide and Intervention, pp. 381–382
- Skills Practice, p. 383
- Practice, p. 384
- Reading to Learn Mathematics, p. 385
- Enrichment, p. 386

Science and Mathematics Lab Manual, pp. 71–76

Transparencies

5-Minute Check Transparency 7-2

Answer Key Transparencies

Technology

Interactive Chalkboard

2 Teach

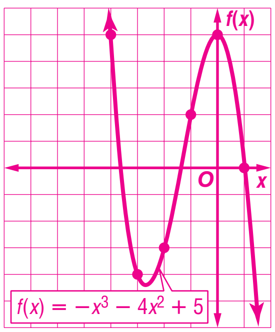
GRAPH POLYNOMIAL FUNCTIONS

In-Class Examples

Power Point®

1 Graph $f(x) = -x^3 - 4x^2 + 5$ by making a table of values.

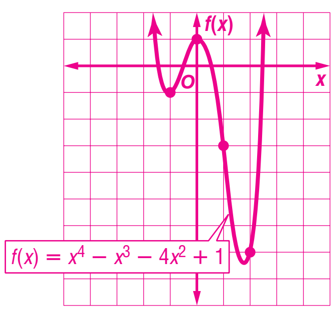
x	$f(x)$
−4	5
−3	−4
−2	−3
−1	2
0	5
1	0
2	−19



2 Determine consecutive values of x between which each real zero of the function $f(x) = x^4 - x^3 - 4x^2 + 1$ is located. Then draw the graph.

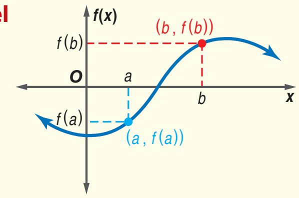
x	$f(x)$	
−2	9	
		} change in signs
−1	−1	
		} change in signs
0	1	
		} change in signs
1	−3	
2	−7	
		} change in signs
3	19	

There are zeros between $x = -2$ and $x = -1$, between $x = -1$ and $x = 0$, between $x = 0$ and $x = 1$, and between $x = 2$ and $x = 3$.



Key Concept — Location Principle

- **Words** Suppose $y = f(x)$ represents a polynomial function and a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b.
- **Model**

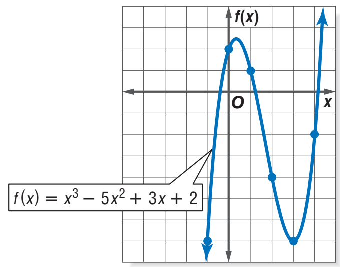


Example 2 Locate Zeros of a Function

Determine consecutive values of x between which each real zero of the function $f(x) = x^3 - 5x^2 + 3x + 2$ is located. Then draw the graph.

Make a table of values. Since $f(x)$ is a third-degree polynomial function, it will have either 1 or 3 real zeros. Look at the values of $f(x)$ to locate the zeros. Then use the points to sketch a graph of the function.

x	$f(x)$	
−2	−32	
−1	−7	} change in signs
0	2	
1	1	} change in signs
2	−4	
3	−7	
4	−2	} change in signs
5	17	

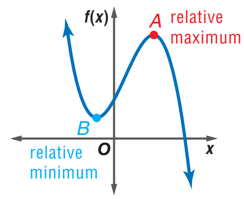


The changes in sign indicate that there are zeros between $x = -1$ and $x = 0$, between $x = 1$ and $x = 2$, and between $x = 4$ and $x = 5$.

MAXIMUM AND MINIMUM POINTS

The graph at the right shows the shape of a general third-degree polynomial function.

Point A on the graph is a **relative maximum** of the cubic function since no other nearby points have a greater y-coordinate. Likewise, point B is a **relative minimum** since no other nearby points have a lesser y-coordinate. These points are often referred to as *turning points*. The graph of a polynomial function of degree n has at most $n - 1$ turning points.



Study Tip

Reading Math

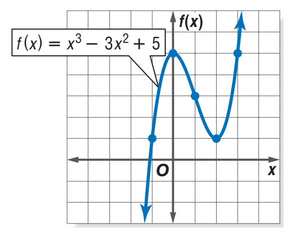
The plurals of maximum and minimum are *maxima* and *minima*.

Example 3 Maximum and Minimum Points

Graph $f(x) = x^3 - 3x^2 + 5$. Estimate the x-coordinates at which the relative maxima and relative minima occur.

Make a table of values and graph the equation.

x	$f(x)$	
−2	−15	} zero between $x = -2$ and $x = -1$
−1	1	
0	5	← indicates a relative maximum
1	3	
2	1	← indicates a relative minimum
3	5	



DAILY INTERVENTION — Unlocking Misconceptions

Modeling Real-World Data Students may incorrectly assume that functions exactly describe every member of a set of real-world data. Stress that a function is just a model of the data, and often it is only a reasonable model for a limited domain of values. Make sure students understand that the function is just an approximation of the real-world data and does not completely describe the data.

Look at the table of values and the graph.

- The values of $f(x)$ change signs between $x = -2$ and $x = -1$, indicating a zero of the function.
- The value of $f(x)$ at $x = 0$ is greater than the surrounding points, so it is a relative maximum.
- The value of $f(x)$ at $x = 2$ is less than the surrounding points, so it is a relative minimum.

The graph of a polynomial function can reveal trends in real-world data.

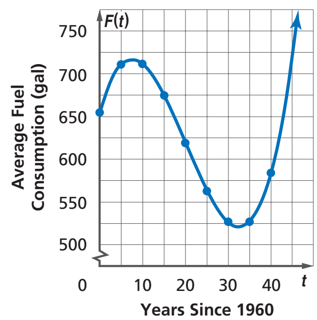
Example 4 Graph a Polynomial Model

ENERGY **The average fuel (in gallons) consumed by individual vehicles in the United States from 1960 to 2000 is modeled by the cubic equation $F(t) = 0.025t^3 - 1.5t^2 + 18.25t + 654$, where t is the number of years since 1960.**

a. Graph the equation.

Make a table of values for the years 1960–2000. Plot the points and connect with a smooth curve. Finding and plotting the points for every fifth year gives a good approximation of the graph.

t	$F(t)$
0	654
5	710.88
10	711.5
15	674.63
20	619
25	563.38
30	526.5
35	527.13
40	584



b. Describe the turning points of the graph and its end behavior.

There is a relative maximum between 1965 and 1970 and a relative minimum between 1990 and 1995. For the end behavior, as t increases, $F(t)$ increases.

c. What trends in fuel consumption does the graph suggest?

Average fuel consumption hit a maximum point around 1970 and then started to decline until 1990. Since 1990, fuel consumption has risen and continues to rise.

A graphing calculator can be helpful in finding the relative maximum and relative minimum of a function.

Graphing Calculator Investigation

Maximum and Minimum Points

You can use a TI-83 Plus to find the coordinates of relative maxima and relative minima. Enter the polynomial function in the Y= list and graph the function. Make sure that all the turning points are visible in the viewing window. Find the coordinates of the minimum and maximum points, respectively.

KEYSTROKES: *Refer to page 293 to review finding maxima and minima.*

(continued on the next page)

More About...

Energy

Gasoline and diesel fuels are the most familiar transportation fuels in this country, but other energy sources are available, including ethanol, a grain alcohol that can be produced from corn or other crops.

Source: U.S. Environmental Protection Agency

Graphing Calculator Investigation

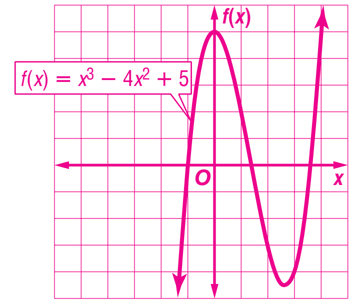
Maximum and Minimum Points Remind students of the procedure for finding relative minima and maxima using the calculator. First, press 2nd [CALC] and select either 3 or 4, depending on whether you are finding a minimum or maximum. Then set the left bound. Use the arrow buttons to move the cursor well to the left of the point you suspect is the minimum or maximum, and press ENTER. Move the cursor well to the right of the suspect point. Press ENTER twice to display the coordinates of the relative maximum/minimum.

MAXIMUM AND MINIMUM POINTS

In-Class Examples

Power Point®

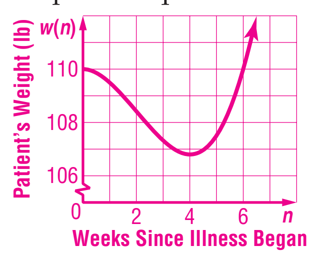
3 Graph $f(x) = x^3 - 4x^2 + 5$. Estimate the x-coordinates at which the relative maxima and relative minima occur.



- **The value of $f(x)$ at $x = 0$ is greater than the surrounding points, so it is a relative maximum.**
- **The value of $f(x)$ at $x \approx 3$ is less than the surrounding points, so it is a relative minimum.**

4 **HEALTH** The weight w, in pounds, of a patient during a 7-week illness is modeled by the cubic equation $w(n) = 0.1n^3 - 0.6n^2 + 110$, where n is the number of weeks since the patient became ill.

a. Graph the equation.



b. Describe the turning points of the graph and its end behavior. **There is a relative minimum point at week 4. For the end behavior, $w(n)$ increases as n increases.**

c. What trends in the patient's weight does the graph suggest? **The patient lost weight for each of 4 weeks after becoming ill. After 4 weeks, the patient gained weight and continues to gain weight.**

3 Practice/Apply

Study Notebook

Have students—

- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 7.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective

- **Graph Polynomial Functions:** 13–26
- **Maximum and Minimum Points:** 13–26

Odd/Even Assignments

Exercises 13–26 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Alert! Exercise 30 involves research on the Internet or other reference materials.

Assignment Guide

Basic: 13–25 odd, 27–32, 36–42, 47–66

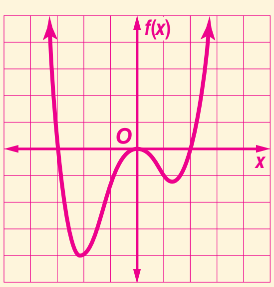
Average: 13–25 odd, 27–42, 47–66 (optional: 43–46)

Advanced: 14–26 even, 27–60 (optional: 61–66)

Answers

1. There must be at least one real zero between two points on a graph when one of the points lies below the x-axis and the other point lies above the x-axis.

3.



Think and Discuss 1–3. See pp. 407A–407H.

1. Graph $f(x) = x^3 - 3x^2 + 4$. Estimate the x-coordinates of the relative maximum and relative minimum points from the graph.
2. Use the maximum and minimum options from the CALC menu to find the exact coordinates of these points. You will need to use the arrow keys to select points to the left and to the right of the point.
3. Graph $f(x) = \frac{1}{2}x^4 - 4x^3 + 7x^2 - 8$. How many relative maximum and relative minimum points does the graph contain? What are the coordinates?

Check for Understanding

Concept Check

1. **Explain** the Location Principle in your own words. See margin.
2. **State** the number of turning points of the graph of a fifth-degree polynomial if it has five distinct real zeros. 4
3. **OPEN ENDED** Sketch a graph of a function that has one relative maximum point and two relative minimum points. See margin.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4, 5	1
6, 7	2
8, 9	3
10–12	4

Graph each polynomial function by making a table of values. 4–5. See pp. 407A–407H.

4. $f(x) = x^3 - x^2 - 4x + 4$
5. $f(x) = x^4 - 7x^2 + x + 5$

Determine consecutive values of x between which each real zero of each function is located. Then draw the graph. 6–7. See pp. 407A–407H for graphs.

6. $f(x) = x^3 - x^2 + 1$ between −1 and 0
7. $f(x) = x^4 - 4x^2 + 2$

7. between −2 and −1, between −1 and 0, between 0 and 1, and between 1 and 2

Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. 8–9. See pp. 407A–407H for graphs.

8. $f(x) = x^3 + 2x^2 - 3x - 5$
9. $f(x) = x^4 - 8x^2 + 10$

8. Sample answer: rel. max. at $x = -2$, rel. min. at $x = 0.5$

9. Sample answer: rel. max. at $x = 0$, rel. min. at $x = -2$ and at $x = 2$

Application

CABLE TV For Exercises 10–12, use the following information.

The number of cable TV systems after 1985 can be modeled by the function $C(t) = -43.2t^2 + 1343t + 790$, where t represents the number of years since 1985.

10. Graph this equation for the years 1985 to 2005. 10–12. See pp. 407A–407H.
11. Describe the turning points of the graph and its end behavior.
12. What trends in cable TV subscriptions does the graph suggest?

Practice and Apply

Homework Help

For Exercises	See Examples
13–26	1, 2, 3
27–35	4

Extra Practice

See page 842.

For Exercises 13–26, complete each of the following.

a. **Graph each function by making a table of values.**

b. **Determine consecutive values of x between which each real zero is located.**

c. **Estimate the x-coordinates at which the relative maxima and relative minima occur.** 13–26. See pp. 407A–407H.

13. $f(x) = -x^3 - 4x^2$
14. $f(x) = x^3 - 2x^2 + 6$
15. $f(x) = x^3 - 3x^2 + 2$
16. $f(x) = x^3 + 5x^2 - 9$
17. $f(x) = -3x^3 + 20x^2 - 36x + 16$
18. $f(x) = x^3 - 4x^2 + 2x - 1$
19. $f(x) = x^4 - 8$
20. $f(x) = x^4 - 10x^2 + 9$
21. $f(x) = -x^4 + 5x^2 - 2x - 1$
22. $f(x) = -x^4 + x^3 + 8x^2 - 3$
23. $f(x) = x^4 - 9x^3 + 25x^2 - 24x + 6$
24. $f(x) = 2x^4 - 4x^3 - 2x^2 + 3x - 5$
25. $f(x) = x^5 + 4x^4 - x^3 - 9x^2 + 3$
26. $f(x) = x^5 - 6x^4 + 4x^3 + 17x^2 - 5x - 6$

DAILY INTERVENTION

Differentiated Instruction ELL

Verbal/Linguistic Have the class work in groups of 3 or 4 students. Instruct students to take turns explaining how to make a table of values for a polynomial function, how to plot several points to begin a graph of the function, how to locate the zeros of the function, and how to estimate the x-coordinates at which the relative maxima and relative minima of the function occur.

27. highest: 1982; lowest: 2000

28. Rel. max. between 1980 and 1985 and between 1990 and 1995, rel. min. between 1975 and 1980 and between 1985 and 1990; as the number of years increases, the percent of the labor force that is unemployed decreases.

30. Sample answer: increase, based on the past fluctuations of the graph

EMPLOYMENT For Exercises 27–30, use the graph that models the unemployment rates from 1975–2000.

27. In what year was the unemployment rate the highest? the lowest?

28. Describe the turning points and end behavior of the graph.

29. If this graph was modeled by a polynomial equation, what is the least degree the equation could have? **5**

30. Do you expect the unemployment rate to increase or decrease from 2001 to 2005? Explain your reasoning.

Online Research **Data Update** What is the current unemployment rate? Visit www.algebra2.com/data_update to learn more.

CHILD DEVELOPMENT For Exercises 31 and 32, use the following information.
The average height (in inches) for boys ages 1 to 20 can be modeled by the equation $B(x) = -0.001x^4 + 0.04x^3 - 0.56x^2 + 5.5x + 25$, where x is the age (in years). The average height for girls ages 1 to 20 is modeled by the equation $G(x) = -0.0002x^4 + 0.006x^3 - 0.14x^2 + 3.7x + 26$.

31. Graph both equations by making a table of values. Use $x = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ as the domain. Round values to the nearest inch. **See pp. 407A–407H.**

32. Compare the graphs. What do the graphs suggest about the growth rate for both boys and girls? **See margin.**

More About...

Child Devolpment
As children develop, their sleeping needs change. Infants sleep about 16–18 hours a day. Toddlers usually sleep 10–12 hours at night and take one or two daytime naps. School-age children need 9–11 hours of sleep, and teens need at least 9 hours of sleep.
Source: www.kidshealth.org

PHYSIOLOGY For Exercises 33–35, use the following information.
During a regular respiratory cycle, the volume of air in liters in the human lungs can be described by the function $V(t) = 0.173t + 0.152t^2 - 0.035t^3$, where t is the time in seconds.

33. Estimate the real zeros of the function by graphing. **0 and between 5 and 6**

34. About how long does a regular respiratory cycle last? **5 s**

35. Estimate the time in seconds from the beginning of this respiratory cycle for the lungs to fill to their maximum volume of air. **3 s**

CRITICAL THINKING For Exercises 36–39, sketch a graph of each polynomial.

36. even-degree polynomial function with one relative maximum and two relative minima **36–39. See pp. 407A–407H for sample graphs.**

37. odd-degree polynomial function with one relative maximum and one relative minimum; the leading coefficient is negative

38. even-degree polynomial function with four relative maxima and three relative minima

39. odd-degree polynomial function with three relative maxima and three relative minima; the leftmost points are negative

40. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson. **See pp. 407A–407H.**

How can graphs of polynomial functions show trends in data?

Include the following in your answer:
- a description of the types of data that are best modeled by polynomial equations rather than linear equations, and
- an explanation of how you would determine when the percent of foreign-born citizens was at its highest and when the percent was at its lowest since 1900.

www.algebra2.com/self_check_quiz

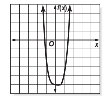
Study Guide and Intervention, p. 381 (shown) and p. 382

Graph Polynomial Functions

Location Principle Suppose $y = f(x)$ represents a polynomial function and a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b.

Example Determine the values of x between which each real zero of the function $f(x) = 2x^4 - x^3 - 5$ is located. Then draw the graph.
Make a table of values. Look at the values of $f(x)$ to locate the zeros. Then use the points to sketch a graph of the function.

x	$f(x)$
−2	35
−1	−2
0	−5
1	−4
2	19

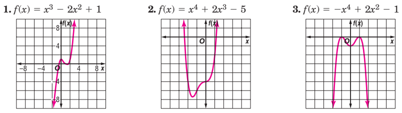


The changes in sign indicate that there are zeros between $x = -2$ and $x = -1$ and between $x = 1$ and $x = 2$.

Exercises

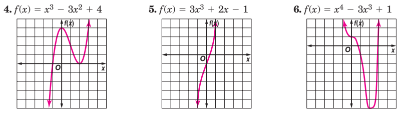
Graph each function by making a table of values. Determine the values of x at which or between which each real zero is located.

1. $f(x) = x^3 - 2x^2 + 1$ 2. $f(x) = x^4 + 2x^3 - 5$ 3. $f(x) = -x^4 + 2x^2 - 1$



between 0 and −1; at 1; between 1 and 2 — between −2 and −3; between 1 and 2 — at ±1

4. $f(x) = x^3 - 3x^2 + 4$ 5. $f(x) = 3x^3 + 2x - 1$ 6. $f(x) = x^4 - 3x^3 + 1$



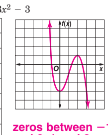
at −1, 2 — between 0 and 1 — between 0 and 1; between 2 and 3

Skills Practice, p. 383 and Practice, p. 384 (shown)

Complete each of the following.
a. Graph each function by making a table of values.
b. Determine consecutive values of x between which each real zero is located.
c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

1. $f(x) = -x^3 + 3x^2 - 3$

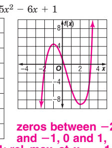
x	$f(x)$
−2	17
−1	1
0	−3
1	−1
2	1
3	−3
4	−19



zeros between −1 and 0, 1 and 2, and 2 and 3; rel. max. at $x = 2$, rel. min. at $x = 0$

2. $f(x) = x^3 - 1.5x^2 - 6x + 1$

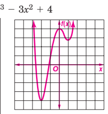
x	$f(x)$
−2	−1
−1	4.5
0	1
1	−5.5
2	−9
3	−3.5
4	17



zeros between −2 and −1, 0 and 1, and 3 and 4; rel. max. at $x = -1$, rel. min. at $x = 2$

3. $f(x) = 0.75x^4 + x^3 - 3x^2 + 4$

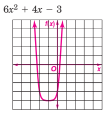
x	$f(x)$
−3	10.75
−2	−4
−1	0.75
0	4
1	2.75
2	12



zeros between −3 and −2, and −2 and −1; rel. max. at $x = 0$, rel. min. at $x = -2$ and $x = 1$

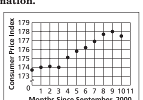
4. $f(x) = x^4 + 4x^3 + 6x^2 + 4x - 3$

x	$f(x)$
−3	12
−2	−3
−1	−4
0	−3
1	12
2	77



zeros between −3 and −2, and 0 and 1; rel. min. at $x = -1$

PRICES For Exercises 5 and 6, use the following information.
The Consumer Price Index (CPI) gives the relative price for a fixed set of goods and services. The CPI from September, 2000 to July, 2001 is shown in the graph.
Source: U. S. Bureau of Labor Statistics



5. Describe the turning points of the graph. **rel. max. in Nov. and June; rel. min in Dec.**

6. If the graph were modeled by a polynomial equation, what is the least degree the equation could have? **4**

7. **LABOR** A town's jobless rate can be modeled by (1, 3.3), (2, 4.9), (3, 5.3), (4, 6.4), (5, 4.5), (6, 5.6), (7, 2.5), (8, 2.7). How many turning points would the graph of a polynomial function through these points have? Describe them. **4: 2 rel. max. and 2 rel. min.**

Reading to Learn Mathematics, p. 385 ELL

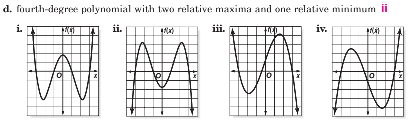
Pre-Activity How can graphs of polynomial functions show trends in data?
Read the introduction to Lesson 7-2 at the top of page 353 in your textbook.
Three points on the graph shown in your textbook are (0, 14), (70, 3.78), and (100, 9). Give the real-world meaning of the coordinates of these points. **Sample answer: In 1900, 14% of the U. S. population was foreign born. In 1970, 3.78% of the population was foreign born. In 2000, 9% of the population was foreign born.**

Reading the Lesson

1. Suppose that $f(x)$ is a third-degree polynomial function and that c and d are real numbers, with $d > c$. Indicate whether each statement is *true* or *false*. (Remember that *true* means *always* true.)
   a. If $f(c) > 0$ and $f(d) < 0$, there is exactly one real zero between c and d. **false**
   b. If $f(c) = f(d) \neq 0$, there are no real zeros between c and d. **false**
   c. If $f(c) < 0$ and $f(d) > 0$, there is at least one real zero between c and d. **true**

2. Match each graph with its description.
   a. third-degree polynomial with one relative maximum and one relative minimum; leading coefficient negative **iii**
   b. fourth-degree polynomial with two relative minima and one relative maximum **i**
   c. third-degree polynomial with one relative maximum and one relative minimum; leading coefficient positive **iv**
   d. fourth-degree polynomial with two relative maxima and one relative minimum **ii**

i. ii. iii. iv.



Helping You Remember

3. The origins of words can help you to remember their meaning and to distinguish between similar words. Look up *maximum* and *minimum* in a dictionary and describe their origins (original language and meaning). **Sample answer: *Maximum* comes from the Latin word *maximus*, meaning *greatest*. *Minimum* comes from the Latin word *minimus*, meaning *least*.**

Answer

32. The growth rate for both boys and girls increases steadily until age 18 and then begins to level off, with boys averaging a height of 71 in. and girls a height of 60 in.

Enrichment, p. 386

Golden Rectangles

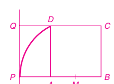
Use a straightedge, a compass, and the instructions below to construct a golden rectangle.

1. Construct square $ABCD$ with sides of 2 centimeters.

2. Construct the midpoint of $\overline{AB}$. Call the midpoint M.

3. Using M as the center, set your compass opening at MC. Construct an arc with center M that intersects $\overline{AB}$. Call the point of intersection P.

4. Construct a line through P that is perpendicular to $\overline{AB}$.



4 Assess

Open-Ended Assessment

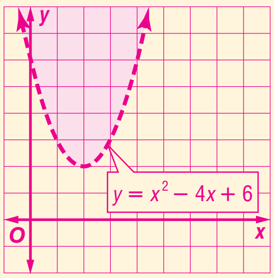
Writing Have students write a paragraph describing how to find the turning points of the graph of a polynomial function.

Getting Ready for Lesson 7-3

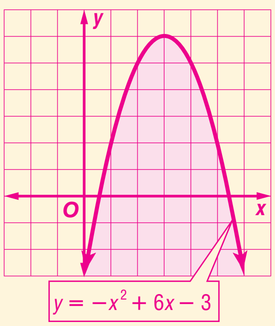
PREREQUISITE SKILL Lesson 7-3 presents solving equations using quadratic techniques. Students will factor polynomials to find the solutions of polynomial equations. Use Exercises 61–66 to determine your students' familiarity with factoring polynomials.

Answers

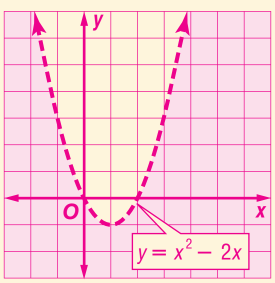
53.



54.



55.

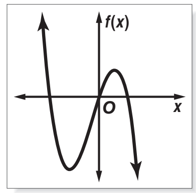


61. $(x + 5)(x - 6)$
62. $(2b - 1)(b - 4)$
63. $(3a + 1)(2a + 5)$
64. $(2m + 3)(2m - 3)$

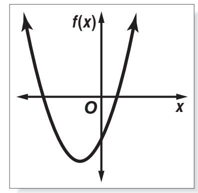
Standardized Test Practice

41. Which of the following could be the graph of $f(x) = x^3 + x^2 - 3x$? **D**

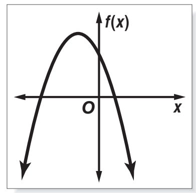
(A)



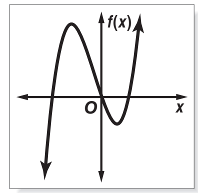
(B)



(C)



(D)



42. The function $f(x) = x^2 - 4x + 3$ has a relative minimum located at which of the following x values? **B**

(A) -2 (B) 2 (C) 3 (D) 4

Graphing Calculator

Use a graphing calculator to estimate the x-coordinates at which the maxima and minima of each function occur. Round to the nearest hundredth.

43. $f(x) = x^3 + x^2 - 7x - 3$ **−1.90; 1.23**
44. $f(x) = -x^3 + 6x^2 - 6x - 5$ **3.41; 0.59**
45. $f(x) = -x^4 + 3x^2 - 8$ **0; −1.22, 1.22**
46. $f(x) = 3x^4 - 7x^3 + 4x - 5$ **0.52; −0.39, 1.62**

Maintain Your Skills

Mixed Review

If $p(x) = 2x^2 - 5x + 4$ and $r(x) = 3x^3 - x^2 - 2$, find each value. *(Lesson 7-1)*

47. $r(2a)$ **$24a^3 - 4a^2 - 2$**
48. $5p(c)$ **$10c^2 - 25c + 20$**
49. $p(2a^2)$ **$8a^4 - 10a^2 + 4$**
50. $r(x - 1)$ **$3x^3 - 10x^2 + 11x - 6$**
51. $p(x^2 + 4)$ **$2x^4 + 11x^2 + 16$**
52. $2[p(x^2 + 1)] - 3r(x - 1)$ **$4x^4 - 9x^3 + 28x^2 - 33x + 20$**

Graph each inequality. *(Lesson 6-7)* **53–55. See margin.**

53. $y > x^2 - 4x + 6$
54. $y \leq -x^2 + 6x - 3$
55. $y < x^2 - 2x$

Solve each matrix equation or system of equations by using inverse matrices. *(Lesson 4-8)*

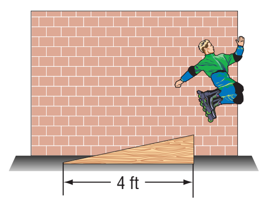
56. $\begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 18 \end{bmatrix}$ **(7, −4)**

57. $\begin{bmatrix} 5 & -7 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ **(−3, −2)**

58. $3j + 2k = 8$
$j - 7k = 18$ **(4, −2)**

59. $5y + 2z = 11$
$10y - 4z = -2$ **(1, 3)**

60. SPORTS Bob and Minya want to build a ramp that they can use while rollerblading. If they want the ramp to have a slope of $\frac{1}{4}$, how tall should they make the ramp? *(Lesson 2-3)* **1 ft**



Getting Ready for the Next Lesson

PREREQUISITE SKILL **Factor each polynomial.**
*(To review **factoring polynomials**, see Lesson 5-4.)*

61. $x^2 - x - 30$
62. $2b^2 - 9b + 4$
63. $6a^2 + 17a + 5$
64. $4m^2 - 9$
65. $t^3 - 27$ **$(t - 3)(t^2 + 3t + 9)$**
66. $r^4 - 1$ **$(r^2 + 1)(r + 1)(r - 1)$**

Graphing Calculator Investigation

A Follow-Up of Lesson 7-2

Modeling Real-World Data

You can use a TI-83 Plus to model data whose curve of best fit is a polynomial function.

Example

The table shows the distance a seismic wave can travel based on its distance from an earthquake's epicenter. Draw a scatter plot and a curve of best fit that relates distance to travel time. Then determine approximately how far from the epicenter the wave will be felt 8.5 minutes after the earthquake occurs.

Source: University of Arizona

Travel Time (min)	1	2	5	7	10	12	13
Distance (km)	400	800	2500	3900	6250	8400	10,000

Step 1 Enter the travel times in L1 and the distances in L2.

KEYSTROKES: *Refer to page 87 to review how to enter lists.*

Step 2 Graph the scatter plot.

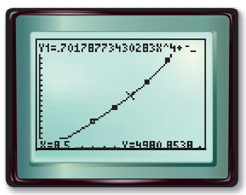
KEYSTROKES: *Refer to page 87 to review how to graph a scatter plot.*

Step 3 Compute and graph the equation for the curve of best fit. A quartic curve is the best fit for these data.

KEYSTROKES: STAT ▶ 7 2nd [L1] , 2nd [L2] ENTER Y= VARS 5 ▶ ▶ 1 GRAPH

Step 4 Use the [CALC] feature to find the value of the function for $x = 8.5$.

KEYSTROKES: *Refer to page 87 to review how to find function values.*



[0, 15] scl: 1 by [0, 10000] scl: 500

After 8.5 minutes, you would expect the wave to be felt approximately 5000 kilometers away.

Exercises 1. See pp. 407A–407H.

Use the table that shows how many minutes out of each eight-hour work day are used to pay one day's worth of taxes.

Year	Minutes
1940	83
1950	117
1960	130
1970	141
1980	145
1990	145
2000	160

Source: Tax Foundation

1. Draw a scatter plot of the data. Then graph several curves of best fit that relate the number of minutes to the year. Try LinReg, QuadReg, and CubicReg.
2. Write the equation for the curve that best fits the data. **See margin.**
3. Based on this equation, how many minutes should you expect to work each day in the year 2010 to pay one day's taxes? **about 184 min**

www.algebra2.com/other_calculator_keystrokes

Answer

2. $(9.4444444 \times 10^{-4})x^3 - 0.1057143x^2 + 4.21031746x - 83.1904762$; The equation is a good fit because $r^2 \approx 0.994$.

Graphing Calculator Investigation

A Follow-Up of Lesson 7-2

Getting Started

Knowing Your Calculator
Students should clear lists L1 and L2 before entering the data from the table in Step 1. This is a more reliable approach than simply "overwriting" old data with new data.

Teach

- Step 3 asserts that a quartic curve will fit the data best. An easy way to verify this is to find quadratic and cubic regression equations and then copy them to the Y= list along with the quartic regression equation. Change the window settings for the *x*-axis to [0, 18.8]. Turn off the scatter plot, and graph all three regression equations on the same screen. Then use the Trace feature to go to each *x* value in the first row of the table. While at each *x* value, use the up/down arrow keys to move between the curves, comparing the *y* value for each regression curve with the *y* value given in the table.
- Have students complete Exercises 1–3.

Assess

In **Exercises 1 and 2**, make sure students have graphed several curves before choosing the one they feel best fits the data. Students should be able to justify their final choice.

In **Exercise 3**, students' answers may vary slightly depending on their best-fit curve.

7-3 Lesson Notes

1 Focus

5-Minute Check Transparency 7-3 Use as a quiz or review of Lesson 7-2.

Mathematical Background notes are available for this lesson on p. 344C.

Building on Prior Knowledge

In Chapter 6, students learned to solve quadratic equations. Students should recognize that the same techniques are used in this lesson to solve higher-degree polynomial equations that can be written using quadratic form.

How can solving polynomial equations help you to find dimensions?

Ask students:

- How does this metal sheet become a box? **by folding up the sides after the squares have been removed from the corners**
- Why are squares, and not rectangles, cut from the corners? **Squares are used so that the sides of the box have the same height. If rectangles are cut from the corners, there are two different heights for the sides and a box cannot be formed.**
- What are the length, width, and height of the box in terms of x? **length: $50 - 2x$; width: $32 - 2x$; height: x**

7-3 Solving Equations Using Quadratic Techniques

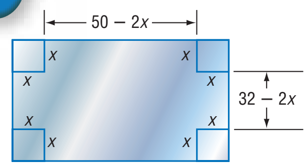
What You'll Learn

- Write expressions in quadratic form.
- Use quadratic techniques to solve equations.

Vocabulary

- quadratic form

How can solving polynomial equations help you to find dimensions?



The Taylor Manufacturing Company makes open metal boxes of various sizes. Each sheet of metal is 50 inches long and 32 inches wide. To make a box, a square is cut from each corner. The volume of the box depends on the side length x of the cut squares. It is given by $V(x) = 4x^3 - 164x^2 + 1600x$. You can solve a polynomial equation to find the dimensions of the square to cut for a box with specific volume.

TEACHING TIP

This method of substituting u for x^2 is called *u substitution.*

QUADRATIC FORM In some cases, you can rewrite a polynomial in x in the form $au^2 + bu + c$. For example, by letting $u = x^2$ the expression $x^4 - 16x^2 + 60$ can be written as $(x^2)^2 - 16(x^2) + 60$ or $u^2 - 16u + 60$. This new, but equivalent, expression is said to be in **quadratic form**.

Key Concept *Quadratic Form*

An expression that is quadratic in form can be written as $au^2 + bu + c$ for any numbers a, b, and c, $a \neq 0$, where u is some expression in x. The expression $au^2 + bu + c$ is called the quadratic form of the original expression.

Example 1 *Write an Expression in Quadratic Form*

Write each expression in quadratic form, if possible.

a. $x^4 + 13x^2 + 36$

$x^4 + 13x^2 + 36 = (x^2)^2 + 13(x^2) + 36$ $\quad (x^2)^2 = x^4$

b. $16x^6 - 625$

$16x^6 - 625 = (4x^3)^2 - 625$ $\quad (x^3)^2 = x^6$

c. $12x^8 - x^2 + 10$

This cannot be written in quadratic form since $x^8 \neq (x^2)^2$.

d. $x - 9x^{\frac{1}{2}} + 8$

$x - 9x^{\frac{1}{2}} + 8 = \left(x^{\frac{1}{2}}\right)^2 - 9\left(x^{\frac{1}{2}}\right) + 8$ $\quad x^1 = \left(x^{\frac{1}{2}}\right)^2$

SOLVE EQUATIONS USING QUADRATIC FORM In Chapter 6, you learned to solve quadratic equations by using the Zero Product Property and the Quadratic Formula. You can extend these techniques to solve higher-degree polynomial equations that can be written using quadratic form or have an expression that contains a quadratic factor.

Resource Manager

Workbook and Reproducible Masters

Chapter 7 Resource Masters

- Study Guide and Intervention, pp. 387–388
- Skills Practice, p. 389
- Practice, p. 390
- Reading to Learn Mathematics, p. 391
- Enrichment, p. 392
- Assessment, p. 443

Transparencies

5-Minute Check Transparency 7-3
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lesson 13
Interactive Chalkboard

Example 2 Solve Polynomial Equations

Solve each equation.

a. $x^4 - 13x^2 + 36 = 0$

$x^4 - 13x^2 + 36 = 0$	Original equation
$(x^2)^2 - 13(x^2) + 36 = 0$	Write the expression on the left in quadratic form.
$(x^2 - 9)(x^2 - 4) = 0$	Factor the trinomial.
$(x - 3)(x + 3)(x - 2)(x + 2) = 0$	Factor each difference of squares.

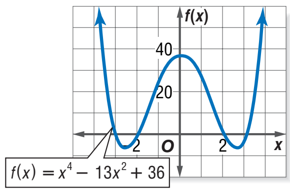
Use the Zero Product Property.

$x - 3 = 0$ or $x + 3 = 0$ or $x - 2 = 0$ or $x + 2 = 0$

$x = 3$ $\quad x = -3$ $\quad x = 2$ $\quad x = -2$

The solutions are $-3, -2, 2,$ and 3.

CHECK The graph of $f(x) = x^4 - 13x^2 + 36$ shows that the graph intersects the x-axis at $-3, -2, 2,$ and 3. ✓



Study Tip

Look Back

To review the formula for factoring the **sum of two cubes**, see Lesson 5-4.

b. $x^3 + 343 = 0$

$x^3 + 343 = 0$	Original equation
$(x)^3 + 7^3 = 0$	This is the sum of two cubes.
$(x + 7)[x^2 - x(7) + 7^2] = 0$	Sum of two cubes formula with $a = x$ and $b = 7$
$(x + 7)(x^2 - 7x + 49) = 0$	Simplify.
$x + 7 = 0$ or $x^2 - 7x + 49 = 0$	Zero Product Property

The solution of the first equation is -7. The second equation can be solved by using the Quadratic Formula.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic Formula
$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(49)}}{2(1)}$	Replace a with 1, b with -7, and c with 49.
$= \frac{7 \pm \sqrt{-147}}{2}$	Simplify.
$= \frac{7 \pm i\sqrt{147}}{2}$ or $\frac{7 \pm 7i\sqrt{3}}{2}$	$\sqrt{147} \times \sqrt{-1} = i\sqrt{147}$

Thus, the solutions of the original equation are $-7, \frac{7 + 7i\sqrt{3}}{2}$, and $\frac{7 - 7i\sqrt{3}}{2}$.

Study Tip

Substitution

To avoid confusion, you can substitute another variable for the expression in parentheses.

For example, $\left(x^{\frac{1}{3}}\right)^2 - 6\left(x^{\frac{1}{3}}\right) + 5 = 0$ could be written as $u^2 - 6u + 5 = 0$. Then, once you have solved the equation for u, substitute $x^{\frac{1}{3}}$ for u and solve for x.

Some equations involving rational exponents can be solved by using a quadratic technique.

Example 3 Solve Equations with Rational Exponents

Solve $x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 5 = 0$.

$x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 5 = 0$	Original equation
$\left(x^{\frac{1}{3}}\right)^2 - 6\left(x^{\frac{1}{3}}\right) + 5 = 0$	Write the expression on the left in quadratic form.

(continued on the next page)

2 Teach

QUADRATIC FORM

In-Class Example Power Point®

1 Write each expression in quadratic form, if possible.

a. $2x^6 + x^3 + 9$ $\mathbf{2(x^3)^2 + (x^3) + 9}$

b. $7x^{10} + 6$ $\mathbf{7(x^5)^2 + 6}$

c. $x^4 + 2x^3 - 1$ **This cannot be written in quadratic form since $x^4 \neq (x^3)^2$.**

d. $x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 4$ $\mathbf{\left(x^{\frac{1}{3}}\right)^2 + 2\left(x^{\frac{1}{3}}\right) - 4}$

SOLVE EQUATIONS USING QUADRATIC FORM

In-Class Examples Power Point®

2 Solve each equation.

a. $x^4 - 29x^2 + 100 = 0$ **−5, −2, 2, 5**

b. $x^3 + 216 = 0$ $\mathbf{-6, 3 + 3i\sqrt{3}, 3 - 3i\sqrt{3}}$

3 Solve $x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0$. **81**

DAILY INTERVENTION

Unlocking Misconceptions

Quadratic Form In Example 1 on p. 360, students may mistakenly conclude that variables must have even powers in order for the expression to be written in quadratic form. Draw students' attention to Example 1d. Clarify that the relationship between the powers of two terms is what indicates whether an expression can be written in quadratic form. In Example 1d, the power of the x term is twice the power of the $x^{\frac{1}{2}}$ term, so the expression can be rewritten in quadratic form.

In-Class Example

Power Point®

4 Solve $x + \sqrt{x} = 12$. **9**

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 7.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises…

Organization by Objective
- **Quadratic Form:** 11–16
- **Solve Equations Using Quadratic Form:** 17–30

Odd/Even Assignments
Exercises 11–30 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 11–27 odd, 32–34, 37–52

Average: 11–31 odd, 32–34, 37–52

Advanced: 12–30 even, 32–48 (optional: 49–52)

All: Practice Quiz 1 (1–5)

Answers

1. Sample answer: $16x^4 - 12x^2 = 0$; $4[4(x^2)^2 - 3x^2] = 0$
2. The solutions of a polynomial equation are the points at which the graph intersects the x-axis.
3. Factor out an x and write the equation in quadratic form so you have $x[(x^2)^2 - 2(x^2) + 1] = 0$. Factor the trinomial and solve for x using the Zero Product Property. The solutions are -1, 0, and 1.

$\left(x^{\frac{1}{3}} - 1\right)\left(x^{\frac{1}{3}} - 5\right) = 0$ Factor the trinomial.

$x^{\frac{1}{3}} - 1 = 0$ or $x^{\frac{1}{3}} - 5 = 0$ Zero Product Property

$x^{\frac{1}{3}} = 1$ $\quad x^{\frac{1}{3}} = 5$ Isolate x on one side of the equation.

$\left(x^{\frac{1}{3}}\right)^3 = 1^3$ $\quad \left(x^{\frac{1}{3}}\right)^3 = 5^3$ Cube each side.

$x = 1$ $\quad x = 125$ Simplify.

CHECK Substitute each value into the original equation.

$$x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 5 = 0 \qquad x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 5 = 0$$

$$1^{\frac{2}{3}} - 6(1)^{\frac{1}{3}} + 5 \stackrel{?}{=} 0 \qquad 125^{\frac{2}{3}} - 6(125)^{\frac{1}{3}} + 5 \stackrel{?}{=} 0$$

$$1 - 6 + 5 \stackrel{?}{=} 0 \qquad 25 - 30 + 5 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark \qquad 0 = 0 \checkmark$$

The solutions are 1 and 125.

To use a quadratic technique, rewrite the equation so one side is equal to zero.

Example 4 Solve Radical Equations

Solve $x - 6\sqrt{x} = 7$.

$x - 6\sqrt{x} = 7$ Original equation

$x - 6\sqrt{x} - 7 = 0$ Rewrite so that one side is zero.

$(\sqrt{x})^2 - 6(\sqrt{x}) - 7 = 0$ Write the expression on the left in quadratic form.

You can use the Quadratic Formula to solve this equation.

$\sqrt{x} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic Formula

$\sqrt{x} = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-7)}}{2(1)}$ Replace a with 1, b with -6, and c with -7.

$\sqrt{x} = \dfrac{6 \pm 8}{2}$ Simplify.

$\sqrt{x} = \dfrac{6 + 8}{2}$ or $\sqrt{x} = \dfrac{6 - 8}{2}$ Write as two equations.

$\sqrt{x} = 7$ $\quad \sqrt{x} = -1$ Simplify.

$x = 49$

Since the principal square root of a number cannot be negative, the equation $\sqrt{x} = -1$ has no solution. Thus, the only solution of the original equation is 49.

Study Tip

Look Back
To review **principal roots**, see Lesson 5-5.

Check for Understanding

Concept Check

1. **OPEN ENDED** Give an example of an equation that is not quadratic but can be written in quadratic form. Then write it in quadratic form. **1–3. See margin.**
2. **Explain** how the graph of the related polynomial function can help you verify the solution to a polynomial equation.
3. **Describe** how to solve $x^5 - 2x^3 + x = 0$.

DAILY INTERVENTION

Differentiated Instruction

Visual/Spatial Have students repeat Example 3, using the substitution method discussed in the Study Tip at the bottom of p. 361. Instruct students to write the given equation in pencil and then use a colored pencil to write the statement "Let $u = x^{\frac{1}{3}}$." Have students continue solving the problem using the colored pencil to help them remember they are working with a substituted variable. After solving for u, when students substitute $x^{\frac{1}{3}}$ for u they should resume using their regular pencil.

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4, 5	1
6–9	2
10	3

Write each expression in quadratic form, if possible.

4. $5y^4 + 7y^3 - 8$ **not possible**
5. $84n^4 - 62n^2$ **$84(n^2)^2 - 62(n^2)$**

Solve each equation. **8. 6, $-3 + 3i\sqrt{3}$, $-3 - 3i\sqrt{3}$**

6. $x^3 + 9x^2 + 20x = 0$ **0, −5, −4**
7. $x^4 - 17x^2 + 16 = 0$ **−4, −1, 4, 1**
8. $x^3 - 216 = 0$
9. $x - 16x^{\frac{1}{2}} = -64$ **64**

Application

10. **POOL** The Shelby University swimming pool is in the shape of a rectangular prism and has a volume of 28,000 cubic feet. The dimensions of the pool are x feet deep by $7x - 6$ feet wide by $9x - 2$ feet long. How deep is the pool? **8 ft**

★ indicates increased difficulty **11. $2(x^2)^2 + 6(x^2) - 10$ 12. not possible 13. $11(n^3)^2 + 44(n^3)$ 15. not possible**

Practice and Apply

Homework Help

For Exercises	See Examples
11–16	1
17–28	2–4
29–36	2

Extra Practice

See page 842.

Write each expression in quadratic form, if possible. **14. $b[7(b^2)^2 - 4(b^2) + 2)]$**

11. $2x^4 + 6x^2 - 10$
12. $a^8 + 10a^2 - 16$
13. $11n^6 + 44n^3$
14. $7b^5 - 4b^3 + 2b$
15. $7x^{\frac{1}{9}} - 3x^{\frac{1}{3}} + 4$
16. $6x^{\frac{2}{5}} - 4x^{\frac{1}{5}} - 16$ **$6\left(x^{\frac{1}{5}}\right)^2 - 4\left(x^{\frac{1}{5}}\right) - 16 = 0$**

Solve each equation. **17–28. See pp. 407A–407H.**

17. $m^4 + 7m^3 + 12m^2 = 0$
18. $a^5 + 6a^4 + 5a^3 = 0$
19. $b^4 = 9$
20. $t^5 - 256t = 0$
21. $d^4 + 32 = 12d^2$
22. $x^4 + 18 = 11x^2$
23. $x^3 + 729 = 0$
24. $y^3 - 512 = 0$
25. $x^{\frac{1}{2}} - 8x^{\frac{1}{4}} + 15 = 0$
26. $p^{\frac{2}{3}} + 11p^{\frac{1}{3}} + 28 = 0$
27. $y - 19\sqrt{y} = -60$
28. $z = 8\sqrt{z} + 240$

★ 29. $s^3 + 4s^2 - s - 4 = 0$ **1, −1, −4**

★ 30. $h^3 - 8h^2 + 3h - 24 = 0$ **8**

31. **GEOMETRY** The width of a rectangular prism is w centimeters. The height is 2 centimeters less than the width. The length is 4 centimeters more than the width. If the volume of the prism is 8 times the measure of the length, find the dimensions of the prism. **w = 4 cm, ℓ = 8 cm, h = 2 cm**

Career Choices

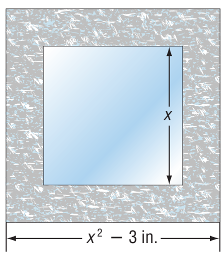
Designer

Designers combine practical knowledge with artistic ability to turn abstract ideas into formal designs. Designers usually specialize in a particular area, such as clothing, or home interiors.

Online Research For information about a career as a designer, visit: www.algebra2.com/careers

DESIGN For Exercises 32–34, use the following information.

Jill is designing a picture frame for an art project. She plans to have a square piece of glass in the center and surround it with a decorated ceramic frame, which will also be a square. The dimensions of the glass and frame are shown in the diagram at the right. Jill determines that she needs 27 square inches of material for the frame.



32. Write a polynomial equation that models the area of the frame. **$x^4 - 7x^2 + 9 = 27$**
33. What are the dimensions of the glass piece? **3 in. × 3 in.**
34. What are the dimensions of the frame? **6 in. × 6 in.**

PACKAGING For Exercises 35 and 36, use the following information.

A computer manufacturer needs to change the dimensions of its foam packaging for a new model of computer. The width of the original piece is three times the height, and the length is equal to the height squared. The volume of the new piece can be represented by the equation $V(h) = 3h^4 + 11h^3 + 18h^2 + 44h + 24$, where h is the height of the original piece. **35. $h^2 + 4$, $3h + 2$, $h + 3$**

★ 35. Factor the equation for the volume of the new piece to determine three expressions that represent the height, length, and width of the new piece.

★ 36. How much did each dimension of the packaging increase for the new foam piece? **The height increased by 3, the width increased by 2, and the length increased by 4.**

www.algebra2.com/self_check_quiz

Study Guide and Intervention, p. 387 (shown) and p. 388

Quadratic Form Certain polynomial expressions in x can be written in the quadratic form $au^2 + bu + c$ for any numbers a, b, and c, $a \neq 0$, where u is an expression in x.

Example Write each polynomial in quadratic form, if possible.

a. $3a^6 - 9a^3 + 12$
Let $u = a^3$.
$3a^6 - 9a^3 + 12 = 3(a^3)^2 - 9(a^3) + 12$

b. $101b - 49\sqrt{b} + 42$
Let $u = \sqrt{b}$.
$101b - 49\sqrt{b} + 42 = 101(\sqrt{b})^2 - 49(\sqrt{b}) + 42$

c. $24a^5 + 12a^3 + 18$
This expression cannot be written in quadratic form, since $a^5 \neq (a^3)^2$.

Exercises

Write each polynomial in quadratic form, if possible.

1. $x^4 + 6x^2 - 8$ **$(x^2)^2 + 6(x^2) - 8$**
2. $4p^4 + 6p^2 + 8$ **$4(p^2)^2 + 6(p^2) + 8$**
3. $x^8 + 2x^4 + 1$ **$(x^4)^2 + 2(x^4) + 1$**
4. $x^{\frac{1}{8}} + 2x^{\frac{1}{16}} + 1$ **$\left(x^{\frac{1}{16}}\right)^2 + 2\left(x^{\frac{1}{16}}\right) + 1$**
5. $6x^4 + 3x^3 + 18$ **not possible**
6. $12x^4 + 10x^2 - 4$ **$12(x^2)^2 + 10(x^2) - 4$**
7. $24x^8 + x^4 + 4$ **$24(x^4)^2 + x^4 + 4$**
8. $18x^6 - 2x^3 + 12$ **$18(x^3)^2 - 2(x^3) + 12$**
9. $100x^4 - 9x^2 - 15$ **$100(x^2)^2 - 9(x^2) - 15$**
10. $25x^8 + 36x^6 - 49$ **not possible**
11. $48x^6 - 32x^3 + 20$ **$48(x^3)^2 - 32(x^3) + 20$**
12. $63x^8 + 5x^4 - 29$ **$63(x^4)^2 + 5(x^4) - 29$**
13. $32x^{10} + 14x^5 - 143$ **$32(x^5)^2 + 14(x^5) - 143$**
14. $50x^3 - 15x\sqrt{x} - 18$ **$50\left(x^{\frac{3}{2}}\right)^2 - 15\left(x^{\frac{3}{2}}\right) - 18$**
15. $60x^6 - 7x^3 + 3$ **$60(x^3)^2 - 7(x^3) + 3$**
16. $10x^{10} - 7x^5 - 7$ **$10(x^5)^2 - 7(x^5) - 7$**

Skills Practice, p. 389 and Practice, p. 390 (shown)

Write each expression in quadratic form, if possible.

1. $10b^4 + 3b^2 - 11$ **$10(b^2)^2 + 3(b^2) - 11$**
2. $-5x^8 + x^2 + 6$ **not possible**
3. $28d^6 + 25d^3$ **$28(d^3)^2 + 25(d^3)$**
4. $4s^8 + 4s^4 + 7$ **$4(s^4)^2 + 4(s^4) + 7$**
5. $500x^4 - x^2$ **$500(x^2)^2 - x^2$**
6. $8b^5 - 8b^3 - 1$ **not possible**
7. $32w^5 - 56w^3 + 8w$ **$8w[4(w^2)^2 - 7(w^2) + 1]$**
8. $e^{\frac{2}{3}} + 7e^{\frac{1}{3}} - 10$ **$\left(e^{\frac{1}{3}}\right)^2 + 7\left(e^{\frac{1}{3}}\right) - 10$**
9. $x^{\frac{1}{5}} + 29x^{\frac{1}{10}} + 2$ **$\left(x^{\frac{1}{10}}\right)^2 + 29\left(x^{\frac{1}{10}}\right) + 2$**

Solve each equation.

10. $y^4 - 7y^3 - 18y^2 = 0$ **−2, 0, 9**
11. $s^5 + 4s^4 - 32s^3 = 0$ **−8, 0, 4**
12. $m^4 - 625 = 0$ **$-5, 5, -5i, 5i$**
13. $n^4 - 49n^2 = 0$ **0, −7, 7**
14. $x^4 - 50x^2 + 49 = 0$ **−1, 1, −7, 7**
15. $t^4 - 21t^2 + 80 = 0$ **$-4, 4, \sqrt{5}, -\sqrt{5}$**
16. $4r^6 - 9r^4 = 0$ **$0, \frac{3}{2}, -\frac{3}{2}$**
17. $x^4 - 24 = -2x^2$ **$-2, 2, -i\sqrt{6}, i\sqrt{6}$**
18. $d^4 = 16d^2 - 48$ **$-2, 2, -2\sqrt{3}, 2\sqrt{3}$**
19. $t^3 - 343 = 0$ **$7, \frac{-7 - 7i\sqrt{3}}{2}, \frac{-7 + 7i\sqrt{3}}{2}$**
20. $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 6 = 0$ **16, 81**
21. $x^{\frac{4}{3}} - 29x^{\frac{2}{3}} + 100 = 0$ **8, 125**
22. $y^3 - 28y^{\frac{3}{2}} + 27 = 0$ **1, 9**
23. $n - 10\sqrt{n} + 25 = 0$ **25**
24. $w - 12\sqrt{w} + 27 = 0$ **9, 81**
25. $x - 2\sqrt{x} - 80 = 0$ **100**

26. **PHYSICS** A proton in a magnetic field follows a path on a coordinate grid modeled by the function $f(x) = x^4 - 2x^2 - 15$. What are the x-coordinates of the points on the grid where the proton crosses the x-axis? **$-\sqrt{5}, \sqrt{5}$**

27. **SURVEYING** Vista county is setting aside a large parcel of land to preserve it as open space. The county has hired Meghan's surveying firm to survey the parcel, which is in the shape of a right triangle. The longer leg of the triangle measures 5 miles less than the square of the shorter leg, and the hypotenuse of the triangle measures 13 miles less than twice the square of the shorter leg. The length of each boundary is a whole number. Find the length of each boundary. **3 mi, 4 mi, 5 mi**

Reading to Learn Mathematics, p. 391 — ELL

Pre-Activity How can solving polynomial equations help you to find dimensions?

Read the introduction to Lesson 7-3 at the top of page 360 in your textbook.

Explain how the formula given for the volume of the box can be obtained from the dimensions shown in the figure.

Sample answer: The volume of a rectangular box is given by the formula $V = \ell wh$. Substitute $50 - 2x$ for ℓ, $32 - 2x$ for w, and x for h to get $V(x) = (50 - 2x)(32 - 2x)(x) = 4x^3 - 164x^2 + 1600x$.

Reading the Lesson

1. Which of the following expressions can be written in quadratic form? **b, c, d, f, g, h, i**
   a. $x^3 + 6x^2 + 9$
   b. $x^4 - 7x^2 + 6$
   c. $m^6 + 4m^3 + 4$
   d. $y - 2y^{\frac{1}{2}} - 15$
   e. $x^5 + x^3 + 1$
   f. $r^4 + 6 - r^8$
   g. $p^{\frac{1}{4}} + 8p^{\frac{1}{2}} + 12$
   h. $r^{\frac{1}{3}} + 2r^{\frac{1}{6}} - 3$
   i. $5\sqrt{z} + 2z - 3$

2. Match each expression from the list on the left with its factorization from the list on the right.
   a. $x^4 - 3x^2 - 40$ **vi**
   b. $x^4 - 10x^2 + 25$ **v**
   c. $x^6 - 9$ **i**
   d. $x - 9$ **ii**
   e. $x^6 + 1$ **iv**
   f. $x + 6\sqrt{x} + 9$ **iii**

   i. $(x^3 + 3)(x^3 - 3)$
   ii. $(\sqrt{x} + 3)(\sqrt{x} - 3)$
   iii. $(\sqrt{x} + 3)^2$
   iv. $(x^2 + 1)(x^4 - x^2 + 1)$
   v. $(x^2 - 5)^2$
   vi. $(x^2 + 5)(x^2 - 8)$

Helping You Remember

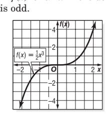
3. What is an easy way to tell whether a trinomial in one variable containing one constant term can be written in quadratic form?

Sample answer: Look at the two terms that are not constants and compare the exponents on the variable. If one of the exponents is twice the other, the trinomial can be written in quadratic form.

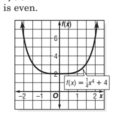
Enrichment, p. 392

Odd and Even Polynomial Functions

Functions whose graphs are symmetric with respect to the origin are called *odd* functions. If $f(-x) = -f(x)$ for all x in the domain of $f(x)$, then $f(x)$ is odd.



Functions whose graphs are symmetric with respect to the y-axis are called *even* functions. If $f(-x) = f(x)$ for all x in the domain of $f(x)$, then $f(x)$ is even.



Example Determine whether $f(x) = x^3 - 3x$ is odd, even, or neither.

$f(x) = x^3 - 3x$

4 Assess

Open-Ended Assessment

Speaking Have students explain how the equation in Example 4 on p. 362 can be solved by first substituting a for $\sqrt{x}$. If students have difficulty getting started, ask them how they could express x in terms of a, given that $a = \sqrt{x}$.

Getting Ready for Lesson 7-4

PREREQUISITE SKILL Lesson 7-4 introduces students to the Remainder and Factor Theorems. Students will use division to find the factors of polynomials. Use Exercises 49–52 to determine your students' familiarity with dividing polynomials by a binomial.

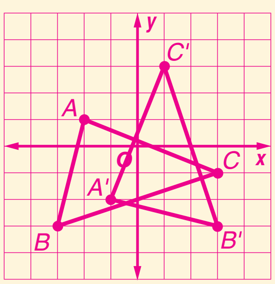
Assessment Options

Practice Quiz 1 The quiz provides students with a brief review of the concepts and skills in Lessons 7-1 through 7-3. Lesson numbers are given to the right of the exercises or instruction lines so students can review concepts not yet mastered.

Quiz (Lessons 7-1 through 7-3) is available on p. 443 of the *Chapter 7 Resource Masters*.

Answers

48.



49. $x^2 + 5x - 4$

50. $4x^2 - 16x + 27 - \frac{64}{x+2}$

51. $x^3 - 6x - 20 - \frac{54}{x-3}$

52. $x^3 + 2x^2 - 10x + 15 - \frac{21}{x+1}$

37. Write the equation in quadratic form, $u^2 - 9u + 8 = 0$, where $u = |a - 3|$. Then factor and use the Zero Product Property to solve for a; 11, 4, 2, and −5.

37. CRITICAL THINKING Explain how you would solve $|a - 3|^2 - 9|a - 3| = -8$. Then solve the equation.

38. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. See pp. 407A–407H.

How can solving polynomial equations help you to find dimensions?

Include the following items in your answer:
- an explanation of how you could determine the dimensions of the cut square if the desired volume was 3600 cubic inches, and
- an explanation of why there can be more than one square that can be cut to produce the same volume.

Standardized Test Practice

39. Which of the following is a solution of $x^4 - 2x^2 - 3 = 0$? **D**

(A) $\sqrt[4]{2}$ (B) 1 (C) −3 (D) $\sqrt{3}$

40. EXTENDED RESPONSE Solve $18x + 9\sqrt{2x} - 4 = 0$ by first rewriting it in quadratic form. Show your work. $\frac{1}{18}$

Maintain Your Skills

Mixed Review

41–42. See pp. 407A–407H.

Graph each function by making a table of values. *(Lesson 7-2)*

41. $f(x) = x^3 - 4x^2 + x + 5$ **42.** $f(x) = x^4 - 6x^3 + 10x^2 - x - 3$

Find $p(7)$ and $p(-3)$ for each function. *(Lesson 7-1)*

43. $p(x) = x^2 - 5x + 3$ 17; 27 **44.** $p(x) = x^3 - 11x - 4$ 262; 2 **45.** $p(x) = \frac{2}{3}x^4 - 3x^3$ $\frac{1715}{3}$; 135

For Exercises 46–48, use the following information.
Triangle ABC with vertices $A(-2, 1)$, $B(-3, -3)$, and $C(3, -1)$ is rotated 90° counterclockwise about the origin. *(Lesson 4-4)*

46. Write the coordinates of the triangle in a vertex matrix. $\begin{bmatrix} -2 & -3 & 3 \\ 1 & -3 & -1 \end{bmatrix}$

47. Find the coordinates of the $A'B'C'$. $A'(-1, -2)$, $B'(3, -3)$, $C'(1, 3)$

48. Graph the preimage and the image. See margin.

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each quotient.**
(To review ***dividing polynomials****, see Lesson 5-3.)* 49–52. See margin.

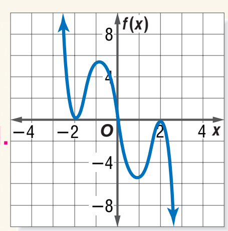
49. $(x^3 + 4x^2 - 9x + 4) \div (x - 1)$ **50.** $(4x^3 - 8x^2 - 5x - 10) \div (x + 2)$

51. $(x^4 - 9x^2 - 2x + 6) \div (x - 3)$ **52.** $(x^4 + 3x^3 - 8x^2 + 5x - 6) \div (x + 1)$

Practice Quiz 1 — Lessons 7-1 through 7-3

1. If $p(x) = 2x^3 - x$, find $p(a - 1)$. *(Lesson 7-1)* $2a^3 - 6a^2 + 5a - 1$

2. Describe the end behavior of the graph at the right. Then determine whether it represents an odd-degree or an even-degree polynomial function and state the number of real zeros. *(Lesson 7-1)* See margin.



3. Graph $y = x^3 + 2x^2 - 4x - 6$. Estimate the x-coordinates at which the relative maxima and relative minima occur. *(Lesson 7-2)* See pp. 407A–407H.

4. Write the expression $18x^{\frac{1}{3}} + 36x^{\frac{2}{3}} + 5$ in quadratic form. *(Lesson 7-3)* See margin.

5. Solve $a^4 = 6a^2 + 27$. *(Lesson 7-3)* $-3, 3, -i\sqrt{3}, i\sqrt{3}$

Answers (Practice Quiz 1)

2. $f(x) \to -\infty$ as $x \to +\infty$, $f(x) \to +\infty$ as $x \to -\infty$; odd; 3

4. $\left(6x^{\frac{1}{3}}\right)^2 + 3\left(6x^{\frac{1}{3}}\right) + 5$ or $36\left(\sqrt[3]{x}\right)^2 + 18\left(\sqrt[3]{x}\right) + 5$

364 Chapter 7 Polynomial Functions

7-4 The Remainder and Factor Theorems

What You'll Learn

- Evaluate functions using synthetic substitution.
- Determine whether a binomial is a factor of a polynomial by using synthetic substitution.

Vocabulary

- synthetic substitution
- depressed polynomial

How can you use the Remainder Theorem to evaluate polynomials?

The number of international travelers to the United States since 1986 can be modeled by the equation $T(x) = 0.02x^3 - 0.6x^2 + 6x + 25.9$, where x is the number of years since 1986 and $T(x)$ is the number of travelers in millions. To estimate the number of travelers in 2006, you can evaluate the function for $x = 20$, or you can use synthetic substitution.

Study Tip

Look Back
To review **dividing polynomials** and **synthetic division**, see Lesson 5-3.

SYNTHETIC SUBSTITUTION Synthetic division is a shorthand method of long division. It can also be used to find the value of a function. Consider the polynomial function $f(a) = 4a^2 - 3a + 6$. Divide the polynomial by $a - 2$.

Method 1 Long Division

$$\begin{array}{r} 4a + 5 \\ a - 2\overline{)4a^2 - 3a + 6} \\ \underline{4a^2 - 8a} \\ 5a + 6 \\ \underline{5a - 10} \\ 16 \end{array}$$

Method 2 Synthetic Division

$$\begin{array}{c|ccc} 2 & 4 & -3 & 6 \\ & & 8 & 10 \\ \hline & 4 & 5 & 16 \end{array}$$

Compare the remainder of 16 to $f(2)$.

$f(2) = 4(2)^2 - 3(2) + 6$ Replace a with 2.

$= 16 - 6 + 6$ Multiply.

$= 16$ Simplify.

Notice that the value of $f(2)$ is the same as the remainder when the polynomial is divided by $a - 2$. This illustrates the **Remainder Theorem**.

Key Concept — *Remainder Theorem*

If a polynomial $f(x)$ is divided by $x - a$, the remainder is the constant $f(a)$, and

Dividend	equals	quotient	times	divisor	plus	remainder.
$f(x)$	$=$	$q(x)$	$\cdot$	$(x - a)$	$+$	$f(a)$,

where $q(x)$ is a polynomial with degree one less than the degree of $f(x)$.

When synthetic division is used to evaluate a function, it is called **synthetic substitution**. It is a convenient way of finding the value of a function, especially when the degree of the polynomial is greater than 2.

7-4 Lesson Notes

1 Focus

5-Minute Check Transparency 7-4 Use as a quiz or review of Lesson 7-3.

Mathematical Background notes are available for this lesson on p. 344D.

How can you use the Remainder Theorem to evaluate polynomials?

Ask students:

- What would be the value of x if you wanted to use the function to estimate the number of travelers in 1987? **1**
- Would you expect the actual number of travelers in 2006 to exactly match the number predicted by the function? Explain. **No. Sample answer: The equation is a model based on data available at this time.**
- Do you think the model is more accurate for the years immediately following 1986 or for years in the future? Explain. **Sample answer: The model closely matches actual data for the years immediately following 1986. For years in the future, the model is less likely to be as accurate.**

Resource Manager

Workbook and Reproducible Masters

Chapter 7 Resource Masters
- Study Guide and Intervention, pp. 393–394
- Skills Practice, p. 395
- Practice, p. 396
- Reading to Learn Mathematics, p. 397
- Enrichment, p. 398

Teaching Algebra With Manipulatives Masters, p. 252

Transparencies

5-Minute Check Transparency 7-4
Answer Key Transparencies

Technology

Interactive Chalkboard

Building on Prior Knowledge

In Lesson 5-3, students learned synthetic division. In this lesson, students will use synthetic division to evaluate a function and to find factors of polynomials.

2 Teach

SYNTHETIC SUBSTITUTION

In-Class Example Power Point®

Teaching Tip Most of this lesson relies heavily on synthetic substitution. Before finishing your discussion of Example 1, be certain that students understand the method. Ask a student volunteer to demonstrate the steps of the synthetic substitution shown in Method 1 of Example 1, and invite students to discuss any problems they have with the technique before moving on.

1 If $f(x) = 3x^4 - 2x^3 + x^2 - 2$, find $f(4)$. **654**

Example 1 *Synthetic Substitution*

If $f(x) = 2x^4 - 5x^2 + 8x - 7$, find $f(6)$.

Method 1 Synthetic Substitution

By the Remainder Theorem, $f(6)$ should be the remainder when you divide the polynomial by $x - 6$.

6	2	0	−5	8	−7
		12	72	402	2460
	2	12	67	410	2453

Notice that there is no x^3 term. A zero is placed in this position as a placeholder.

The remainder is 2453. Thus, by using synthetic substitution, $f(6) = 2453$.

Method 2 Direct Substitution

Replace x with 6.

$f(x) = 2x^4 - 5x^2 + 8x - 7$ Original function

$f(6) = 2(6)^4 - 5(6)^2 + 8(6) - 7$ Replace x with 6.

$= 2592 - 180 + 48 - 7$ or 2453 Simplify.

By using direct substitution, $f(6) = 2453$.

FACTORS OF POLYNOMIALS Divide $f(x) = x^4 + x^3 - 17x^2 - 20x + 32$ by $x - 4$.

4	1	1	−17	−20	32
		4	20	12	−32
	1	5	3	−8	0

Study Tip

Depressed Polynomial
A depressed polynomial has a degree that is one less than the original polynomial.

The quotient of $f(x)$ and $x - 4$ is $x^3 + 5x^2 + 3x - 8$. When you divide a polynomial by one of its binomial factors, the quotient is called a **depressed polynomial**. From the results of the division and by using the Remainder Theorem, we can make the following statement.

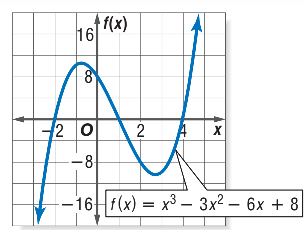
$$\underbrace{x^4 + x^3 - 17x^2 - 20x + 32}_{\text{Dividend}} \underbrace{=}_{\text{equals}} \underbrace{(x^3 + 5x^2 + 3x - 8)}_{\text{quotient}} \underbrace{\cdot}_{\text{times}} \underbrace{(x - 4)}_{\text{divisor}} \underbrace{+}_{\text{plus}} \underbrace{0}_{\text{remainder.}}$$

Since the remainder is 0, $f(4) = 0$. This means that $x - 4$ is a factor of $x^4 + x^3 - 17x^2 - 20x + 32$. This illustrates the **Factor Theorem**, which is a special case of the Remainder Theorem.

Key Concept *Factor Theorem*

The binomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

Suppose you wanted to find the factors of $x^3 - 3x^2 - 6x + 8$. One approach is to graph the related function, $f(x) = x^3 - 3x^2 - 6x + 8$. From the graph at the right, you can see that the graph of $f(x)$ crosses the x-axis at −2, 1, and 4. These are the zeros of the function. Using these zeros and the Zero Product Property, we can express the polynomial in factored form.



$$f(x) = [x - (-2)](x - 1)(x - 4)$$
$$= (x + 2)(x - 1)(x - 4)$$

This method of factoring a polynomial has its limitations. Most polynomial functions are not easily graphed and once graphed, the exact zeros are often difficult to determine.

The Factor Theorem can help you find all factors of a polynomial.

Example 2 Use the Factor Theorem

Show that $x + 3$ is a factor of $x^3 + 6x^2 - x - 30$. Then find the remaining factors of the polynomial.

The binomial $x + 3$ is a factor of the polynomial if -3 is a zero of the related polynomial function. Use the Factor Theorem and synthetic division.

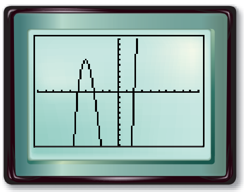
−3	1	6	−1	−30
		−3	−9	30
	1	3	−10	0

Since the remainder is 0, $x + 3$ is a factor of the polynomial. The polynomial $x^3 + 6x^2 - x - 30$ can be factored as $(x + 3)(x^2 + 3x - 10)$. The polynomial $x^2 + 3x - 10$ is the depressed polynomial. Check to see if this polynomial can be factored.

$x^2 + 3x - 10 = (x - 2)(x + 5)$ Factor the trinomial.

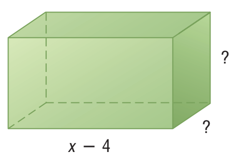
So, $x^3 + 6x^2 - x - 30 = (x + 3)(x - 2)(x + 5)$.

CHECK You can see that the graph of the related function $f(x) = x^3 + 6x^2 - x - 30$ crosses the x-axis at -5, -3, and 2. Thus, $f(x) = [x - (-5)][x - (-3)](x - 2)$. ✓



Example 3 Find All Factors of a Polynomial

GEOMETRY **The volume of the rectangular prism is given by $V(x) = x^3 + 3x^2 - 36x + 32$. Find the missing measures.**



The volume of a rectangular prism is $\ell \times w \times h$.

You know that one measure is $x - 4$, so $x - 4$ is a factor of $V(x)$.

4	1	3	−36	32
		4	28	−32
	1	7	−8	0

The quotient is $x^2 + 7x - 8$. Use this to factor $V(x)$.

$V(x) = x^3 + 3x^2 - 36x + 32$ Volume function

$= (x - 4)(x^2 + 7x - 8)$ Factor.

$= (x - 4)(x + 8)(x - 1)$ Factor the trinomial $x^2 + 7x - 8$.

So, the missing measures of the prism are $x + 8$ and $x - 1$.

Study Tip

Factoring
The factors of a polynomial do not have to be binomials. For example, the factors of $x^3 + x^2 - x + 15$ are $x + 3$ and $x^2 - 2x + 5$.

www.algebra2.com/extra_examples

FACTORS OF POLYNOMIALS

Teaching Tip Remind students that not all polynomials can be factored. Emphasize that the factors indicate where the graph of the function crosses the x-axis. If the graph of a polynomial function has no x-intercepts, then the polynomial cannot be factored. Students can graph the function $f(x) = x^4 - x^3 - x^2 + 2$ to see an example of a polynomial that cannot be factored.

In-Class Examples

Power Point®

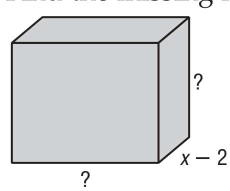
2 Show that $x - 3$ is a factor of $x^3 + 4x^2 - 15x - 18$. Then find the remaining factors of the polynomial.

3	1	4	−15	−18
		3	21	18
	1	7	6	0

So, $x^3 + 4x^2 - 15x - 18 = (x - 3)(x^2 + 7x + 6)$. Since $x^2 + 7x + 6 = (x + 1)(x + 6)$, $x^3 + 4x^2 - 15x - 18 = (x - 3)(x + 1)(x + 6)$.

Teaching Tip Point out that the Factor Theorem does not say anything about which numbers to try. Techniques for identifying potential factors will be introduced later in the chapter.

3 **GEOMETRY** The volume of the rectangular prism is given by $V(x) = x^3 + 7x^2 + 2x - 40$. Find the missing measures.



The missing measures of the rectangular prism are $x + 4$ and $x + 5$.

DAILY INTERVENTION

Differentiated Instruction

Intrapersonal Have students describe two or three things about this lesson that they found difficult to understand. Then have them address each item by writing an explanation that will help them review the material later or that they can refer to if they become confused in later lessons. Suggest that students add these notes to their Study Notebooks.

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 7.
- write a note explaining how to determine whether a given binomial is a factor of a given polynomial.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective
- **Synthetic Substitution:** 13–20
- **Factors of Polynomials:** 21–30

Odd/Even Assignments
Exercises 13–30 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 13–31 odd, 37, 38, 41, 46–62

Average: 13–35 odd, 37–41, 46–62

Advanced: 14–36 even, 39–59 (optional: 60–62)

Answers

25. $x + 3,\ x - \frac{1}{2}$ or $2x - 1$
26. $x - 1,\ x + \frac{4}{3}$ or $3x + 4$
27. $x + 7,\ x - 4$
28. $x - 1,\ x + 6$
29. $x - 1,\ x^2 + 2x + 3$
30. $2x - 3,\ 2x + 3,\ 4x^2 + 9$

Check for Understanding

Concept Check

1. **OPEN ENDED** Give an example of a polynomial function that has a remainder of 5 when divided by $x - 4$. **Sample answer: $f(x) = x^2 - 2x - 3$**
2. **State** the degree of the depressed polynomial that is the result of dividing $x^5 + 3x^4 - 16x - 48$ by one of its first-degree binomial factors. **4**
3. **Write** the dividend, divisor, quotient, and remainder represented by the synthetic division at the right.

−2	1	0	6	32
		−2	4	−20
	1	−2	10	12

3. dividend: $x^3 + 6x + 32$; divisor: $x + 2$; quotient: $x^2 - 2x + 10$; remainder: 12

Guided Practice

GUIDED PRACTICE KEY

Exercises	Examples
4, 5	1
6–9	2
10	3

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function.

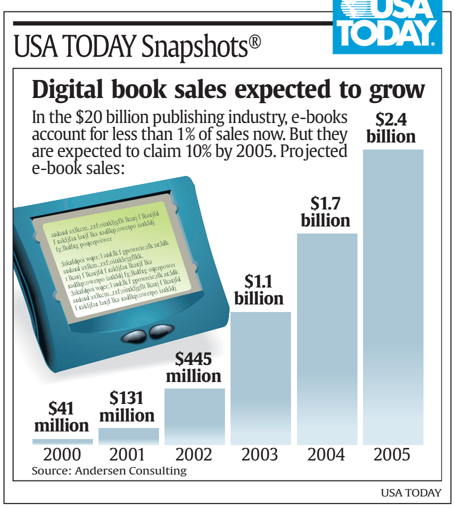
4. $f(x) = x^3 - 2x^2 - x + 1$ **7, −91**
5. $f(x) = 5x^4 - 6x^2 + 2$ **353, 1186**

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

6. $x^3 - x^2 - 5x - 3;\ x + 1$ **$x + 1,\ x - 3$**
7. $x^3 - 3x + 2;\ x - 1$ **$x - 1,\ x + 2$**
8. $6x^3 - 25x^2 + 2x + 8;\ 3x - 2$ **$2x + 1,\ x - 4$**
9. $x^4 + 2x^3 - 8x - 16;\ x + 2$ **$x - 2,\ x^2 + 2x + 4$**

Application

For Exercises 10–12, use the graph at the right. The projected sales of e-books can be modeled by the function $S(x) = -17x^3 + 200x^2 - 113x + 44$, where x is the number of years since 2000. **10. $2.894 billion**



10. Use synthetic substitution to estimate the sales for 2006.
11. Evaluate $S(6)$. **$2.894 billion**
12. Which method—synthetic division or direct substitution—do you prefer to use to evaluate polynomials? Explain your answer. **Sample answer: Direct substitution, because it can be done quickly with a calculator.**

★ **indicates increased difficulty**

Practice and Apply

Homework Help

For Exercises	See Examples
13–20	1
21–36	2
37–44	3

Extra Practice
See page 843.

Use synthetic substitution to find $g(3)$ and $g(-4)$ for each function. **18. 267, 680**

13. $g(x) = x^2 - 8x + 6$ **−9, 54**
14. $g(x) = x^3 + 2x^2 - 3x + 1$ **37, −19**
15. $g(x) = x^3 - 5x + 2$ **14, −42**
16. $g(x) = x^4 - 6x - 8$ **55, 272**
17. $g(x) = 2x^3 - 8x^2 - 2x + 5$ **−19, −243**
18. $g(x) = 3x^4 + x^3 - 2x^2 + x + 12$
19. $g(x) = x^5 + 8x^3 + 2x - 15$ **450, −1559**
20. $g(x) = x^6 - 4x^4 + 3x^2 - 10$ **422, 3110**

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. **24. $x - 3,\ x - 1$**

21. $x^3 + 2x^2 - x - 2;\ x - 1$ **$x + 1,\ x + 2$**
22. $x^3 - x^2 - 10x - 8;\ x + 1$ **$x - 4,\ x + 2$**
23. $x^3 + x^2 - 16x - 16;\ x + 4$ **$x - 4,\ x + 1$**
24. $x^3 - 6x^2 + 11x - 6;\ x - 2$

Online Lesson Plans

USA TODAY Education's Online site offers resources and interactive features connected to each day's newspaper. *Experience TODAY*, USA TODAY's daily lesson plan, is available on the site and delivered daily to subscribers. This plan provides instruction for integrating USA TODAY graphics and key editorial features into your mathematics classroom. Log on to www.education.usatoday.com.

368 Chapter 7 Polynomial Functions

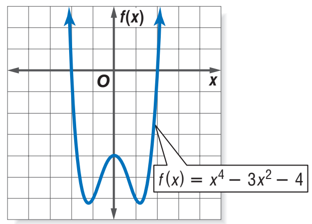
25–30. See margin.

25. $2x^3 - 5x^2 - 28x + 15; x - 5$
26. $3x^3 + 10x^2 - x - 12; x + 3$
27. $2x^3 + 7x^2 - 53x - 28; 2x + 1$
28. $2x^3 + 17x^2 + 23x - 42; 2x + 7$
29. $x^4 + 2x^3 + 2x^2 - 2x - 3; x + 1$
30. $16x^5 - 32x^4 - 81x + 162; x - 2$

WebQuest

Changes in world population can be modeled by a polynomial function. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

31. Use the graph of the polynomial function at the right to determine at least one binomial factor of the polynomial. Then find all the factors of the polynomial. $x - 2, x + 2, x^2 + 1$



32. Use synthetic substitution to show that $x - 8$ is a factor of $x^3 - 4x^2 - 29x - 24$. Then find any remaining factors. See pp. 407A–407H; $(x + 3)(x + 1)$.

Find values of k so that each remainder is 3.

★ **33.** $(x^2 - x + k) \div (x - 1)$ 3
★ **34.** $(x^2 + kx - 17) \div (x - 2)$ 8
★ **35.** $(x^2 + 5x + 7) \div (x + k)$ 1, 4
★ **36.** $(x^3 + 4x^2 + x + k) \div (x + 2)$ −3

ENGINEERING **For Exercises 37 and 38, use the following information.**
When a certain type of plastic is cut into sections, the length of each section determines its strength. The function $f(x) = x^4 - 14x^3 + 69x^2 - 140x + 100$ can describe the relative strength of a section of length x feet. Sections of plastic x feet long, where $f(x) = 0$, are extremely weak. After testing the plastic, engineers discovered that sections 5 feet long were extremely weak.

37. Show that $x - 5$ is a factor of the polynomial function. See pp. 407A–407H.

38. Are there other lengths of plastic that are extremely weak? Explain your reasoning. Yes, 2-ft lengths; the binomial $x - 2$ is a factor of the polynomial since $f(2) = 0$.

More About

Architecture

The Sears Tower elevators operate as fast as 1600 feet per minute—among the fastest in the world.

Source: www.the-skydeck.com

ARCHITECTURE **For Exercises 39 and 40, use the following information.**
Elevators traveling from one floor to the next do not travel at a constant speed. Suppose the speed of an elevator in feet per second is given by the function $f(t) = -0.5t^4 + 4t^3 - 12t^2 + 16t$, where t is the time in seconds.

39. Find the speed of the elevator at 1, 2, and 3 seconds. 7.5 ft/s, 8 ft/s, 7.5 ft/s

40. It takes 4 seconds for the elevator to go from one floor to the next. Use synthetic substitution to find $f(4)$. Explain what this means. 0; The elevator is stopped.

41. **CRITICAL THINKING** Consider the polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a + b + c + d + e = 0$. Show that this polynomial is divisible by $x - 1$. See margin.

PERSONAL FINANCE **For Exercises 42–45, use the following information.**
Zach has purchased some home theater equipment for $2000, which he is financing through the store. He plans to pay $340 per month and wants to have the balance paid off after six months. The formula $B(x) = 2000x^6 - 340(x^5 + x^4 + x^3 + x^2 + x + 1)$ represents his balance after six months if x represents 1 plus the monthly interest rate (expressed as a decimal).

42. Find his balance after 6 months if the annual interest rate is 12%. (*Hint*: The monthly interest rate is the annual rate divided by 12, so $x = 1.01$.) $31.36

43. Find his balance after 6 months if the annual interest rate is 9.6%. $16.70

44. How would the formula change if Zach wanted to pay the balance in five months? $B(x) = 2000x^5 - 340(x^4 + x^3 + x^2 + x + 1)$

45. Suppose he finances his purchase at 10.8% and plans to pay $410 every month. Will his balance be paid in full after five months? No, he will still owe $4.40.

www.algebra2.com/self_check_quiz

Answer

41. By the Remainder Theorem, the remainder when $f(x)$ is divided by $x - 1$ is equivalent to $f(1)$, or $a + b + c + d + e$. Since $a + b + c + d + e = 0$, the remainder when $f(x)$ is divided by $x - 1$ is 0. Therefore, $x - 1$ is a factor of $f(x)$.

Study Guide and Intervention, p. 393 (shown) and p. 394

Synthetic Substitution

Remainder Theorem	The remainder, when you divide the polynomial $f(x)$ by $(x - a)$, is the constant $f(a)$. $f(x) = q(x) \cdot (x - a) + f(a)$, where $q(x)$ is a polynomial with degree one less than the degree of $f(x)$.

Example 1 **If $f(x) = 3x^4 + 2x^3 - 5x^2 + x - 2$, find $f(-2)$.**

Method 1 Synthetic Substitution
By the Remainder Theorem, $f(-2)$ should be the remainder when you divide the polynomial by $x + 2$.

−2	3	2	−5	1	−2
		−6	8	−6	10
	3	−4	3	−5	8

The remainder is 8, so $f(-2) = 8$.

Method 2 Direct Substitution
Replace x with −2.
$f(x) = 3x^4 + 2x^3 - 5x^2 + x - 2$
$f(-2) = 3(-2)^4 + 2(-2)^3 - 5(-2)^2 + (-2) - 2$
$= 48 - 16 - 20 - 2 - 2$ or 8
So $f(-2) = 8$.

Example 2 **If $f(x) = 5x^3 + 2x - 1$, find $f(3)$.**
Again, by the Remainder Theorem, $f(3)$ should be the remainder when you divide the polynomial by $x - 3$.

3	5	0	2	−1
		15	45	141
	5	15	47	140

The remainder is 140, so $f(3) = 140$.

Exercises

Use synthetic substitution to find $f(-5)$ and $f\left(\frac{1}{2}\right)$ for each function.

1. $f(x) = -3x^2 + 5x - 1$ $-101; \frac{3}{4}$
2. $f(x) = 4x^2 + 6x - 7$ 63; −3
3. $f(x) = -x^3 + 3x^2 - 5$ $195; -\frac{35}{8}$
4. $f(x) = x^4 + 11x^2 - 1$ $899; \frac{29}{16}$

Use synthetic substitution to find $f(4)$ and $f(-3)$ for each function.

5. $f(x) = 2x^3 + x^2 - 5x + 3$ 127; −27
6. $f(x) = 3x^3 - 4x + 2$ 178; −67
7. $f(x) = 5x^3 - 4x^2 + 2$ 258; −169
8. $f(x) = 2x^4 - 4x^3 + 3x^2 + x - 6$ 302; 288
9. $f(x) = 5x^4 + 3x^3 - 4x^2 - 2x + 4$ 1404; 298
10. $f(x) = 3x^4 - 2x^3 - x^2 + 2x - 5$ 627; 277
11. $f(x) = 2x^4 - 4x^3 - x^2 - 6x + 3$ 219; 282
12. $f(x) = 4x^4 - 4x^3 + 3x^2 - 2x - 3$ 805; 462

Skills Practice, p. 395 and Practice, p. 396 (shown)

Use synthetic substitution to find $f(-3)$ and $f(4)$ for each function.

1. $f(x) = x^2 + 2x + 3$ 6, 27
2. $f(x) = x^2 - 5x + 10$ 34, 6
3. $f(x) = x^2 - 5x - 4$ 20, −8
4. $f(x) = x^3 - x^2 - 2x + 3$ −27, 43
5. $f(x) = x^3 + 2x^2 + 5$ −4, 101
6. $f(x) = x^3 - 6x^2 + 2x$ −87, −24
7. $f(x) = x^3 - 2x^2 - 2x + 8$ −31, 32
8. $f(x) = x^3 - x^2 + 4x - 4$ −52, 60
9. $f(x) = x^3 + 3x^2 + 2x - 50$ −56, 70
10. $f(x) = x^4 + x^3 - 3x^2 - x + 12$ 42, 280
11. $f(x) = x^4 - 2x^2 - x + 7$ 73, 227
12. $f(x) = 2x^4 - 3x^3 + 4x^2 - 2x + 1$ 286, 377
13. $f(x) = 2x^4 - x^3 + 2x^2 - 26$ 181, 454
14. $f(x) = 3x^4 - 4x^3 + 3x^2 - 5x - 3$ 390, 537
15. $f(x) = x^5 + 7x^3 - 4x - 10$ −430, 1446
16. $f(x) = x^6 + 2x^5 - x^4 + x^3 - 9x^2 + 20$ 74, 5828

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

17. $x^3 + 3x^2 - 6x - 8; x - 2$ $x + 1, x + 4$
18. $x^3 + 7x^2 + 7x - 15; x - 1$ $x + 3, x + 5$
19. $x^3 - 9x^2 + 27x - 27; x - 3$ $x - 3, x - 3$
20. $x^3 - x^2 - 8x + 12; x + 3$ $x - 2, x - 2$
21. $x^3 + 5x^2 - 2x - 24; x - 2$ $x + 3, x + 4$
22. $x^3 - x^2 - 14x + 24; x + 4$ $x - 3, x - 2$
23. $3x^3 - 4x^2 - 17x + 6; x + 2$ $x - 3, 3x - 1$
24. $4x^3 - 12x^2 - x + 3; x - 3$ $2x - 1, 2x + 1$
25. $18x^3 + 9x^2 - 2x - 1; 2x + 1$ $3x + 1, 3x - 1$
26. $6x^3 + 5x^2 - 3x - 2; 3x - 2$ $2x + 1, x + 1$
27. $x^5 + x^4 - 5x^3 - 5x^2 + 4x + 4; x + 1$ $x - 1, x + 1, x - 2, x + 2$
28. $x^5 - 2x^4 + 4x^3 - 8x^2 - 5x + 10; x - 2$ $x - 1, x + 1, x^2 + 5$

29. POPULATION The projected population in thousands for a city over the next several years can be estimated by the function $P(x) = x^3 + 2x^2 - 8x + 520$, where x is the number of years since 2000. Use synthetic substitution to estimate the population for 2005. 655,000

30. VOLUME The volume of water in a rectangular swimming pool can be modeled by the polynomial $2x^3 - 9x^2 + 7x + 6$. If the depth of the pool is given by the polynomial $2x + 1$, what polynomials express the length and width of the pool? $x - 3$ and $x - 2$

Reading to Learn Mathematics, p. 397 ELL

Pre-Activity **How can you use the Remainder Theorem to evaluate polynomials?**

Read the introduction to Lesson 7-4 at the top of page 365 in your textbook.

Show how you would use the model in the introduction to estimate the number of international travelers (in millions) to the United States in the year 2000. (Show how you would substitute numbers, but do not actually calculate the result.)
Sample answer: $0.02(14)^3 - 0.6(14)^2 + 6(14) + 25.9$

Reading the Lesson

1. Consider the following synthetic division.

1	3	2	−6	4
		3	5	−1
	3	5	−1	3

a. Using the division symbol ÷, write the division problem that is represented by this synthetic division. (Do not include the answer.) $(3x^3 + 2x^2 - 6x + 4) \div (x - 1)$

b. Identify each of the following for this division.
dividend $3x^3 + 2x^2 - 6x + 4$ divisor $x - 1$
quotient $3x^3 + 5x - 1$ remainder 3

c. If $f(x) = 3x^3 + 2x^2 - 6x + 4$, what is $f(1)$? 3

2. Consider the following synthetic division.

−3	1	0	0	27
		−3	9	−27
	1	−3	9	0

a. This division shows that $x + 3$ is a factor of $x^3 + 27$.

b. The division shows that −3 is a zero of the polynomial function $f(x) = x^3 + 27$.

c. The division shows that the point (−3, 0) is on the graph of the polynomial function $f(x) = x^3 + 27$.

Helping You Remember

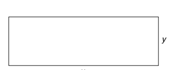
3. Think of a mnemonic for remembering the sentence, "Dividend equals quotient times divisor plus remainder."
Sample answer: Definitely every quiet teacher deserves proper rewards.

Enrichment, p. 398

Using Maximum Values

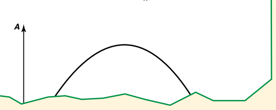
Many times maximum solutions are needed for different situations. For instance, what is the area of the largest rectangular field that can be enclosed with 2000 feet of fencing?

Let x and y denote the length and width of the field, respectively.



Perimeter: $2x + 2y = 2000 \rightarrow y = 1000 - x$
Area: $A = xy = x(1000 - x) = -x^2 + 1000x$

This problem is equivalent to finding the highest point on the graph of $A(x) = -x^2 + 1000x$ shown on the right.



Complete the square for $-x^2 + 1000x$.

$A = -(x^2 - 1000x + 500^2) + 500^2$

4 Assess

Open-Ended Assessment

Speaking Ask students to offer a verbal comparison of the use of synthetic substitution and the use of direct substitution to determine factors of a polynomial.

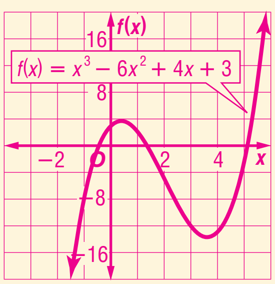
Getting Ready for Lesson 7-5

PREREQUISITE SKILL In Lesson 7-5, students will find the zeros of polynomial functions by using the Quadratic Formula. Use Exercises 60–62 to determine your students' familiarity with using the Quadratic Formula.

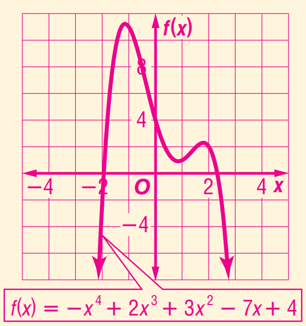
Answers

46. Using the Remainder Theorem you can evaluate a polynomial for a value of a by dividing the polynomial by $x - a$ using synthetic division. Answers should include the following.
   - It is easier to use the Remainder Theorem when you have polynomials of degree 2 and lower or when you have access to a calculator.
   - The estimated number of international traveler to the U.S. in 2006 is 65.9 million.

52.



53.



46. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

How can you use the Remainder Theorem to evaluate polynomials?

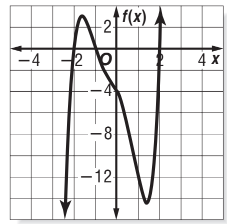
Include the following items in your answer:
- an explanation of when it is easier to use the Remainder Theorem to evaluate a polynomial rather than substitution, and
- evaluate the expression for the number of international travelers to the U.S. for $x = 20$.

Standardized Test Practice

47. Determine the zeros of the function $f(x) = x^2 + 7x + 12$ by factoring. **D**

Ⓐ 7, 12 Ⓑ 3, 4 Ⓒ −5, 5 Ⓓ −4, −3

48. **SHORT RESPONSE** Using the graph of the polynomial function at the right, find all the factors of the polynomial $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$.
$x - 2$, $x + 2$, $x + 1$, $x^2 + 1$



Maintain Your Skills

Mixed Review

Write each expression in quadratic form, if possible. *(Lesson 7-3)*

49. $x^4 - 8x^2 + 4$ **$(x^2)^2 - 8(x^2) + 4$**
50. $9d^6 + 5d^3 - 2$ **$9(d^3)^2 + 5(d^3) - 2$**
51. $r^4 - 5r^3 + 18r$ **not possible**

Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. *(Lesson 7-2)* **52–53. See margin for graphs.**

52. $f(x) = x^3 - 6x^2 + 4x + 3$
53. $f(x) = -x^4 + 2x^3 + 3x^2 - 7x + 4$

52. Sample answer: rel. max. at $x = 0.5$, rel. min. at $x = 3.5$
53. Sample answer: rel. max. at $x = -1$ and $x = 1.5$, rel. min. at $x = 1$

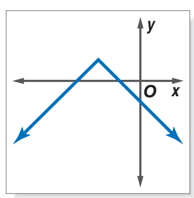
54. **PHYSICS** A model airplane is fixed on a string so that it flies around in a circle. The formula $F_c = m\left(\frac{4\pi^2 r}{T^2}\right)$ describes the force required to keep the airplane going in a circle, where m represents the mass of the plane, r represents the radius of the circle, and T represents the time for a revolution. Solve this formula for T. Write in simplest radical form. *(Lesson 5-8)* **$T = \frac{2\pi\sqrt{mrF_c}}{F_c}$**

Solve each matrix equation. *(Lesson 4-1)*

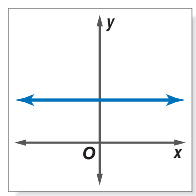
55. $\begin{bmatrix} 7x \\ 12 \end{bmatrix} = \begin{bmatrix} 28 \\ -6y \end{bmatrix}$ **(4, −2)**
56. $\begin{bmatrix} 5a + 2b \\ a - 7b \end{bmatrix} = \begin{bmatrix} -17 \\ 4 \end{bmatrix}$ **(−3, −1)**

Identify each function as S for step, C for constant, A for absolute value, or P for piecewise. *(Lesson 2-6)*

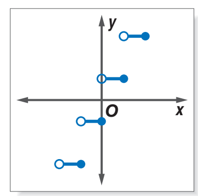
57. **A**



58. **C**



59. **S**



Getting Ready for the Next Lesson

PREREQUISITE SKILL Find the exact solutions of each equation by using the Quadratic Formula. *(For review of the **Quadratic Formula**, see Lesson 6-5.)*

60. $x^2 + 7x + 8 = 0$ **$\frac{-7 \pm \sqrt{17}}{2}$**
61. $3x^2 - 9x + 2 = 0$ **$\frac{9 \pm \sqrt{57}}{6}$**
62. $2x^2 + 3x + 2 = 0$ **$\frac{-3 \pm i\sqrt{7}}{4}$**

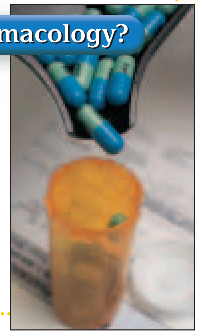
7-5 Roots and Zeros

What You'll Learn

- Determine the number and type of roots for a polynomial equation.
- Find the zeros of a polynomial function.

How can the roots of an equation be used in pharmacology?

When doctors prescribe medication, they give patients instructions as to how much to take and how often it should be taken. The amount of medication in your body varies with time. Suppose the equation $M(t) = 0.5t^4 + 3.5t^3 - 100t^2 + 350t$ models the number of milligrams of a certain medication in the bloodstream t hours after it has been taken. The doctor can use the roots of this equation to determine how often the patient should take the medication to maintain a certain concentration in the body.



TYPES OF ROOTS You have already learned that a zero of a function $f(x)$ is any value c such that $f(c) = 0$. When the function is graphed, the real zeros of the function are the x-intercepts of the graph.

Concept Summary — *Zeros, Factors, and Roots*

Let $f(x) = a_nx^n + \ldots + a_1x + a_0$ be a polynomial function. Then

- c is a zero of the polynomial function $f(x)$,
- $x - c$ is a factor of the polynomial $f(x)$, and
- c is a root or solution of the polynomial equation $f(x) = 0$.

In addition, if c is a real number, then $(c, 0)$ is an intercept of the graph of $f(x)$.

Study Tip

Look Back
For review of **complex numbers**, see Lesson 5-9.

When you solve a polynomial equation with degree greater than zero, it may have one or more real roots, or no real roots (the roots are imaginary numbers). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero will have at least one root in the set of complex numbers. This is the **Fundamental Theorem of Algebra**.

Key Concept — *Fundamental Theorem of Algebra*

Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.

Example 1 *Determine Number and Type of Roots*

Solve each equation. State the number and type of roots.

a. $x + 3 = 0$

$x + 3 = 0$ Original equation

$x = -3$ Subtract 3 from each side.

This equation has exactly one real root, -3.

7-5 Lesson Notes

1 Focus

5-Minute Check Transparency 7-5 Use as a quiz or review of Lesson 7-4.

Mathematical Background notes are available for this lesson on p. 344D.

Building on Prior Knowledge

In Chapter 6, students learned several methods for finding the roots of quadratic equations. In this lesson, students will incorporate those techniques into finding roots of polynomial equations.

How can the roots of an equation be used in pharmacology?

Ask students:

- Would the given equation be valid for any value of t? Explain. **No; the value of t must be positive because the number of hours cannot be negative. Also, once the number of milligrams, $M(t)$, reaches zero as the value of t increases, any greater values of t will be meaningless also.**
- What would a root of this equation tell the doctor? **There is no more medication in the patient's bloodstream.**

Resource Manager

Workbook and Reproducible Masters

Chapter 7 Resource Masters

- Study Guide and Intervention, pp. 399–400
- Skills Practice, p. 401
- Practice, p. 402
- Reading to Learn Mathematics, p. 403
- Enrichment, p. 404
- Assessment, pp. 443, 445

Transparencies

5-Minute Check Transparency 7-5
Answer Key Transparencies

Technology

Alge2PASS: Tutorial Plus, Lesson 14
Interactive Chalkboard

2 Teach

TYPES OF ROOTS

In-Class Example Power Point®

1 Solve each equation. State the number and type of roots.

a. $a - 10 = 0$ This equation has exactly one real root, 10.

b. $x^2 + 2x - 48 = 0$ This equation has two real roots, 6 and −8.

c. $3a^3 + 18a = 0$ This equation has one real root, 0, and two imaginary roots $i\sqrt{6}$ and $-i\sqrt{6}$.

d. $y^4 - 16 = 0$ This equation has two real roots, 2 and −2, and two imaginary roots, $2i$ and $-2i$.

Study Tip

Reading Math
In addition to double roots, equations can have triple or quadruple roots. In general, these roots are referred to as *repeated roots*.

b. $x^2 - 8x + 16 = 0$

$x^2 - 8x + 16 = 0$ Original equation

$(x - 4)^2 = 0$ Factor the left side as a perfect square trinomial.

$x = 4$ Solve for x using the Square Root Property.

Since $x - 4$ is twice a factor of $x^2 - 8x + 16$, 4 is a double root. So this equation has two real roots, 4 and 4.

c. $x^3 + 2x = 0$

$x^3 + 2x = 0$ Original equation

$x(x^2 + 2) = 0$ Factor out the GCF.

Use the Zero Product Property.

$x = 0$ or $x^2 + 2 = 0$

$x^2 = -2$ Subtract two from each side.

$x = \pm\sqrt{-2}$ or $\pm i\sqrt{2}$ Square Root Property

This equation has one real root, 0, and two imaginary roots, $i\sqrt{2}$ and $-i\sqrt{2}$.

d. $x^4 - 1 = 0$

$$x^4 - 1 = 0$$
$$(x^2 + 1)(x^2 - 1) = 0$$
$$(x^2 + 1)(x + 1)(x - 1) = 0$$
$$x^2 + 1 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x^2 = -1 \qquad x = -1 \qquad x = 1$$
$$x = \pm\sqrt{-1} \text{ or } \pm i$$

This equation has two real roots, 1 and −1, and two imaginary roots, i and $-i$.

Compare the degree of each equation and the number of roots of each equation in Example 1. The following corollary of the Fundamental Theorem of Algebra is an even more powerful tool for problem solving.

Key Concept *Corollary*

A polynomial equation of the form $P(x) = 0$ of degree n with complex coefficients has exactly n roots in the set of complex numbers.

Similarly, a polynomial function of nth degree has exactly n zeros.

French mathematician René Descartes made more discoveries about zeros of polynomial functions. His rule of signs is given below.

Key Concept *Descartes' Rule of Signs*

If $P(x)$ is a polynomial with real coefficients whose terms are arranged in descending powers of the variable,

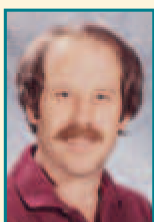
- the number of positive real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and
- the number of negative real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this number by an even number.

More About...

Descartes

René Descartes (1596–1650) was a French mathematician and philosopher. One of his best-known quotations comes from his *Discourse on Method*: "I think, therefore I am."

Source: *A History of Mathematics*



Teacher to Teacher

Warren Zarrell — James Monroe H.S., North Hills, CA

"My students have difficulty finding $p(-x)$, as found in Example 2. I tell them to change the sign of every odd degree term in the polynomial."

372 Chapter 7 Polynomial Functions

Example 2 Find Numbers of Positive and Negative Zeros

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$.

Since $p(x)$ has degree 5, it has five zeros. However, some of them may be imaginary. Use Descartes' Rule of Signs to determine the number and type of real zeros. Count the number of changes in sign for the coefficients of $p(x)$.

$$p(x) = x^5 \quad - \quad 6x^4 \quad - \quad 3x^3 \quad + \quad 7x^2 \quad - \quad 8x \quad + \quad 1$$

yes (+ to −), no (− to −), yes (− to +), yes (+ to −), yes (− to +)

Since there are 4 sign changes, there are 4, 2, or 0 positive real zeros.

Find $p(-x)$ and count the number of changes in signs for its coefficients.

$$p(x) = (-x)^5 - 6(-x)^4 - 3(-x)^3 + 7(-x)^2 - 8(-x) + 1$$
$$= -x^5 - 6x^4 + 3x^3 + 7x^2 + 8x + 1$$

no (+ to −), yes (− to +), no (+ to +), no (+ to +), no (+ to +)

Since there is 1 sign change, there is exactly 1 negative real zero.

Thus, the function $p(x)$ has either 4, 2, or 0 positive real zeros and exactly 1 negative real zero. Make a chart of the possible combinations of real and imaginary zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros	Total Number of Zeros
4	1	0	4 + 1 + 0 = 5
2	1	2	2 + 1 + 2 = 5
0	1	4	0 + 1 + 4 = 5

Study Tip

Zero at the Origin

Recall that the number 0 has no sign. Therefore, if 0 is a zero of a function, the sum of the number of positive real zeros, negative real zeros, and imaginary zeros is reduced by how many times 0 is a zero of the function.

FIND ZEROS

We can find all of the zeros of a function using some of the strategies you have already learned.

Example 3 Use Synthetic Substitution to Find Zeros

Find all of the zeros of $f(x) = x^3 - 4x^2 + 6x - 4$.

Since $f(x)$ has degree 3, the function has three zeros. To determine the possible number and type of real zeros, examine the number of sign changes for $f(x)$ and $f(-x)$.

$f(x) = x^3 - 4x^2 + 6x - 4$ yes, yes, yes

$f(-x) = -x^3 - 4x^2 - 6x - 4$ no, no, no

Since there are 3 sign changes for the coefficients of $f(x)$, the function has 3 or 1 positive real zeros. Since there are no sign changes for the coefficient of $f(-x)$, $f(x)$ has no negative real zeros. Thus, $f(x)$ has either 3 real zeros, or 1 real zero and 2 imaginary zeros.

To find these zeros, first list some possibilities and then eliminate those that are not zeros. Since none of the zeros are negative and evaluating the function for 0 results in −4, begin by evaluating $f(x)$ for positive integral values from 1 to 4. You can use a shortened form of synthetic substitution to find $f(a)$ for several values of a.

www.algebra2.com/extra_examples

In-Class Example (Power Point®)

Teaching Tip Point out to students the method for determining the number of imaginary zeros for a polynomial function. In Example 2, the polynomial has degree 5, so it has a maximum of 5 real zeros. You find the numbers of positive and negative real zeros, and subtract the sum of these two numbers from 5 to find the number of imaginary zeros. Remind students that imaginary zeros come in conjugate pairs, so the number of imaginary zeros must be an even number.

2 State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $p(x) = -x^6 + 4x^3 - 2x^2 - x - 1$. **The function has either 2 or 0 positive real zeros, 2 or 0 negative real zeros, and 6, 4, or 2 imaginary zeros.**

FIND ZEROS

In-Class Example (Power Point®)

3 Find all of the zeros of $f(x) = x^3 - x^2 + 2x + 4$. **The function has one real zero at $x = -1$, and two imaginary zeros at $x = 1 + i\sqrt{3}$ and $x = 1 - i\sqrt{3}$.**

DAILY INTERVENTION — Differentiated Instruction

Kinesthetic As you work Example 3 and Guided Practice Exercises 8–11 in class, provide each student with approximately 20 slips of paper. As students begin the process of finding the zeros of each polynomial function, have them first determine from the degree of the polynomial the number of zeros they need to find. Students should then count off slips of paper, one for each zero. As students work the problem and find the zeros, they should record the information about each zero (positive, negative, imaginary) on one of the slips of paper.

In-Class Example

Power Point®

4 Write a polynomial function of least degree with integral coefficients whose zeros include 4 and $4 - i$.

$f(x) = x^3 - 12x^2 + 49x - 68$ is a polynomial function of least degree with integral coefficients whose zeros are 4, $4 - i$, and $4 + i$.

Study Tip

Finding Zeros
While direct substitution could be used to find each real zero of a polynomial, using synthetic substitution provides you with a depressed polynomial that can be used to find any imaginary zeros.

x	1	−4	6	−4
1	1	−3	3	−1
2	1	−2	2	**0**
3	1	−1	3	5
4	1	0	6	20

Each row in the table shows the coefficients of the depressed polynomial and the remainder.

From the table, we can see that one zero occurs at $x = 2$. Since the depressed polynomial of this zero, $x^2 - 2x + 2$, is quadratic, use the Quadratic Formula to find the roots of the related quadratic equation, $x^2 - 2x + 2 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

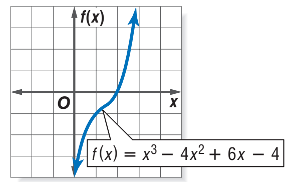
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$ Replace a with 1, b with −2, and c with 2.

$$= \frac{2 \pm \sqrt{-4}}{2}$$ Simplify.

$$= \frac{2 \pm 2i}{2}$$ $\sqrt{4} \times \sqrt{-1} = 2i$

$$= 1 \pm i$$ Simplify.

Thus, the function has one real zero at $x = 2$ and two imaginary zeros at $x = 1 + i$ and $x = 1 - i$. The graph of the function verifies that there is only one real zero.



In Chapter 6, you learned that solutions of a quadratic equation that contains imaginary numbers come in pairs. This applies to the zeros of polynomial functions as well. For any polynomial function, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the **Complex Conjugates Theorem**.

Key Concept — Complex Conjugates Theorem

Suppose a and b are real numbers with $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

Standardized Test Practice (A) (B) (C) (D)

Example 4 Use Zeros to Write a Polynomial Function

Short-Response Test Item

Write a polynomial function of least degree with integral coefficients whose zeros include 3 and $2 - i$.

Read the Test Item

- If $2 - i$ is a zero, then $2 + i$ is also a zero according to the Complex Conjugates Theorem. So, $x - 3$, $x - (2 - i)$, and $x - (2 + i)$ are factors of the polynomial function.

Solve the Test Item

- Write the polynomial function as a product of its factors.

  $f(x) = (x - 3)[x - (2 - i)][x - (2 + i)]$

Standardized Test Practice (A) (B) (C) (D)

Example 4 Ask students to indicate with a show of hands how many of them have made mistakes in mathematics exercises because they could not read their own handwriting. Stress that throughout this course, students must work using neat and careful handwriting. It is extremely easy to misread coefficients and exponents, or misread i as the number one. In addition, when answering short-response items on standardized tests, students must be aware that if their handwriting is illegible or difficult to read then their answers will not be graded or they may be penalized.

374 Chapter 7 Polynomial Functions

The Princeton Review

Test-Taking Tip

Knowing quadratic identities like the difference of two squares and perfect square trinomials can save you time on standardized tests.

- Multiply the factors to find the polynomial function.

$f(x) = (x - 3)[x - (2 - i)][x - (2 + i)]$	Write an equation.
$= (x - 3)[(x - 2) + i][(x - 2) - i]$	Regroup terms.
$= (x - 3)[(x - 2)^2 - i^2]$	Rewrite as the difference of two squares.
$= (x - 3)[x^2 - 4x + 4 - (-1)]$	Square $x - 2$ and replace i^2 with -1.
$= (x - 3)(x^2 - 4x + 5)$	Simplify.
$= x^3 - 4x^2 + 5x - 3x^2 + 12x - 15$	Multiply using the Distributive Property.
$= x^3 - 7x^2 + 17x - 15$	Combine like terms.

$f(x) = x^3 - 7x^2 + 17x - 15$ is a polynomial function of least degree with integral coefficients whose zeros are 3, $2 - i$, and $2 + i$.

1. Sample answer: $p(x) = x^3 - 6x^2 + x + 1$; $p(x)$ has either 2 or 0 positive real zeros, 1 negative real zero, and 2 or 0 imaginary zeros.

Check for Understanding

Concept Check

1. **OPEN ENDED** Write a polynomial function $p(x)$ whose coefficients have two sign changes. Then describe the nature of its zeros.
2. **Explain** why an odd-degree function must always have at least one real root. **See margin.**
3. **State** the least degree a polynomial equation with real coefficients can have if it has roots at $x = 5 + i$, $x = 3 - 2i$, and a double root at $x = 0$. **6**

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4, 5	1
6, 7	2
8–11	3
12	4

Solve each equation. State the number and type of roots. **5. −7, 0, and 3; 3 real**

4. $x^2 + 4 = 0$ **$\pm 2i$; 2 imaginary**
5. $x^3 + 4x^2 - 21x = 0$

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

6. $f(x) = 5x^3 + 8x^2 - 4x + 3$ **2 or 0; 1; 2 or 0**
7. $r(x) = x^5 - x^3 - x + 1$ **2 or 0; 1; 2 or 4**

Find all of the zeros of each function.

8. -4, $1 + 2i$, $1 - 2i$
9. 2, $1 + i$, $1 - i$

8. $p(x) = x^3 + 2x^2 - 3x + 20$
9. $f(x) = x^3 - 4x^2 + 6x - 4$
10. $v(x) = x^3 - 3x^2 + 4x - 12$ **$2i$, $-2i$, 3**
11. $f(x) = x^3 - 3x^2 + 9x + 13$ **$2 + 3i$, $2 - 3i$, -1**

Standardized Test Practice

12. **SHORT RESPONSE** Write a polynomial function of least degree with integral coefficients whose zeros include 2 and $4i$. **$f(x) = x^3 - 2x^2 + 16x - 32$**

★ indicates increased difficulty **15. 0, $3i$, $-3i$; 1 real, 2 imaginary 16. $3i$, $3i$, $-3i$, and $-3i$; 4 imaginary**

Practice and Apply

Homework Help

For Exercises	See Examples
13–18	1
19–24, 41	2
25–34, 44–48	3
35–40, 42, 43	4

Extra Practice

See page 843.

Solve each equation. State the number and type of roots.

13. $3x + 8 = 0$ **$-\frac{8}{3}$; 1 real**
14. $2x^2 - 5x + 12 = 0$ **$\frac{5 \pm i\sqrt{71}}{4}$; 2 imaginary**
15. $x^3 + 9x = 0$
16. $x^4 + 81 = 0$
17. $x^4 - 16 = 0$ **2, −2, $2i$, and $-2i$; 2 real, 2 imaginary**
18. $x^5 - 8x^3 + 16x = 0$ **−2, −2, 0, 2, and 2, 5 real**

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function. **19–24. See margin.**

19. $f(x) = x^3 - 6x^2 + 1$
20. $g(x) = 5x^3 + 8x^2 - 4x + 3$
21. $h(x) = 4x^3 - 6x^2 + 8x - 5$
22. $q(x) = x^4 + 5x^3 + 2x^2 - 7x - 9$
23. $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$
24. $f(x) = x^{10} - x^8 + x^6 - x^4 + x^2 - 1$

www.algebra2.com/self_check_quiz

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 7.
- summarize what they know so far about identifying types of zeros and finding some of them.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

About the Exercises...

Organization by Objective
- **Types of Roots:** 13–24
- **Find Zeros:** 25–40

Odd/Even Assignments
Exercises 13–40 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 13–41 odd, 49–70
Average: 13–41 odd, 42–45, 49–70
Advanced: 14–40 even, 44–66 (optional: 67–70)

Answers

2. An odd-degree function approaches positive infinity in one direction and negative infinity in the other direction, so the graph must cross the x-axis at least once, giving it at least one real root.
19. 2 or 0; 1; 2 or 0
20. 2 or 0; 1; 2 or 0
21. 3 or 1; 0; 2 or 0
22. 1; 3 or 1; 2 or 0
23. 4, 2, or 0; 1; 4, 2, or 0
24. 5, 3, or 1; 5, 3, or 1; 0, 2, 4, 6, or 8

DAILY INTERVENTION

Unlocking Misconceptions

Finding Zeros Students may incorrectly assume that they now know how to find all zeros. However, in this lesson they are using the guess-and-check technique to test possible zeros. In later lessons, students will learn further techniques for finding the zeros of polynomial functions.

Study Guide and Intervention, p. 399 (shown) and p. 400

Types of Roots The following statements are equivalent for any polynomial function $f(x)$.

- c is a zero of the polynomial function $f(x)$.
- $(x - c)$ is a factor of the polynomial $f(x)$.
- c is a root or solution of the polynomial equation $f(x) = 0$.

If c is real, then $(c, 0)$ is an intercept of the graph of $f(x)$.

Fundamental Theorem of Algebra	Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.
Corollary to the Fundamental Theorem of Algebras	A polynomial equation of the form $P(x) = 0$ of degree n with complex coefficients has exactly n roots in the set of complex numbers.
Descartes' Rule of Signs	If $P(x)$ is a polynomial with real coefficients whose terms are arranged in descending powers of the variable, • the number of positive real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and • the number of negative real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this number by an even number.

Example 1 **Solve the equation $6x^3 + 3x = 0$ and state the number and type of roots.**

$6x^3 + 3x = 0$

$3x(2x^2 + 1) = 0$

Use the Zero Product Property.

$3x = 0$ or $2x^2 + 1 = 0$

$x = 0$ or $2x^2 = -1$

$x = \pm\frac{i\sqrt{2}}{2}$

The equation has one real root, 0, and two imaginary roots, $\pm\frac{i\sqrt{2}}{2}$.

Example 2 **State the number of positive real zeros, negative real zeros, and imaginary zeros for $p(x) = 4x^4 - 3x^3 + x^2 + 2x - 5$.**

Since $p(x)$ has degree 4, it has 4 zeros.

Use Descartes' Rule of Signs to determine the number and type of real zeros. Since there are three sign changes, there are 3 or 1 positive real zeros.

Find $p(-x)$ and count the number of changes in sign for its coefficients.

$p(-x) = 4(-x)^4 - 3(-x)^3 + (-x)^2 + 2(-x) - 5$

$= 4x^4 + 3x^3 + x^2 - 2x - 5$

Since there is one sign change, there is exactly 1 negative real zero.

Exercises

Solve each equation and state the number and type of roots.

1. $x^2 + 4x - 21 = 0$ **3, −7; 2 real**
2. $2x^3 - 50x = 0$ **0, ±5; 3 real**
3. $12x^3 + 100x = 0$ **$0, \pm\frac{5i\sqrt{3}}{3}$; 1 real, 2 imaginary**

State the number of positive real zeros, negative real zeros, and imaginary zeros for each function.

4. $f(x) = 3x^3 + x^2 - 8x - 12$ **1; 2 or 0; 0 or 2**
5. $f(x) = 2x^4 - x^3 - 3x + 7$ **2 or 0; 0; 2 or 4**
6. $f(x) = 3x^5 - x^4 - x^3 + 6x^2 - 5$ **3 or 1; 2 or 0; 0, 2, or 4**

Skills Practice, p. 401 and Practice, p. 402 (shown)

Solve each equation. State the number and type of roots.

1. $-9x - 15 = 0$ **$-\frac{5}{3}$; 1 real**
2. $x^4 - 5x^2 + 4 = 0$ **−1, 1, −2, 2; 4 real**
3. $x^5 = 81x$ **0, −3, 3, −3*i*, 3*i*; 3 real, 2 imaginary**
4. $x^3 + x^2 - 3x - 3 = 0$ **$-1, -\sqrt{3}, \sqrt{3}$; 3 real**
5. $x^3 + 6x + 20 = 0$ **−2, 1 ± 3*i*; 1 real, 2 imaginary**
6. $x^4 - x^3 - x^2 - x - 2 = 0$ **2, −1, −*i*, *i*; 2 real, 2 imaginary**

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

7. $f(x) = 4x^3 - 2x^2 + x + 3$ **2 or 0; 1; 2 or 0**
8. $p(x) = 2x^4 - 2x^3 + 2x^2 - x - 1$ **3 or 1; 1; 2 or 0**
9. $q(x) = 3x^4 + x^3 - 3x^2 + 7x + 5$ **2 or 0; 2 or 0; 4, 2, or 0**
10. $h(x) = 7x^4 + 3x^3 - 2x^2 - x + 1$ **2 or 0; 2 or 0; 4, 2, or 0**

Find all the zeros of each function.

11. $h(x) = 2x^3 + 3x^2 - 65x + 84$ **$-7, \frac{3}{2}, 4$**
12. $p(x) = x^3 - 3x^2 + 9x - 7$ **$1, 1 + i\sqrt{6}, 1 - i\sqrt{6}$**
13. $h(x) = x^3 - 7x^2 + 17x - 15$ **3, 2 + *i*, 2 − *i***
14. $q(x) = x^4 + 50x^2 + 49$ **−*i*, *i*, −7*i*, 7*i***
15. $g(x) = x^4 + 4x^3 - 3x^2 - 14x - 8$ **−1, −1, 2, −4**
16. $f(x) = x^4 - 6x^3 + 6x^2 + 24x - 40$ **−2, 2, 3 − *i*, 3 + *i***

Write a polynomial function of least degree with integral coefficients that has the given zeros.

17. $-5, 3i$ **$f(x) = x^3 + 5x^2 + 9x + 45$**
18. $-2, 3 + i$ **$f(x) = x^3 - 4x^2 - 2x + 20$**
19. $-1, 4, 3i$ **$f(x) = x^4 - 3x^3 + 5x^2 - 27x - 36$**
20. $2, 5, 1 + i$ **$f(x) = x^4 - 9x^3 + 26x^2 - 34x + 20$**

21. **CRAFTS** Stephan has a set of plans to build a wooden box. He wants to reduce the volume of the box to 105 cubic inches. He would like to reduce the length of each dimension in the plan by the same amount. The plans call for the box to be 10 inches by 8 inches by 6 inches. Write and solve a polynomial equation to find out how much Stephen should take from each dimension. **$(10 - x)(8 - x)(6 - x) = 105$; 3 in.**

Reading to Learn Mathematics, p. 403 ELL

Pre-Activity **How can the roots of an equation be used in pharmacology?**

Read the introduction to Lesson 7-5 at the top of page 371 in your textbook.

Using the model given in the introduction, write a polynomial equation with 0 on one side that can be solved to find the time or times at which there is 100 milligrams of medication in a patient's bloodstream.
$0.5t^4 + 3.5t^3 - 100t^2 + 350t - 100 = 0$

Reading the Lesson

1. Indicate whether each statement is *true* or *false*.
   a. Every polynomial equation of degree greater than one has at least one root in the set of real numbers. **false**
   b. If c is a root of the polynomial equation $f(x) = 0$, then $(x - c)$ is a factor of the polynomial $f(x)$. **true**
   c. If $(x + c)$ is a factor of the polynomial $f(x)$, then c is a zero of the polynomial function f. **false**
   d. A polynomial function f of degree n has exactly $(n - 1)$ complex zeros. **false**
2. Let $f(x) = x^6 - 2x^5 + 3x^4 - 4x^3 + 5x^2 + 6x - 7$.
   a. What are the possible numbers of positive real zeros of f? **5, 3, or 1**
   b. Write $f(-x)$ in simplified form (with no parentheses).
   $x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 - 6x - 7$
   What are the possible numbers of negative real zeros of f? **1**
   c. Complete the following chart to show the possible combinations of positive real zeros, negative real zeros, and imaginary zeros of the polynomial function f.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros	Total Number of Zeros
5	1	0	6
3	1	2	6
1	1	4	6

Helping You Remember

3. It is easier to remember mathematical concepts and results if you relate them to each other. How can the Complex Conjugates Theorem help you remember the part of Descartes' Rule of Signs that says, "or is less than this number by an even number."
Sample answer: For a polynomial function in which the polynomial has real coefficients, imaginary zeros come in conjugate pairs. Therefore, there must be an even number of imaginary zeros. For each pair of imaginary zeros, the number of positive or negative zeros decreases by 2.

25. −2, −2 + 3*i*, −2*i* − 3*i*
26. 4, 1 + *i*, 1 − *i*
27. $2i, -2i, \frac{i}{2}, -\frac{i}{2}$
28. 5*i*, −5*i*, 7
29. $-\frac{3}{2}$, 1 + 4*i*, 1 − 4*i*
30. $\frac{1}{2}$, 4 + 5*i*, 4 − 5*i*
31. 4 − *i*, 4 + *i*, −3
32. 3 − *i*, 3 + *i*, 4, −1

Find all of the zeros of each function.

25. $g(x) = x^3 + 6x^2 + 21x + 26$
26. $h(x) = x^3 - 6x^2 + 10x - 8$
27. $h(x) = 4x^4 + 17x^2 + 4$
28. $f(x) = x^3 - 7x^2 + 25x - 175$
29. $g(x) = 2x^3 - x^2 + 28x + 51$
30. $q(x) = 2x^3 - 17x^2 + 90x - 41$
31. $f(x) = x^3 - 5x^2 - 7x + 51$
32. $p(x) = x^4 - 9x^3 + 24x^2 - 6x - 40$
33. $r(x) = x^4 - 6x^3 + 12x^2 + 6x - 13$ **3 − 2*i*, 3 + 2*i*, −1, 1**
34. $h(x) = x^4 - 15x^3 + 70x^2 - 70x - 156$ **5 − *i*, 5 + *i*, −1, 6**

Write a polynomial function of least degree with integral coefficients that has the given zeros. **35–40. See margin.**

35. $-4, 1, 5$
36. $-2, 2, 4, 6$
37. $4i, 3, -3$
38. $2i, 3i, 1$
39. $9, 1 + 2i$
40. $6, 2 + 2i$

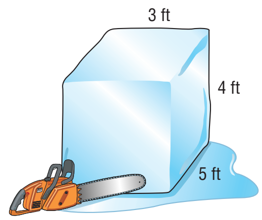
Space Exploration

A space shuttle is a reusable vehicle, launched like a rocket, which can put people and equipment in orbit around Earth. The first space shuttle was launched in 1981.

Source: kidsastronomy.about.com

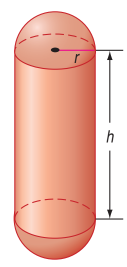
41. Sketch the graph of a polynomial function that has the indicated number and type of zeros. **See pp. 407A–407H.**
   a. 3 real, 2 imaginary
   b. 4 real
   c. 2 imaginary

SCULPTING **For Exercises 42 and 43, use the following information.**
Antonio is preparing to make an ice sculpture. He has a block of ice that he wants to reduce in size by shaving off the same amount from the length, width, and height. He wants to reduce the volume of the ice block to 24 cubic feet. **42. $(3 - x)(4 - x)(5 - x) = 24$**



42. Write a polynomial equation to model this situation.
43. How much should he take from each dimension? **1 ft**

SPACE EXPLORATION **For Exercises 44 and 45, use the following information.**
The space shuttle has an external tank for the fuel that the main engines need for the launch. This tank is shaped like a capsule, a cylinder with a hemispherical dome at either end. The cylindrical part of the tank has an approximate volume of 336π cubic meters and a height of 17 meters more than the radius of the tank. (*Hint:* $V(r) = \pi r^2 h$)



44. Write an equation that represents the volume of the cylinder. **$V(r) = \pi r^3 + 17\pi r^2$**
45. What are the dimensions of the tank? **radius = 4 m, height = 21 m**

MEDICINE **For Exercises 46–48, use the following information.**
Doctors can measure cardiac output in patients at high risk for a heart attack by monitoring the concentration of dye injected into a vein near the heart. A normal heart's dye concentration is given by $d(x) = -0.006x^4 - 0.15x^3 - 0.05x^2 + 1.8x$, where x is the time in seconds.

★ 46. How many positive real zeros, negative real zeros, and imaginary zeros exist for this function? (*Hint:* Notice that 0, which is neither positive nor negative, is a zero of this function since $d(0) = 0$.) **1; 2 or 0; 2 or 0**

★ 47. Approximate all real zeros to the nearest tenth by graphing the function using a graphing calculator. **See margin for graph; −24.1, −4.0, 0, and 3.1.**

★ 48. What is the meaning of the roots in this problem? **Nonnegative roots represent times when there is no concentration of dye registering on the monitor.**

49. **CRITICAL THINKING** Find a counterexample to disprove the following statement.
The polynomial function of least degree with integral coefficients with zeros at $x = 4$, $x = -1$, and $x = 3$, is unique.

49. Sample answer: $f(x) = x^3 - 6x^2 + 5x + 12$ and $g(x) = 2x^3 - 12x^2 + 10x + 24$; each have zeros at $x = 4$, $x = -2$, and $x = 3$.

Enrichment, p. 404

The Bisection Method for Approximating Real Zeros

The **bisection method** can be used to approximate zeros of polynomial functions like $f(x) = x^3 + x^2 - 3x - 3$.

Since $f(1) = -4$ and $f(2) = 3$, there is at least one real zero between 1 and 2. The midpoint of this interval is $\frac{1+2}{2} = 1.5$. Since $f(1.5) = -1.875$, the zero is between 1.5 and 2. The midpoint of this interval is $\frac{1.5+2}{2} = 1.75$. Since $f(1.75)$ is about 0.172, the zero is between 1.5 and 1.75. The midpoint of this interval is $\frac{1.5+1.75}{2} = 1.625$ and $f(1.625)$ is about −0.94. The zero is between 1.625 and 1.75. The midpoint of this interval is $\frac{1.625+1.75}{2} = 1.6875$. Since $f(1.6875)$ is about −0.41, the zero is between 1.6875 and 1.75. Therefore, the zero is 1.7 to the nearest tenth.

The diagram below summarizes the results obtained by the bisection method.

sign of $f(x)$: − − − − + +

Answers

35. $f(x) = x^3 - 2x^2 - 19x + 20$
36. $f(x) = x^4 - 10x^3 + 20x^2 + 40x - 96$
37. $f(x) = x^4 + 7x^2 - 144$
38. $f(x) = x^5 - x^4 + 13x^3 - 13x^2 + 36x - 36$
39. $f(x) = x^3 - 11x^2 + 23x - 45$
40. $f(x) = x^3 - 10x^2 + 32x - 48$

50. **CRITICAL THINKING** If a sixth-degree polynomial equation has exactly five distinct real roots, what can be said of one of its roots? Draw a graph of this situation. **One root is a double root; see margin for sample graph.**

51. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See pp. 407A–407H.**

How can the roots of an equation be used in pharmacology?

Include the following items in your answer:

- an explanation of what the roots of this equation represent, and
- an explanation of what the roots of this equation reveal about how often a patient should take this medication.

Standardized Test Practice

52. The equation $x^4 - 1 = 0$ has exactly ___?___ complex root(s). **A**

Ⓐ 4 Ⓑ 0 Ⓒ 2 Ⓓ 1

53. How many negative real zeros does $f(x) = x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$ have? **C**

Ⓐ 3 Ⓑ 2 Ⓒ 1 Ⓓ 0

Maintain Your Skills

Mixed Review

Use synthetic substitution to find $f(-3)$ and $f(4)$ for each function. *(Lesson 7-4)*

54. $f(x) = x^3 - 5x^2 + 16x - 7$ **−127, 41**

55. $f(x) = x^4 + 11x^3 - 3x^2 + 2x - 5$ **−254, 915**

56. **RETAIL** The store Bunches of Boxes and Bags assembles boxes for mailing. The store manager found that the volume of a box made from a rectangular piece of cardboard with a square of length x inches cut from each corner is $4x^3 - 168x^2 + 1728x$ cubic inches. If the piece of cardboard is 48 inches long, what is the width? *(Lesson 7-3)* **36 in.**

Determine whether each function has a maximum or a minimum value. Then find the maximum or minimum value of each function. *(Lesson 6-1)*

57. $f(x) = x^2 - 8x + 3$ **min.; −13**

58. $f(x) = -3x^2 - 18x + 5$ **max.; 32**

59. $f(x) = -7 + 4x^2$ **min.; −7**

Factor completely. If the polynomial is not factorable, write *prime*. *(Lesson 5-4)*

60. $15a^2b^2 - 5ab^2c^2$ $\mathbf{5ab^2(3a - c^2)}$

61. $12p^2 - 64p + 45$ $\mathbf{(6p - 5)(2p - 9)}$

62. $4y^3 + 24y^2 + 36y$ $\mathbf{4y(y + 3)^2}$

Use matrices A, B, C, and D to find the following. *(Lesson 4-2)*

$$A = \begin{bmatrix} -4 & 4 \\ 2 & -3 \\ 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 6 & -2 \end{bmatrix} \quad C = \begin{bmatrix} -4 & -5 \\ -3 & 1 \\ 2 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ -3 & 4 \end{bmatrix}$$

63. $A + D$ 64. $B - C$ 65. $3B - 2A$

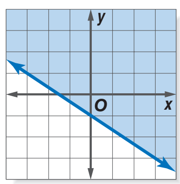
63. $\begin{bmatrix} -3 & 2 \\ 3 & -4 \\ -2 & 9 \end{bmatrix}$

64. $\begin{bmatrix} 11 & 5 \\ 7 & 0 \\ 4 & -5 \end{bmatrix}$

65. $\begin{bmatrix} 29 & -8 \\ 8 & 9 \\ 16 & -16 \end{bmatrix}$

66. Write an inequality for the graph at the right. *(Lesson 2-7)*

$\mathbf{y \geq -\frac{2}{3}x - 1}$



Getting Ready for the Next Lesson

BASIC SKILL **Find all values of $\pm\frac{a}{b}$ given each replacement set.**

67. $a = \{1, 5\}; b = \{1, 2\}$

68. $a = \{1, 2\}; b = \{1, 2, 7, 14\}$

69. $a = \{1, 3\}; b = \{1, 3, 9\}$

70. $a = \{1, 2, 4\}; b = \{1, 2, 4, 8, 16\}$

67–70. See margin.

4 Assess

Open-Ended Assessment

Writing Have students summarize the method for determining the number of real zeros for a polynomial function, and how to determine how many of them are positive, negative, and imaginary.

Getting Ready for Lesson 7-6

BASIC SKILL In Lesson 7-6, students will be introduced to the Rational Zero Theorem. They will factor integral coefficients in order to list every possible rational zero of a polynomial. Use Exercises 67–70 to determine your students' familiarity with finding all possible rational values of an expression $\pm\frac{a}{b}$, given all the possible values of a and b.

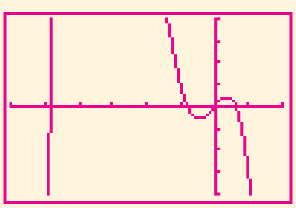
Assessment Options

Quiz (Lessons 7-4 and 7-5) is available on p. 443 of the *Chapter 7 Resource Masters*.

Mid-Chapter Test (Lessons 7-1 through 7-5) is available on p. 445 of the *Chapter 7 Resource Masters*.

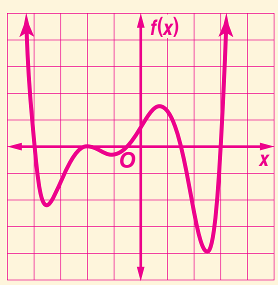
Answers

47.



[−30, 10] scl: 5 by [−20, 20] scl: 5

50. Sample graph:



67. $\pm\frac{1}{2}, \pm 1, \pm\frac{5}{2}, \pm 5$

68. $\pm\frac{1}{14}, \pm\frac{1}{7}, \pm\frac{2}{7}, \pm\frac{1}{2}, \pm 1, \pm 2$

69. $\pm\frac{1}{9}, \pm\frac{1}{3}, \pm 1, \pm 3$

70. $\pm\frac{1}{16}, \pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 4$

7-6 Lesson Notes

1 Focus

5-Minute Check Transparency 7-6 Use as a quiz or review of Lesson 7-5.

Mathematical Background notes are available for this lesson on p. 344D.

How can the Rational Zero Theorem solve problems involving large numbers?

Ask students:

- Why does the polynomial equation contain an expression in which the variable h appears three times? **The actual width and length of the overhead compartment are not known; only their relationships to the height of the compartment are known.**
- How do you know that $h + 8$ is the length? **The length is 8 inches longer than the height, which is represented by the variable h.**
- Could the height be 5 inches? Explain. **No. If the height were 5 inches, then the width would be 0 inches.**

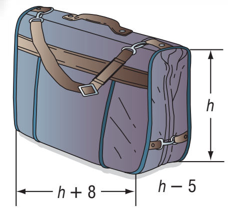
7-6 Rational Zero Theorem

What You'll Learn

- Identify the possible rational zeros of a polynomial function.
- Find all the rational zeros of a polynomial function.

How can the Rational Zero Theorem solve problems involving large numbers?

On an airplane, carry-on baggage must fit into the overhead compartment above the passenger's seat. The length of the compartment is 8 inches longer than the height, and the width is 5 inches shorter than the height. The volume of the compartment is 2772 cubic inches. You can solve the polynomial equation $h(h + 8)(h - 5) = 2772$, where h is the height, $h + 8$ is the length, and $h - 5$ is the width, to find the dimensions of the overhead compartment in which your luggage must fit.



IDENTIFY RATIONAL ZEROS Usually it is not practical to test all possible zeros of a polynomial function using only synthetic substitution. The **Rational Zero Theorem** can help you choose some possible zeros to test.

Key Concept — Rational Zero Theorem

- **Words** Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n$ represent a polynomial function with integral coefficients. If $\frac{p}{q}$ is a rational number in simplest form and is a zero of $y = f(x)$, then p is a factor of a_n and q is a factor of a_0.
- **Example** Let $f(x) = 2x^3 + 3x^2 - 17x + 12$. If $\frac{3}{2}$ is a zero of $f(x)$, then 3 is a factor of 12 and 2 is a factor of 2.

In addition, if the coefficient of the x term with the highest degree is 1, we have the following corollary.

Key Concept — Corollary (Integral Zero Theorem)

If the coefficients of a polynomial function are integers such that $a_0 = 1$ and $a_n \neq 0$, any rational zeros of the function must be factors of a_n.

Example 1 Identify Possible Zeros

List all of the possible rational zeros of each function.

a. $f(x) = 2x^3 - 11x^2 + 12x + 9$

If $\frac{p}{q}$ is a rational zero, then p is a factor of 9 and q is a factor of 2. The possible values of p are ±1, ±3, and ±9. The possible values for q are ±1 and ±2. So, $\frac{p}{q} = \pm1, \pm3, \pm9, \pm\frac{1}{2}, \pm\frac{3}{2}$, and $\pm\frac{9}{2}$.

Resource Manager

Workbook and Reproducible Masters

Chapter 7 Resource Masters
- Study Guide and Intervention, pp. 405–406
- Skills Practice, p. 407
- Practice, p. 408
- Reading to Learn Mathematics, p. 409
- Enrichment, p. 410

Graphing Calculator and Spreadsheet Masters, p. 39

Transparencies

5-Minute Check Transparency 7-6
Answer Key Transparencies

Technology

Interactive Chalkboard

b. $f(x) = x^3 - 9x^2 - x + 105$

Since the coefficient of x^3 is 1, the possible rational zeros must be a factor of the constant term 105. So, the possible rational zeros are the integers ±1, ±3, ±5, ±7, ±15, ±21, ±35, and ±105.

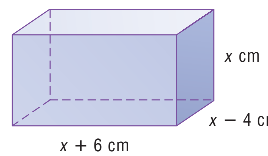
FIND RATIONAL ZEROS Once you have written the possible rational zeros, you can test each number using synthetic substitution.

Example 2 Use the Rational Zero Theorem

GEOMETRY The volume of a rectangular solid is 675 cubic centimeters. The width is 4 centimeters less than the height, and the length is 6 centimeters more than the height. Find the dimensions of the solid.

Let x = the height, $x - 4$ = the width, and $x + 6$ = the length.

Write an equation for the volume.



$x(x - 4)(x + 6) = 675$ Formula for volume

$x^3 + 2x^2 - 24x = 675$ Multiply.

$x^3 + 2x^2 - 24x - 675 = 0$ Subtract 675.

The leading coefficient is 1, so the possible integer zeros are factors of 675, ±1, ±3, ±5, ±9, ±15, ±25, ±27, ±45, ±75, ±135, ±225, and ±675. Since length can only be positive, we only need to check positive zeros. From Descartes' Rule of Signs, we also know there is only one positive real zero. Make a table and test possible real zeros.

p	1	2	−24	−675
1	1	3	−21	−696
3	1	5	−9	−702
5	1	7	11	−620
9	1	11	75	0

One zero is 9. Since there is only one positive real zero, we do not have to test the other numbers. The other dimensions are 9 − 4 or 5 centimeters and 9 + 6 or 15 centimeters.

CHECK Verify that the dimensions are correct. $5 \times 9 \times 15 = 675$ ✓

You usually do not need to test all of the possible zeros. Once you find a zero, you can try to factor the depressed polynomial to find any other zeros.

Study Tip

Descartes' Rule of Signs

Examine the signs of the coefficients in the equation, + + − −. There is one change of sign, so there is only one positive real zero.

Example 3 Find All Zeros

Find all of the zeros of $f(x) = 2x^4 - 13x^3 + 23x^2 - 52x + 60$.

From the corollary to the Fundamental Theorem of Algebra, we know there are exactly 4 complex roots. According to Descartes' Rule of Signs, there are 4, 2, or 0 positive real roots and 0 negative real roots. The possible rational zeros are ±1, ±2, ±3, ±4, ±5, ±6, ±10, ±12, ±15, ±20, ±30, ±60, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}$, and $\pm\frac{15}{2}$. Make a table and test some possible rational zeros.

$\frac{p}{q}$	2	−13	23	−52	60
1	2	−11	34	−18	20
2	2	−9	5	−42	−24
3	2	−7	2	−46	−78
5	2	−3	8	−12	0

(continued on the next page)

www.algebra2.com/extra_examples

2 Teach

IDENTIFY RATIONAL ZEROS

In-Class Example Power Point®

Teaching Tip While discussing Example 1, point out that ±1 will always be possible rational zeros. Also make sure students clearly understand that these are just *possible* zeros. Until each potential zero has been tested by synthetic substitution, it should not be referred to as a zero.

1 List all of the possible rational zeros of each function.

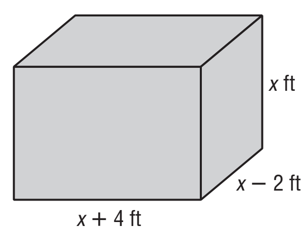
a. $f(x) = 3x^4 - x^3 + 4$
$\pm1, \pm2, \pm4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$

b. $f(x) = x^4 + 7x^3 - 15$
±1, ±3, ±5, ±15

FIND RATIONAL ZEROS

In-Class Examples Power Point®

2 **GEOMETRY** The volume of a rectangular solid is 1120 cubic feet. The width is 2 feet less than the height, and the length is 4 feet more than the height. Find the dimensions of the solid.



length: 14 ft, width: 8 ft, height: 10 ft

3 Find all of the zeros of $f(x) = x^4 + x^3 - 19x^2 + 11x + 30$.
−5, −1, 2, 3

DAILY INTERVENTION

Differentiated Instruction

Logical Organize the students in groups of four or five. Have the students in each group split the work shown in Example 3 into four or five steps, depending on the size of their group. Each student then prepares and gives an explanation to the group of their part of the Example. In particular, students should explain any mathematical processes, what the result of their step is to be, and how the result relates to the next step in the process.

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 7.
- write notes summarizing the Rational Zero Theorem and the Integral Zero Theorem.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

DAILY INTERVENTION

FIND THE ERROR Suggest students begin by assuming that *both* Lauren and Luis are incorrect. While students may spot Lauren's mistake quickly by looking at the possible integral zeros she has listed, ask students not to conclude that Luis must be correct without actually checking the possibilities he listed.

About the Exercises...

Organization by Objective
- **Identify Rational Zeros:** 12–17
- **Find Rational Zeros:** 18–33

Odd/Even Assignments
Exercises 12–33 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 13–33 odd, 37, 38, 42–61

Average: 13–33 odd, 34–38, 42–61

Advanced: 12–32 even, 34–36, 39–55 (optional: 56–61)

All: Practice Quiz 2 (1–5)

Since $f(5) = 0$, you know that $x = 5$ is a zero. The depressed polynomial is $2x^3 - 3x^2 + 8x - 12$.

Factor $2x^3 - 3x^2 + 8x - 12$.

$2x^3 - 3x^2 + 8x - 12 = 0$	Write the depressed polynomial.
$2x^3 + 8x - 3x^2 - 12 = 0$	Regroup terms.
$2x(x^2 + 4) - 3(x^2 + 4) = 0$	Factor by grouping.
$(x^2 + 4)(2x - 3) = 0$	Distributive Property

$x^2 + 4 = 0$ or $2x - 3 = 0$ Zero Product Property

$x^2 = -4$ $\quad 2x = 3$

$x = \pm 2i$ $\quad x = \frac{3}{2}$

There is another real zero at $x = \frac{3}{2}$ and two imaginary zeros at $x = 2i$ and $x = -2i$.

The zeros of this function are $5, \frac{3}{2}, 2i$ and $-2i$.

Check for Understanding

Concept Check

1. **Explain** why it is useful to use the Rational Zero Theorem when finding the zeros of a polynomial function. **Sample answer: You limit the number of possible solutions.**

2. **OPEN ENDED** Write a polynomial function that has possible rational zeros of $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$. **Sample answer: $2x^2 + x + 3$**

3. **FIND THE ERROR** Lauren and Luis are listing the possible rational zeros of $f(x) = 4x^5 + 4x^4 - 3x^3 + 2x^2 - 5x + 6$.

Lauren	Luis
$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}$, $\pm 2, \pm\frac{2}{3}, \pm 4, \pm\frac{4}{3}$	$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2$, $\pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm 6$

Who is correct? Explain your reasoning. **Luis; Lauren found numbers in the form $\frac{q}{p}$, not $\frac{p}{q}$ as Luis did according to the Rational Zero Theorem.**

Guided Practice

List all of the possible rational zeros of each function.

4. $p(x) = x^4 - 10$ **$\pm 1, \pm 2, \pm 5, \pm 10$**

5. $d(x) = 6x^3 + 6x^2 - 15x - 2$ **$\pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}$**

Find all of the rational zeros of each function.

6. $p(x) = x^3 - 5x^2 - 22x + 56$ **$-4, 2, 7$**

7. $f(x) = x^3 - x^2 - 34x - 56$ **$-2, -4, 7$**

8. $t(x) = x^4 - 13x^2 + 36$ **$2, -2, 3, -3$**

9. $f(x) = 2x^3 - 7x^2 - 8x + 28$ **$-2, 2, \frac{7}{2}$**

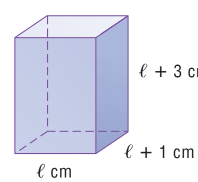
10. Find all of the zeros of $f(x) = 6x^3 + 5x^2 - 9x + 2$. **$\frac{2}{3}, \frac{-3 \pm \sqrt{17}}{4}$**

GUIDED PRACTICE KEY

Exercises	Examples
4, 5	1
6–9, 11	2
10	3

Application

11. **GEOMETRY** The volume of the rectangular solid is 1430 cubic centimeters. Find the dimensions of the solid. **10 cm × 11 cm × 13 cm**



Practice and Apply

Homework Help

For Exercises	See Examples
12–17	1
18–29, 34–41	2
30–33	3

Extra Practice

See page 843.

List all of the possible rational zeros of each function.

12. $f(x) = x^3 + 6x + 2$ **±1, ±2**
13. $h(x) = x^3 + 8x + 6$ **±1, ±2, ±3, ±6**
14. $f(x) = 3x^4 + 15$
15. $n(x) = x^5 + 6x^3 - 12x + 18$
16. $p(x) = 3x^3 - 5x^2 - 11x + 3$
17. $h(x) = 9x^6 - 5x^3 + 27$

14. $\pm1, \pm3, \pm5, \pm15, \pm\frac{1}{3}, \pm\frac{5}{3}$
15. $\pm1, \pm2, \pm3, \pm6, \pm9, \pm18$
16. $\pm1, \pm\frac{1}{3}, \pm3$
17. $\pm1, \pm\frac{1}{3}, \pm\frac{1}{9}, \pm3, \pm9, \pm27$

Find all of the rational zeros of each function. **18. −6, −5, 10**

18. $f(x) = x^3 + x^2 - 80x - 300$
19. $p(x) = x^3 - 3x - 2$ **−1, −1, 2**
20. $h(x) = x^4 + x^2 - 2$ **1, −1**
21. $g(x) = x^4 - 3x^3 - 53x^2 - 9x$ **0, 9**
22. $f(x) = 2x^5 - x^4 - 2x + 1$ $\frac{1}{2}$, **−1, 1**
23. $f(x) = x^5 - 6x^3 + 8x$ **0, 2, −2**
24. $g(x) = x^4 - 3x^3 + x^2 - 3x$ **0, 3**
25. $p(x) = x^4 + 10x^3 + 33x^2 + 38x + 8$
26. $p(x) = x^3 + 3x^2 - 25x + 21$ **−7, 1, 3**
27. $h(x) = 6x^3 + 11x^2 - 3x - 2$
28. $h(x) = 10x^3 - 17x^2 - 7x + 2$
29. $g(x) = 48x^4 - 52x^3 + 13x - 3$

25. $-2, -4$ **27.** $\frac{1}{2}, -\frac{1}{3}, -2$ **28.** $-\frac{1}{2}, \frac{1}{5}, 2$ **29.** $-\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}$

Find all of the zeros of each function. **30–33. See margin.**

30. $p(x) = 6x^4 + 22x^3 + 11x^2 - 38x - 40$
31. $g(x) = 5x^4 - 29x^3 + 55x^2 - 28x$
32. $h(x) = 9x^5 - 94x^3 + 27x^2 + 40x - 12$
33. $p(x) = x^5 - 2x^4 - 12x^3 - 12x^2 - 13x - 10$

FOOD For Exercises 34–36, use the following information. **35.** $2, -3 \pm i\sqrt{3}$; **2**

Terri's Ice Cream Parlor makes gourmet ice cream cones. The volume of each cone is 8π cubic inches. The height is 4 inches more than the radius of the cone's opening.

34. Write a polynomial equation that represents the volume of an ice cream cone. Use the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$. $V = \frac{1}{3}\pi r^3 + \frac{4}{3}\pi r^2$
35. What are the possible values of r? Which of these values are reasonable?
36. Find the dimensions of the cone. r **= 2 in.,** h **= 6 in.**

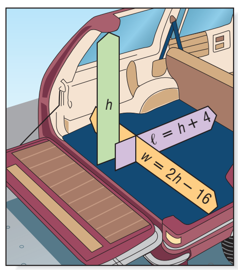
Food

The largest ice cream sundae, weighing 24.91 tons, was made in Edmonton, Alberta, in July 1988.

Source: *The Guinness Book of Records.*

AUTOMOBILES For Exercises 37 and 38, use the following information.

The length of the cargo space in a sport-utility vehicle is 4 inches greater than the height of the space. The width is sixteen inches less than twice the height. The cargo space has a total volume of 55,296 cubic inches.



37. Write a polynomial function that represents the volume of the cargo space.
38. Find the dimensions of the cargo space.

37. $V = 2h^3 - 8h^2 - 64h$ **38.** $\ell = 36$ **in.,** $w = 48$ **in.,** $h = 32$ **in.**

AMUSEMENT PARKS For Exercises 39–41, use the following information.

An amusement park owner wants to add a new wilderness water ride that includes a mountain that is shaped roughly like a pyramid. Before building the new attraction, engineers must build and test a scale model. **39.** $V = \frac{1}{3}\ell^3 - 3\ell^2$

39. If the height of the scale model is 9 inches less than its length and its base is a square, write a polynomial function that describes the volume of the model in terms of its length. Use the formula for the volume of a pyramid, $V = \frac{1}{3}Bh$.
40. If the volume of the model is 6300 cubic inches, write an equation for the situation.
41. What are the dimensions of the scale model? $\ell = 30$ **in.,** $w = 30$ **in.,** $h = 21$ **in.**

40. $6300 = \frac{1}{3}\ell^3 - 3\ell^2$

42. **CRITICAL THINKING** Suppose k and $2k$ are zeros of $f(x) = x^3 + 4x^2 + 9kx - 90$. Find k and all three zeros of $f(x)$. $k = -3$; **−3, −6, 5**

www.algebra2.com/self_check_quiz

Answers

30. $-2, \frac{4}{3}, \frac{-3 \pm i}{2}$

31. $\frac{4}{5}, 0, \frac{5 \pm i\sqrt{3}}{2}$

32. $3, \frac{2}{3}, -\frac{2}{3}, \frac{-3 \pm \sqrt{13}}{2}$

33. $-1, -2, 5, i, -i$

Enrichment, p. 410

Infinite Continued Fractions

Some infinite expressions are actually equal to real numbers! The infinite continued fraction at the right is one example.

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}$$

If you use x to stand for the infinite fraction, then the entire denominator of the first fraction on the right is also equal to x. This observation leads to the following equation:

$$x = 1 + \frac{1}{x}$$

Write a decimal for each continued fraction.

1. $1 + \frac{1}{1}$ **2**
2. $1 + \cfrac{1}{1 + \frac{1}{1}}$ **1.5**
3. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}$ $1.66\overline{6}$

Study Guide and Intervention, p. 405 (shown) and p. 406

Identify Rational Zeros

Rational Zero Theorem	Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-2}x^2 + a_{n-1}x + a^n$ represent a polynomial function with integral coefficients. If $\frac{p}{q}$ is a rational number in simplest form and is a zero of $y = f(x)$, then p is a factor of a_n and q is a factor of a_0.
Corollary (Integral Zero Theorem)	If the coefficients of a polynomial are integers such that $a_0 = 1$ and $a_n \neq 0$, any rational zeros of the function must be factors of a_n.

Example **List all of the possible rational zeros of each function.**

a. $f(x) = 3x^4 - 2x^2 + 6x - 10$

If $\frac{p}{q}$ is a rational root, then p is a factor of −10 and q is a factor of 3. The possible values for p are ±1, ±2, ±5, and ±10. The possible values for q are ±1 and ±3. So all of the possible rational zeros are $\frac{p}{q} = \pm1, \pm2, \pm5, \pm10, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}$, and $\pm\frac{10}{3}$.

b. $q(x) = x^3 - 10x^2 + 14x - 36$

Since the coefficient of x^3 is 1, the possible rational zeros must be the factors of the constant term −36. So the possible rational zeros are ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18, and ±36.

Exercises

List all of the possible rational zeros of each function.

1. $f(x) = x^3 + 3x^2 - x + 8$ **±1, ±2, ±4, ±8**
2. $g(x) = x^5 - 7x^4 + 3x^2 + x - 20$ **±1, ±2, ±4, ±5, ±10, ±20**
3. $h(x) = x^4 - 7x^3 - 4x^2 + x - 49$ **±1, ±7, ±49**
4. $p(x) = 2x^4 - 5x^3 + 8x^2 + 3x - 5$ $\pm1, \pm5, \pm\frac{1}{2}, \pm\frac{5}{2}$
5. $q(x) = 3x^4 - 5x^3 + 10x + 12$ $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$
6. $r(x) = 4x^5 - 2x + 18$ $\pm1, \pm2, \pm3, \pm6, \pm9, \pm18, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{9}{4}$
7. $f(x) = x^7 - 6x^5 - 3x^4 + x^3 + 4x^2 - 120$ **±1, ±2, ±3, ±4, ±5, ±6, ±8, ±10, ±12, ±15, ±20, ±24, ±30, ±40, ±60, ±120**
8. $g(x) = 5x^6 - 3x^4 + 5x^3 + 2x^2 - 15$ $\pm1, \pm3, \pm5, \pm15, \pm\frac{1}{5}, \pm\frac{3}{5}$
9. $h(x) = 6x^5 - 3x^4 + 12x^3 + 18x^2 - 9x + 21$ $\pm1, \pm3, \pm7, \pm21, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{7}{2}, \pm\frac{21}{2}, \pm\frac{1}{3}, \pm\frac{7}{3}, \pm\frac{1}{6}, \pm\frac{7}{6}$
10. $p(x) = 2x^7 - 3x^6 + 11x^5 - 20x^2 + 11$ $\pm1, \pm11, \pm\frac{1}{2}, \pm\frac{11}{2}$

Skills Practice, p. 407 and Practice, p. 408 (shown)

List all of the possible rational zeros of each function.

1. $h(x) = x^3 - 5x^2 + 2x + 12$ **±1, ±2, ±3, ±4, ±6, ±12**
2. $s(x) = x^4 - 8x^3 + 7x - 14$ **±1, ±2, ±7, ±14**
3. $f(x) = 3x^5 - 5x^2 + x + 6$ $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm1, \pm2, \pm3, \pm6$
4. $p(x) = 3x^2 + x + 7$ $\pm\frac{1}{3}, \pm\frac{7}{3}, \pm1, \pm7$
5. $g(x) = 5x^3 + x^2 - x + 8$ $\pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{4}{5}, \pm\frac{8}{5}, \pm1, \pm2, \pm4, \pm8$
6. $q(x) = 6x^5 + x^3 - 3$ $\pm\frac{1}{6}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm1, \pm3$

Find all of the rational zeros of each function.

7. $q(x) = x^3 + 3x^2 - 6x - 8 = 0$ **−4, −1, 2**
8. $v(x) = x^3 - 9x^2 + 27x - 27$ **3**
9. $c(x) = x^3 - x^2 - 8x + 12$ **−3, 2**
10. $f(x) = x^4 - 49x^2$ **0, −7, 7**
11. $h(x) = x^3 - 7x^2 + 17x - 15$ **3**
12. $b(x) = x^3 + 6x + 20$ **−2**
13. $f(x) = x^3 - 6x^2 + 4x - 24$ **6**
14. $g(x) = 2x^3 + 3x^2 - 4x - 4$ **−2**
15. $h(x) = 2x^3 - 7x^2 - 21x + 54 = 0$ $-3, 2, \frac{9}{2}$
16. $z(x) = x^4 - 3x^3 + 5x^2 - 27x - 36$ **−1, 4**
17. $d(x) = x^4 + x^3 + 16$ **no rational zeros**
18. $n(x) = x^4 - 2x^3 - 3$ **−1**
19. $p(x) = 2x^4 - 7x^3 + 4x^2 + 7x - 6$ $-1, 1, \frac{3}{2}, 2$
20. $q(x) = 6x^4 + 29x^3 + 40x^2 + 7x - 12$ $-\frac{3}{2}, -\frac{4}{3}$

Find all of the zeros of each function.

21. $f(x) = 2x^4 + 7x^3 - 2x^2 - 19x - 12$ $-1, -3, \frac{1 + \sqrt{33}}{4}, \frac{1 - \sqrt{33}}{4}$
22. $q(x) = x^4 - 4x^3 + x^2 + 16x - 20$ $-2, 2, 2 + i, 2 - i$
23. $h(x) = x^6 - 8x^3$ $0, 2, -1 + i\sqrt{3}, -1 - i\sqrt{3}$
24. $g(x) = x^6 - 1$ $-1, 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}, \frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}$
25. **TRAVEL** The height of a box that Joan is shipping is 3 inches less than the width of the box. The length is 2 inches more than twice the width. The volume of the box is 1540 in^3. What are the dimensions of the box? **22 in. by 10 in. by 7 in.**
26. **GEOMETRY** The height of a square pyramid is 3 meters shorter than the side of its base. If the volume of the pyramid is 432 m^3, how tall is it? Use the formula $V = \frac{1}{3}Bh$. **9 m**

Reading to Learn Mathematics, p. 409 ELL

Pre-Activity **How can the Rational Zero Theorem solve problems involving large numbers?**

Read the introduction to Lesson 7-6 at the top of page 378 in your textbook.

Rewrite the polynomial equation $w(w + 8)(w - 5) = 2772$ in the form $f(x) = 0$, where $f(x)$ is a polynomial written in descending powers of x. $w^3 + 3w^2 - 40w - 2772 = 0$

Reading the Lesson

1. For each of the following polynomial functions, list all the possible values of p, all the possible values of q, and all the possible rational zeros $\frac{p}{q}$.

   a. $f(x) = x^3 - 2x^2 - 11x + 12$

   possible values of p: **±1, ±2, ±3, ±4, ±6, ±12**

   possible values of q: **±1**

   possible values of $\frac{p}{q}$: **±1, ±2, ±3, ±4, ±6, ±12**

   b. $f(x) = 2x^4 + 9x^3 - 23x^2 - 81x + 45$

   possible values of p: **±1, ±3, ±5, ±9, ±15, ±45**

   possible values of q: **±1, ±2**

   possible values of $\frac{p}{q}$: $\pm1, \pm3, \pm5, \pm9, \pm15, \pm45, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{15}{2}, \pm\frac{45}{2}$

2. Explain in your own words how Descartes' Rule of Signs, the Rational Zero Theorem, and synthetic division can be used together to find all of the rational zeros of a polynomial function with integer coefficients.

   Sample answer: Use Descartes' Rule to find the possible numbers of positive and negative real zeros. Use the Rational Zero Theorem to list all possible rational zeros. Use synthetic division to test which of the numbers on the list of possible rational zeros are actually zeros of the polynomial function. (Descartes' Rule may help you to limit the possibilities.)

Helping You Remember

3. Some students have trouble remembering which numbers go in the numerators and which go in the denominators when forming a list of possible rational zeros of a polynomial function. How can you use the linear polynomial equation $ax + b = 0$, where a and b are nonzero integers, to remember this?

   Sample answer: The solution of the equation is $-\frac{b}{a}$**. The numerator** b **is a factor of the constant term in** $ax + b$**. The denominator** a **is a factor of the leading coefficient in** $ax + b$**.**

4 Assess

Open-Ended Assessment

Speaking Have students explain how to find all the possible rational zeros of a polynomial function. Ask them to demonstrate the technique using a polynomial function of degree 3 or higher while explaining the process.

Getting Ready for Lesson 7-7

PREREQUISITE SKILL Students will perform arithmetic operations on functions and find the composition of functions in Lesson 7-7. These skills will rely on students' ability to correctly perform operations on polynomials. Use Exercises 56–61 to determine your students' familiarity with performing operations with polynomials.

Assessment Options

Practice Quiz 2 The quiz provides students with a brief review of the concepts and skills in Lessons 7-4 through 7-6. Lesson numbers are given to the right of the exercises or instruction lines so students can review concepts not yet mastered.

Answer

43. The Rational Zero Theorem helps factor large numbers by eliminating some possible zeros because it is not practical to test all of them using synthetic substitution. Answers should include the following.

- **The polynomial equation that represents the volume of the compartment is $V = w^3 + 3w^2 - 40w$.**
- **Reasonable measures of the width of the compartment are, in inches, 1, 2, 3, 4, 6, 7, 9, 12, 14, 18, 21, 22, 28, 33, 36, 42, 44, 63, 66, 77, and 84. The solution shows that $w = 14$ in., $\ell = 22$ in., and $d = 9$ in.**

43. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson. **See margin.**

How can the Rational Zero Theorem solve problems involving large numbers?

Include the following items in your answer:
- the polynomial equation that represents the volume of the compartment, and
- a list of all reasonable measures of the width of the compartment, assuming that the width is a whole number.

Standardized Test Practice

44. Using the Rational Zero Theorem, determine which of the following is a zero of the function $f(x) = 12x^5 - 5x^3 + 2x - 9$. **D**

(A) -6 (B) $\frac{3}{8}$ (C) $-\frac{2}{3}$ (D) 1

45. **OPEN ENDED** Write a polynomial with $-5, -2, 1, 3$, and 4 as roots.
Sample answer: $x^5 - x^4 - 27x^3 + 41x^2 + 106x - 120$

Maintain Your Skills

Mixed Review **Given a function and one of its zeros, find all of the zeros of the function.** *(Lesson 7-5)*

46. $g(x) = x^3 + 4x^2 - 27x - 90; -3$ **46. $-6, -3, 5$**

47. $h(x) = x^3 - 11x + 20; 2 + i$ **47. $-4, 2 + i, 2 - i$**

48. $f(x) = x^3 + 5x^2 + 9x + 45; -5$ **$-5, 3i, -3i$**

49. $g(x) = x^3 - 3x^2 - 41x + 203; -7$ **$-7, 5 + 2i, 5 - 2i$**

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. *(Lesson 7-4)*

50. $20x^3 - 29x^2 - 25x + 6; x - 2$ **$4x + 3, 5x - 1$**

51. $3x^4 - 21x^3 + 38x^2 - 14x + 24; x - 3$ **$x - 4, 3x^2 + 2$**

Simplify. *(Lesson 5-5)*

52. $\sqrt{245}$ **$7\sqrt{5}$**

53. $\pm\sqrt{18x^3y^2}$ **$\pm 3xy\sqrt{2x}$**

54. $\sqrt{16x^2 - 40x + 25}$ **$|4x - 5|$**

55. **GEOMETRY** The perimeter of a right triangle is 24 centimeters. Three times the length of the longer leg minus two times the length of the shorter leg exceeds the hypotenuse by 2 centimeters. What are the lengths of all three sides? *(Lesson 3-5)* **6 cm, 8 cm, 10 cm**

Getting Ready for the Next Lesson **PREREQUISITE SKILL Simplify.**
*(To review **operations with polynomials**, see Lessons 5-2 and 5-3.)*

56. $(x^2 - 7) + (x^3 + 3x^2 + 1)$ **56. $x^3 + 4x^2 - 6$**

57. $(8x^2 - 3x) - (4x^2 + 5x - 3)$ **57. $4x^2 - 8x + 3$**

58. $(x + 2)(x^2 + 3x - 5)$ **58. $x^3 + 5x^2 + x - 10$**

59. $(x^3 + 3x^2 - 3x + 1)(x - 5)^2$ **59. $x^5 - 7x^4 + 8x^3 + 106x^2 - 85x + 25$**

60. $(x^2 - 2x - 30) \div (x + 7)$ **60. $x - 9 + \frac{33}{x + 7}$**

61. $(x^3 + 2x^2 - 3x + 1) \div (x + 1)$ **61. $x^2 + x - 4 + \frac{5}{x + 1}$**

Practice Quiz 2 — *Lessons 7-4 through 7-6*

Use synthetic substitution to find $f(-2)$ and $f(3)$ for each function. *(Lesson 7-4)*

1. $f(x) = 7x^5 - 25x^4 + 17x^3 - 32x^2 + 10x - 22$ **$-930, -145$**
2. $f(x) = 3x^4 - 12x^3 - 21x^2 + 30x$ **$0, -180$**
3. Write the polynomial equation of degree 4 with leading coefficient 1 that has roots at $-2, -1, 3$, and 4. *(Lesson 7-5)* **$x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$**

Find all of the rational zeros of each function. *(Lesson 7-6)*

4. $f(x) = 5x^3 - 29x^2 + 55x - 28$ **$\frac{4}{5}$**
5. $g(x) = 4x^3 + 16x^2 - x - 24$ **$-\frac{3}{2}$**

382 Chapter 7 Polynomial Functions

7-7 Operations on Functions

What You'll Learn

- Find the sum, difference, product, and quotient of functions.
- Find the composition of functions.

Vocabulary

- composition of functions

Why is it important to combine functions in business?

Carol Coffmon owns a garden store where she sells birdhouses. The revenue from the sale of the birdhouses is given by $r(x) = 125x$. The function for the cost of making the birdhouses is given by $c(x) = 65x + 5400$. Her profit p is the revenue minus the cost or $p = r - c$. So the profit function $p(x)$ can be defined as $p(x) = (r - c)(x)$. If you have two functions, you can form a new function by performing arithmetic operations on them.

ARITHMETIC OPERATIONS Let $f(x)$ and $g(x)$ be any two functions. You can add, subtract, multiply, and divide functions according to the following rules.

Key Concept *Operations with Function*

Operation	Definition	Examples if $f(x) = x + 2$, $g(x) = 3x$
Sum	$(f + g)(x) = f(x) + g(x)$	$(x + 2) + 3x = 4x + 2$
Difference	$(f - g)(x) = f(x) - g(x)$	$(x + 2) - 3x = -2x + 2$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$(x + 2)3x = 3x^2 + 6x$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$	$\frac{x + 2}{3x}$

Example 1 Add and Subtract Functions

Given $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$, find each function.

a. $(f + g)(x)$

$(f + g)(x) = f(x) + g(x)$ — Addition of functions

$= (x^2 - 3x + 1) + (4x + 5)$ — $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$

$= x^2 + x + 6$ — Simplify.

b. $(f - g)(x)$

$(f - g)(x) = f(x) - g(x)$ — Subtraction of functions

$= (x^2 - 3x + 1) - (4x + 5)$ — $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$

$= x^2 - 7x - 4$ — Simplify.

Notice that the functions f and g have the same domain of all real numbers. The functions $f + g$ and $f - g$ also have domains that include all real numbers. For each new function, the domain consists of the intersection of the domains of $f(x)$ and $g(x)$. The domain of the quotient function is further restricted by excluded values that make the denominator equal to zero.

7-7 Lesson Notes

1 Focus

5-Minute Check Transparency 7-7 Use as a quiz or review of Lesson 7-6.

Mathematical Background notes are available for this lesson on p. 344D.

Why is it important to combine functions in business?

Ask students:

- What is the difference between revenue and profit? **Revenue is the amount of money generated by the sales of the birdhouses. Profit is the amount that revenue exceeds the cost (or expenses) of producing the birdhouses.**
- Is profit always a positive value? Explain. **No; profit is negative when expenses exceed revenue, in which case it is called *loss*.**

2 Teach

ARITHMETIC OPERATIONS

In-Class Example PowerPoint®

1 Given $f(x) = 3x^2 + 7x$ and $g(x) = 2x^2 - x - 1$, find each function.

a. $(f + g)(x)$ $5x^2 + 6x - 1$

b. $(f - g)(x)$ $x^2 + 8x + 1$

Resource Manager

Workbook and Reproducible Masters

Chapter 7 Resource Masters

- Study Guide and Intervention, pp. 411–412
- Skills Practice, p. 413
- Practice, p. 414
- Reading to Learn Mathematics, p. 415
- Enrichment, p. 416
- Assessment, p. 444

Graphing Calculator and Spreadsheet Masters, p. 40

School-to-Career Masters, p. 13

Teaching Algebra With Manipulatives Masters, pp. 253–255

Transparencies

5-Minute Check Transparency 7-7
Real-World Transparency 7
Answer Key Transparencies

Technology

Interactive Chalkboard

In-Class Example

Power Point®

2 Given $f(x) = 3x^2 - 2x + 1$ and $g(x) = x - 4$, find each function.

a. $(f \cdot g)(x)$ $\mathbf{3x^3 - 14x^2 + 9x - 4}$

b. $\left(\frac{f}{g}\right)(x)$ $\mathbf{\frac{3x^2 - 2x + 1}{x - 4}, x \neq 4}$

COMPOSITION OF FUNCTIONS

Tips for New Teachers

Intervention Some students may read $f \circ g$ as the word *fog*. Listen for students making this verbal error. Stress that students must learn to read this correctly because the correct wording will help them understand the meaning. Lead students to understand the similarity in meaning between $f(x)$ (read "f of x") and $f \circ g$ (read "f of g of x"). Reinforce this understanding by showing how $f \circ g$ can also be written as $f[g(x)]$. You can also relate $f[g(x)]$ to an expression containing nested parentheses, such as $(1 + (3 \cdot 5(4)))$ in which the expressions in parentheses are evaluated from the innermost parentheses to the outermost.

Example 2 Multiply and Divide Functions

Given $f(x) = x^2 + 5x - 1$ and $g(x) = 3x - 2$, find each function.

a. $(f \cdot g)(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$ — Product of functions

$= (x^2 + 5x - 1)(3x - 2)$ — $f(x) = x^2 + 5x - 1$ and $g(x) = 3x - 2$

$= x^2(3x - 2) + 5x(3x - 2) - 1(3x - 2)$ — Distributive Property

$= 3x^3 - 2x^2 + 15x^2 - 10x - 3x + 2$ — Distributive Property

$= 3x^3 + 13x^2 - 13x + 2$ — Simplify.

b. $\left(\frac{f}{g}\right)(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ — Division of functions

$= \frac{x^2 + 5x - 1}{3x - 2}, x \neq \frac{2}{3}$ — $f(x) = x^2 + 5x - 1$ and $g(x) = 3x - 2$

Because $x = \frac{2}{3}$ makes $3x - 2 = 0$, $\frac{2}{3}$ is excluded from the domain of $\left(\frac{f}{g}\right)(x)$.

Study Tip

Reading Math
$[f \circ g](x)$ and $f[g(x)]$ are both read *f of g of x*.

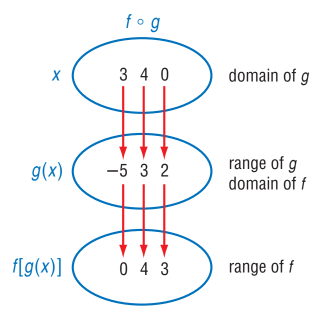
COMPOSITION OF FUNCTIONS Functions can also be combined using **composition of functions**. In a composition, a function is performed, and then a second function is performed on the result of the first function. The composition of f and g is denoted by $f \circ g$.

Key Concept — *Composition of Functions*

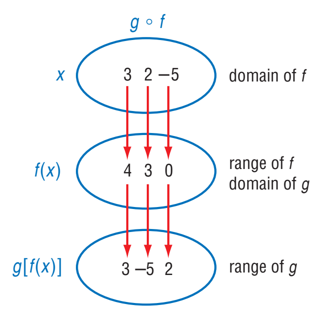
Suppose f and g are functions such that the range of g is a subset of the domain of f. Then the composite function $f \circ g$ can be described by the equation

$$[f \circ g](x) = f[g(x)].$$

The composition of functions can be shown by mappings. Suppose $f = \{(3, 4), (2, 3), (-5, 0)\}$ and $g = \{(3, -5), (4, 3), (0, 2)\}$. The composition of these functions is shown below.



$f \circ g = \{(3, 0), (4, 4), (0, 3)\}$



$g \circ f = \{(3, 3), (2, -5), (-5, 2)\}$

The composition of two functions may not exist. Given two functions f and g, $[f \circ g](x)$ is defined only if the range of $g(x)$ is a subset of the domain of $f(x)$. Similarly, $[g \circ f](x)$ is defined only if the range of $f(x)$ is a subset of the domain of $g(x)$.

384 Chapter 7 Polynomial Functions

Example 3 Evaluate Composition of Relations

If $f(x) = \{(7, 8), (5, 3), (9, 8), (11, 4)\}$ and $g(x) = \{(5, 7), (3, 5), (7, 9), (9, 11)\}$, find $f \circ g$ and $g \circ f$.

To find $f \circ g$, evaluate $g(x)$ first. Then use the range of g as the domain of f and evaluate $f(x)$.

$f[g(5)] = f(7)$ or 8 — $g(5) = 7$

$f[g(3)] = f(5)$ or 3 — $g(3) = 5$

$f[g(7)] = f(9)$ or 8 — $g(7) = 9$

$f[g(9)] = f(11)$ or 4 — $g(9) = 11$

$f \circ g = \{(5, 8), (3, 3), (7, 8), (9, 4)\}$

To find $g \circ f$, evaluate $f(x)$ first. Then use the range of f as the domain of g and evaluate $g(x)$.

$g[f(7)] = g(8)$ — $g(8)$ is undefined.

$g[f(5)] = g(3)$ or 5 — $f(5) = 3$

$g[f(9)] = g(8)$ — $g(8)$ is undefined.

$g[f(11)] = g(4)$ — $g(4)$ is undefined.

Since 8 and 4 are not in the domain of g, $g \circ f$ is undefined for $x = 7$, $x = 9$, and $x = 11$. However, $g[f(5)] = 5$ so $g \circ f = \{(5, 5)\}$.

Notice that in most instances $f \circ g \neq g \circ f$. Therefore, the order in which you compose two functions is very important.

Example 4 Simplify Composition of Functions

a. Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = x + 3$ and $g(x) = x^2 + x - 1$.

$[f \circ g](x) = f[g(x)]$	Composition of functions
$= f(x^2 + x - 1)$	Replace $g(x)$ with $x^2 + x - 1$.
$= (x^2 + x - 1) + 3$	Substitute $x^2 + x - 1$ for x in $f(x)$.
$= x^2 + x + 2$	Simplify.

$[g \circ f](x) = g[f(x)]$	Composition of functions
$= g(x + 3)$	Replace $f(x)$ with $x + 3$.
$= (x + 3)^2 + (x + 3) - 1$	Substitute $x + 3$ for x in $g(x)$.
$= x^2 + 6x + 9 + x + 3 - 1$	Evaluate $(x + 3)^2$.
$= x^2 + 7x + 11$	Simplify.

So, $[f \circ g](x) = x^2 + x + 2$ and $[g \circ f](x) = x^2 + 7x + 11$.

b. Evaluate $[f \circ g](x)$ and $[g \circ f](x)$ for $x = 2$.

$[f \circ g](x) = x^2 + x + 2$	Function from part a
$[f \circ g](2) = (2)^2 + 2 + 2$	Replace x with 2.
$= 8$	Simplify.

$[g \circ f](x) = x^2 + 7x + 11$	Function from part a
$[g \circ f](2) = (2)^2 + 7(2) + 11$	Replace x with 2.
$= 29$	Simplify.

So, $[f \circ g](2) = 8$ and $[g \circ f](2) = 29$.

www.algebra2.com/extra_examples

In-Class Examples

Power Point®

3 If $f(x) = \{(2, 6), (9, 4), (7, 7), (0, -1)\}$ and $g(x) = \{(7, 0), (-1, 7), (4, 9), (8, 2)\}$, find $f \circ g$ and $g \circ f$. **$f \circ g = \{(7, -1), (-1, 7), (4, 4), (8, 6)\}$; $g \circ f = \{(9, 9), (7, 0), (0, 7)\}$**

4

a. Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = 3x^2 - x + 4$ and $g(x) = 2x - 1$. **$[f \circ g](x) = 12x^2 - 14x + 8$; $[g \circ f](x) = 6x^2 - 2x + 7$**

b. Evaluate $[f \circ g](x)$ and $[g \circ f](x)$ for $x = -2$. **$[f \circ g](-2) = 84$; $[g \circ f](-2) = 35$**

DAILY INTERVENTION

Differentiated Instruction

Naturalist Invite students to think of events in nature whose occurrence students can relate to as being similar to the order in which the composition of functions must be carried out. Students interested in science might think of the stages of metamorphosis in insects, during which a larva changes to a pupa and finally to an adult. Each stage is dependent on the one before and the order of the stages is fixed.

In-Class Example
Power Point®

5 **TAXES** Tracie Long has \$100 deducted from every paycheck for retirement. She can have this deduction taken before state taxes are applied, which reduces her taxable income. Her state income tax rate is 4%. If Tracie earns \$1500 every pay period, find the difference in her net income if she has the retirement deduction taken before or after state taxes. **Her net pay is \$4 more by having her retirement deduction taken before state taxes.**

3 Practice/Apply

Study Notebook

Have students—
- add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 7.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

DAILY INTERVENTION

FIND THE ERROR Point out that the order in which the functions are applied is related to the proximity of the function name to the variable x. Since f is closer to x here, it is the first function to be evaluated. Invite students to suggest other ways they can avoid confusing $f \circ g$ with $g \circ f$.

Study Tip

Combining Functions
By combining functions, you can make the evaluation of the functions more efficient.

Example 5 Use Composition of Functions

TAXES **Tyrone Davis has \$180 deducted from every paycheck for retirement. He can have these deductions taken before taxes are applied, which reduces his taxable income. His federal income tax rate is 18%. If Tyrone earns \$2200 every pay period, find the difference in his net income if he has the retirement deduction taken before taxes or after taxes.**

Explore Let x = Tyrone's income per paycheck, $r(x)$ = his income after the deduction for retirement, and $t(x)$ = his income after the deduction for federal income tax.

Plan Write equations for $r(x)$ and $t(x)$.
\$180 is deducted from every paycheck for retirement: $r(x) = x - 180$.

Tyrone's tax rate is 18%: $t(x) = x - 0.18x$.

Solve If Tyrone has his retirement deducted *before* taxes, then his net income is represented by $[t \circ r](2200)$.

$[t \circ r](2200) = t(2200 - 180)$ — Replace x with 2200 in $r(x) = x - 180$.
$= t(2020)$
$= 2020 - 0.18(2020)$ — Replace x with 2020 in $t(x) = x - 0.18x$.
$= 1656.40$

If Tyrone has his retirement deducted *after* taxes, then his net income is represented by $[r \circ t](2200)$.

$[r \circ t](2200) = r[2200 - 0.18(2200)]$ — Replace x with 2200 in $t(x) = x - 0.18x$.
$= r(1804)$
$= 1804 - 180$ — Replace x with 1804 in $r(x) = x - 180$.
$= 1624$

$[t \circ r](2200) = 1656.40$ and $[r \circ t](2200) = 1624$. The difference is \$1656.40 − \$1624 or \$32.40. So, his net pay is \$32.40 more by having his retirement deducted before taxes.

Examine The answer makes sense. Since the taxes are being applied to a smaller amount, less taxes will be deducted from his paycheck.

Check for Understanding

Concept Check

1. Sometimes; sample answer: If $f(x) = x - 2$, $g(x) = x + 8$, then $f \circ g = x + 6$ and $g \circ f = x + 6$.

2. Sample answer: $g(x) = \{(-2, 1), (-1, 2), (4, 3)\}$, $f(x) = \{(1, 7), (2, 9), (3, 3)\}$

1. **Determine** whether the following statement is *always*, *sometimes*, or *never* true. Support your answer with an example.
Given two functions f and g, $f \circ g = g \circ f$.

2. **OPEN ENDED** Write a set of ordered pairs for functions f and g, given that $f \circ g = \{(4, 3), (-1, 9), (-2, 7)\}$.

3. **FIND THE ERROR** Danette and Marquan are finding $[g \circ f](3)$ for $f(x) = x^2 + 4x + 5$ and $g(x) = x - 7$.

Danette
$[g \circ f](3) = g[(3)^2 + 4(3) + 5]$
$= g(26)$
$= 26 - 7$
$= 19$

Marquan
$[g \circ f](3) = f(3 - 7)$
$= f(-4)$
$= (-4)^2 + 4(-4) + 5$
$= 5$

Who is correct? Explain your reasoning. **See margin.**

Answer

3. Danette; $[g \circ f](x) = g[f(x)]$ means to evaluate the f function first and then the g function. Marquan evaluated the functions in the wrong order.

386 Chapter 7 Polynomial Functions

Guided Practice

GUIDED PRACTICE KEY

Exercises	Examples
4–7	1, 2
8, 9	3
10–14	4
15, 16	5

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. **4–5. See margin.**

4. $f(x) = 3x + 4$
$g(x) = 5 + x$

5. $f(x) = x^2 + 3$
$g(x) = x - 4$

For each set of ordered pairs, find $f \circ g$ and $g \circ f$, if they exist.

6. $f = \{(-1, 9), (4, 7)\}$
$g = \{(-5, 4), (7, 12), (4, -1)\}$
$\{(-5, 7), (4, 9)\}$; $\{(4, 12)\}$

7. $f = \{(0, -7), (1, 2), (2, -1)\}$
$g = \{(-1, 10), (2, 0)\}$
$\{(2, -7)\}$; $\{(1, 0), (2, 10)\}$

Find $[g \circ h](x)$ and $[h \circ g](x)$.

8. $g(x) = 2x$
$h(x) = 3x - 4$ **$6x - 8$; $6x - 4$**

9. $g(x) = x + 5$
$h(x) = x^2 + 6$ **$x^2 + 11$; $x^2 + 10x + 31$**

If $f(x) = 3x$, $g(x) = x + 7$, and $h(x) = x^2$, find each value.

10. $f[g(3)]$ **30**

11. $g[h(-2)]$ **11**

12. $h[h(1)]$ **1**

Application

SHOPPING **For Exercises 13–15, use the following information.**
Mai-Lin is shopping for computer software. She finds a CD-ROM program that costs $49.99, but is on sale at a 25% discount. She also has a $5 coupon she can use on the product.

13. Express the price of the CD after the discount and the price of the CD after the coupon using function notation. Let x represent the price of the CD, $p(x)$ represent the price after the 25% discount, and $c(x)$ represent the price after the coupon. **$p(x) = \frac{3}{4}x$; $c(x) = x - 5$**

14. Find $c[p(x)]$ and explain what this value represents.

15. Find $p[c(x)]$ and explain what this value represents.

16. Which method results in the lower sale price? Explain your reasoning.
See margin.

14. $32.50; price of CD when 25% discount is taken and then the coupon is subtracted

15. $33.75; price of CD when coupon is subtracted and then 25% discount is taken

Practice and Apply

Homework Help

For Exercises	See Examples
17–22	1, 2
23–28	3
29–46	4
47–55	5

Extra Practice

See page 844.

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

17. $f(x) = x + 9$
$g(x) = x - 9$

18. $f(x) = 2x - 3$
$g(x) = 4x + 9$

19. $f(x) = 2x^2$
$g(x) = 8 - x$

20. $f(x) = x^2 + 6x + 9$
$g(x) = 2x + 6$

21. $f(x) = x^2 - 1$
$g(x) = \frac{x}{x + 1}$

22. $f(x) = x^2 - x - 6$
$g(x) = \frac{x - 3}{x + 2}$

17–22. See margin.

For each set of ordered pairs, find $f \circ g$ and $g \circ f$ if they exist.

23. $f = \{(1, 1), (0, -3)\}$
$g = \{(1, 0), (-3, 1), (2, 1)\}$

24. $f = \{(1, 2), (3, 4), (5, 4)\}$
$g = \{(2, 5), (4, 3)\}$

25. $f = \{(3, 8), (4, 0), (6, 3), (7, -1)\}$
$g = \{(0, 4), (8, 6), (3, 6), (-1, 8)\}$

26. $f = \{(4, 5), (6, 5), (8, 12), (10, 12)\}$
$g = \{4, 6), (2, 4), (6, 8), (8, 10)\}$

27. $f = \{(2, 5), (3, 9), (-4, 1)\}$
$g = \{(5, -4), (8, 3), (2, -2)\}$
$\{(5, 1), (8, 9)\}$; $\{(2, -4)\}$

28. $f = \{(7, 0), (-5, 3), (8, 3), (-9, 2)\}$
$g = \{(2, -5), (1, 0), (2, -9), (3, 6)\}$
$\{(2, 3), (2, 2)\}$; $\{(-5, 6), (8, 6), (-9, -5)\}$

Find $[g \circ h](x)$ and $[h \circ g](x)$.

29. $g(x) = 4x$
$h(x) = 2x - 1$ **$8x - 4$; $8x - 1$**

30. $g(x) = -5x$
$h(x) = -3x + 1$ **$15x - 5$; $15x + 1$**

31. $g(x) = x + 2$
$h(x) = x^2$

32. $g(x) = x - 4$
$h(x) = 3x^2$

33. $g(x) = 2x$
$h(x) = x^3 + x^2 + x + 1$

34. $g(x) = x + 1$
$h(x) = 2x^2 - 5x + 8$

32. $3x^2 - 4$; $3x^2 - 24x + 48$ **33. $2x^3 + 2x^2 + 2x + 2$; $8x^3 + 4x^2 + 2x + 1$**

23. $\{(1, -3), (-3, 1), (2, 1)\}$; $\{(1, 0), (0, 1)\}$
24. $\{(2, 4), (4, 4)\}$; $\{(1, 5), (3, 3), (5, 3)\}$
25. $\{(0, 0), (8, 3), (3, 3)\}$; $\{(3, 6), (4, 4), (6, 6), (7, 8)\}$
26. $\{(4, 5), (2, 5), (6, 12), (8, 12)\}$; does not exist
31. $x^2 + 2$; $x^2 + 4x + 4$
34. $2x^2 - 5x + 9$; $2x^2 - x + 5$

www.algebra2.com/self_check_quiz

About the Exercises...

Organization by Objective

- **Arithmetic Operations:** 17–22
- **Composition of Functions:** 23–46

Odd/Even Assignments

Exercises 17–46 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 17–45 odd, 47, 48, 56–81

Average: 17–45 odd, 47, 48, 55–81

Advanced: 18–46 even, 49–54, 56–75 (optional: 76–81)

Answers

4. $4x + 9$; $2x - 1$; $3x^2 + 19x + 20$; $\frac{3x + 4}{x + 5}$, $x \neq -5$

5. $x^2 + x - 1$; $x^2 - x + 7$; $x^3 - 4x^2 + 3x - 12$; $\frac{x^2 + 3}{x - 4}$, $x \neq 4$

16. Discount first, then coupon; sample answer: 25% of 49.99 is greater than 25% of 44.99.

17. $2x$; 18; $x^2 - 81$; $\frac{x + 9}{x - 9}$, $x \neq 9$

18. $6x + 6$; $-2x - 12$; $8x^2 + 6x - 27$; $\frac{2x - 3}{4x + 9}$, $x \neq -\frac{9}{4}$

19. $2x^2 - x + 8$; $2x^2 + x - 8$; $-2x^3 + 16x^2$; $\frac{2x^2}{8 - x}$, $x \neq 8$

20. $x^2 + 8x + 15$; $x^2 + 4x + 3$; $2x^3 + 18x^2 + 54x + 54$; $\frac{x + 3}{2}$, $x \neq -3$

21. $\frac{x^3 + x^2 - 1}{x + 1}$, $x \neq -1$; $\frac{x^3 + x^2 - 2x - 1}{x + 1}$, $x \neq -1$; $x^2 - x$, $x \neq -1$; $\frac{x^3 + x^2 - x - 1}{x}$, $x \neq 0$

22. $\frac{x^3 + x^2 - 7x - 15}{x + 2}$, $x \neq -2$; $\frac{x^3 + x^2 - 9x - 9}{x + 2}$, $x \neq -2$; $x^2 - 6x + 9$, $x \neq -2$; $x^2 + 4x + 4$, $x \neq -2, 3$

Lesson 7-7 Operations on Functions 387

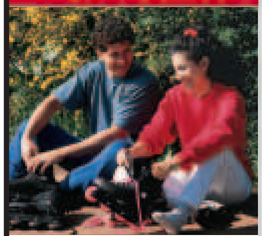
If $f(x) = 4x$, $g(x) = 2x - 1$, and $h(x) = x^2 + 1$, find each value.

35. $f[g(-1)]$ **−12**
36. $h[g(4)]$ **50**
37. $g[f(5)]$ **39**
38. $f[h(-4)]$ **68**
39. $g[g(7)]$ **25**
40. $f[f(-3)]$ **−48**
41. $h\left[f\left(\frac{1}{4}\right)\right]$ **2**
42. $g\left[h\left(-\frac{1}{2}\right)\right]$ **$1\frac{1}{2}$**
43. $[g \circ (f \circ h)](3)$ **79**
44. $[f \circ (h \circ g)](3)$ **104**
45. $[h \circ (g \circ f)](2)$ **226**
46. $[f \circ (g \circ h)](2)$ **36**

POPULATION GROWTH **For Exercises 47 and 48, use the following information.**
From 1990 to 1999, the number of births $b(x)$ in the U.S. can be modeled by the function $b(x) = -27x + 4103$, and the number of deaths $d(x)$ can be modeled by the function $d(x) = 23x + 2164$, where x is the number of years since 1990 and $b(x)$ and $d(x)$ are in thousands.

47. The net increase in population P is the number of births per year minus the number of deaths per year or $P = b - d$. Write an expression that can be used to model the population increase in the U.S. from 1990 to 1999 in function notation. **$P(x) = -50x + 1939$**

48. Assume that births and deaths continue at the same rates. Estimate the net increase in population in 2010. **939,000**



More About . . .

Shopping
Americans spent over \$500 million on inline skates and equipment in 2000.
Source: National Sporting Goods Association

SHOPPING **For Exercises 49–51, use the following information.**
Liluye wants to buy a pair of inline skates that are on sale for 30% off the original price of \$149. The sales tax is 5.75%.

49. Express the price of the inline skates after the discount and the price of the inline skates after the sales tax using function notation. Let x represent the price of the inline skates, $p(x)$ represent the price after the 30% discount, and $s(x)$ represent the price after the sales tax. **$p(x) = 0.70x$; $s(x) = 1.0575x$**

50. Which composition of functions represents the price of the inline skates, $p[s(x)]$ or $s[p(x)]$? Explain your reasoning.

50. $s[p(x)]$; The 30% would be taken off first, and then the sales tax would be calculated on this price.

51. How much will Liluye pay for the inline skates? **\$110.30**

TEMPERATURE **For Exercises 52–54, use the following information.**
There are three temperature scales: Fahrenheit (°F), Celsius (°C), and Kelvin (K). The function $K(C) = C + 273$ can be used to convert Celsius temperatures to Kelvin. The function $C(F) = \frac{5}{9}(F - 32)$ can be used to convert Fahrenheit temperatures to Celsius.

52. Write a composition of functions that could be used to convert Fahrenheit temperatures to Kelvin. **$[K \circ C](F) = \frac{5}{9}(F - 32) + 273$**

53. Find the temperature in Kelvin for the boiling point of water and the freezing point of water if water boils at 212°F and freezes at 32°F. **373 K; 272 K**

54. While performing an experiment, Kimi found the temperature of a solution at different intervals. She needs to record the change in temperature in degrees Kelvin, but only has a thermometer with a Fahrenheit scale. What will she record when the temperature of the solution goes from 158°F to 256°F? **309.67 K**

55. **FINANCE** Kachina pays \$50 each month on a credit card that charges 1.6% interest monthly. She has a balance of \$700. The balance at the beginning of the nth month is given by $f(n) = f(n - 1) + 0.016\,f(n - 1) - 50$. Find the balance at the beginning of the first five months. No additional charges are made on the card. (*Hint:* $f(1) = 700$) **\$700, \$661.20, \$621.78, \$581.73, \$541.04**

Study Guide and Intervention, p. 411 (shown) and p. 412

Arithmetic Operations

Operations with Functions	Sum	$(f + g)(x) = f(x) + g(x)$
	Difference	$(f - g)(x) = f(x) - g(x)$
	Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
	Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

Example **Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for $f(x) = x^2 + 3x - 4$ and $g(x) = 3x - 2$.**

$(f + g)(x) = f(x) + g(x)$ — Addition of functions
$= (x^2 + 3x - 4) + (3x - 2)$ — $f(x) = x^2 + 3x - 4$, $g(x) = 3x - 2$
$= x^2 + 6x - 6$ — Simplify.

$(f - g)(x) = f(x) - g(x)$ — Subtraction of functions
$= (x^2 + 3x - 4) - (3x - 2)$ — $f(x) = x^2 + 3x - 4$, $g(x) = 3x - 2$
$= x^2 - 2$ — Simplify.

$(f \cdot g)(x) = f(x) \cdot g(x)$ — Multiplication of functions
$= (x^2 + 3x - 4)(3x - 2)$ — $f(x) = x^2 + 3x - 4$, $g(x) = 3x - 2$
$= x^2(3x - 2) + 3x(3x - 2) - 4(3x - 2)$ — Distributive Property
$= 3x^3 - 2x^2 + 9x^2 - 6x - 12x + 8$ — Distributive Property
$= 3x^3 + 7x^2 - 18x + 8$ — Simplify.

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ — Division of functions
$= \frac{x^2 + 3x - 4}{3x - 2}, x \neq \frac{2}{3}$ — $f(x) = x^2 + 3x - 4$ and $g(x) = 3x - 2$

Exercises

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

1. $f(x) = 8x - 3$; $g(x) = 4x + 5$
$12x + 2$; $4x - 8$; $32x^2 + 28x - 15$; $\frac{8x - 3}{4x + 5}, x \neq -\frac{5}{4}$

2. $f(x) = x^2 + x - 6$; $g(x) = x - 2$
$x^2 + 2x - 8$; $x^2 - 4$; $x^3 - x^2 - 8x + 12$; $x + 3, x \neq 2$

3. $f(x) = 3x^2 - x + 5$; $g(x) = 2x - 3$
$3x^2 + x + 2$; $3x^2 - 3x + 8$; $6x^3 - 11x^2 + 13x - 15$; $\frac{3x^2 - x + 5}{2x - 3}, x \neq \frac{3}{2}$

4. $f(x) = 2x - 1$; $g(x) = 3x^2 + 11x - 4$
$3x^2 + 13x - 5$; $-3x^2 - 9x + 3$; $6x^3 + 19x^2 - 19x + 4$; $\frac{2x - 1}{(3x - 1)(x + 4)}, x \neq \frac{1}{3}, -4$

5. $f(x) = x^2 - 1$; $g(x) = \frac{1}{x + 1}$
$x^2 - 1 + \frac{1}{x + 1}$; $x^2 - 1 - \frac{1}{x + 1}$; $x - 1$; $x^3 + x^2 - x - 1, x \neq -1$

Skills Practice, p. 413 and Practice, p. 414 (shown)

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

1. $f(x) = 2x + 1$, $g(x) = x - 3$
$3x - 2$; $x + 4$; $2x^2 - 5x - 3$; $\frac{2x + 1}{x - 3}, x \neq 3$

2. $f(x) = 8x^2$, $g(x) = \frac{1}{x^2}$
$\frac{8x^4 + 1}{x^2}, x \neq 0$; $\frac{8x^4 - 1}{x^2}, x \neq 0$; $8, x \neq 0$; $8x^4, x \neq 0$

3. $f(x) = x^2 + 7x + 12$, $g(x) = x^2 - 9$
$2x^2 + 7x + 3$; $7x + 21$; $x^4 + 7x^3 + 3x^2 - 63x - 108$; $\frac{x + 4}{x - 3}, x \neq \pm 3$

For each set of ordered pairs, find $f \circ g$ and $g \circ f$ if they exist.

4. $f = \{(-9, -1), (-1, 0), (3, 4)\}$
$g = \{(0, -9), (-1, 3), (4, -1)\}$
$\{(0, -1), (-1, 4), (4, 0)\}$; $\{(-9, 3), (-1, -9), (3, -1)\}$

5. $f = \{(-4, 3), (0, -2), (1, -2)\}$
$g = \{(-2, 0), (3, 1)\}$
$\{(-2, -2), (3, -2)\}$; $\{(-4, 1), (0, 0), (1, 0)\}$

6. $f = \{(-4, -5), (0, 3), (1, 6)\}$
$g = \{(6, 1), (-5, 0), (3, -4)\}$
$\{(6, 6), (-5, 3), (3, -5)\}$; $\{(-4, 0), (0, -4), (1, 1)\}$

7. $f = \{(0, -3), (1, -3), (6, 8)\}$
$g = \{(8, 2), (-3, 0), (-3, 1)\}$
does not exist; $\{(0, 0), (1, 0), (6, 2)\}$

Find $[g \circ h](x)$ and $[h \circ g](x)$.

8. $g(x) = 3x$, $h(x) = x - 4$ **$3x - 12$; $3x - 4$**
9. $g(x) = -8x$, $h(x) = 2x + 3$ **$-16x - 24$; $-16x + 3$**
10. $g(x) = x + 6$, $h(x) = 3x^2$ **$3x^2 + 6$; $3x^2 + 36x + 108$**
11. $g(x) = x + 3$, $h(x) = 2x^2$ **$2x^2 + 3$; $2x^2 + 12x + 18$**
12. $g(x) = -2x$, $h(x) = x^2 + 3x + 2$ **$-2x^2 - 6x - 4$; $4x^2 - 6x + 2$**
13. $g(x) = x - 2$, $h(x) = 3x^2 + 1$ **$3x^2 - 1$; $3x^2 - 12x + 13$**

If $f(x) = x^2$, $g(x) = 5x$, and $h(x) = x + 4$, find each value.

14. $f[g(1)]$ **25**
15. $g[h(-2)]$ **10**
16. $h[f(4)]$ **20**
17. $f[h(-9)]$ **25**
18. $h[g(-3)]$ **−11**
19. $g[f(8)]$ **320**
20. $h[f(20)]$ **404**
21. $[f \circ (h \circ g)](-1)$ **1**
22. $[f \circ (g \circ h)](4)$ **1600**

23. **BUSINESS** The function $f(x) = 1000 - 0.01x^2$ models the manufacturing cost per item when x items are produced, and $g(x) = 150 - 0.001x^2$ models the service cost per item. Write a function $C(x)$ for the total manufacturing and service cost per item.
$C(x) = 1150 - 0.011x^2$

24. **MEASUREMENT** The formula $f = \frac{n}{12}$ converts inches n to feet f, and $m = \frac{f}{5280}$ converts feet to miles m. Write a composition of functions that converts inches to miles.
$[m \circ f]n = \frac{n}{63{,}360}$

Reading to Learn Mathematics, p. 415 ELL

Pre-Activity Why is it important to combine functions in business?

Read the introduction to Lesson 7-7 at the top of page 383 in your textbook.

Describe two ways to calculate Ms. Coffmon's profit from the sale of 50 birdhouses. (Do not actually calculate her profit.) **Sample answer: 1. Find the revenue by substituting 50 for x in the expression $125x$. Next, find the cost by substituting 50 for x in the expression $65x + 5400$. Finally, subtract the cost from the revenue to find the profit. 2. Form the profit function $p(x) = r(x) - c(x) = 125x - (65x + 5400) = 60x - 5400$. Substitute 50 for x in the expression $60x - 5400$.**

Reading the Lesson

1. Determine whether each statement is *true* or *false*. (Remember that *true* means *always* true.)
   a. If f and g are polynomial functions, then $f + g$ is a polynomial function. **true**
   b. If f and g are polynomial functions, then $\frac{f}{g}$ is a polynomial function. **false**
   c. If f and g are polynomial functions, the domain of the function $f \cdot g$ is the set of all real numbers. **true**
   d. If $f(x) = 3x + 2$ and $g(x) = x - 4$, the domain of the function $\frac{f}{g}$ is the set of all real numbers. **false**
   e. If f and g are polynomial functions, then $(f \circ g)(x) = (g \circ f)(x)$. **false**
   f. If f and g are polynomial functions, then $(f \cdot g)(x) = (g \cdot f)(x)$ **true**

2. Let $f(x) = 2x - 5$ and $g(x) = x^2 + 1$.
   a. Explain in words how you would find $(f \circ g)(-3)$. (Do not actually do any calculations.) **Sample answer: Square −3 and add 1. Take the number you get, multiply it by 2, and subtract 5.**
   b. Explain in words how you would find $(g \circ f)(-3)$. (Do not actually do any calculations.) **Sample answer: Multiply −3 by 2 and subtract 5. Take the number you get, square it, and add 1.**

Helping You Remember

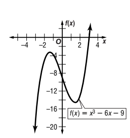
3. Some students have trouble remembering the correct order in which to apply the two original functions when evaluating a composite function. Write three sentences, each of which explains how to do this in a slightly different way. (Hint: Use the word *closest* in the first sentence, the words *inside* and *outside* in the second, and the words *left* and *right* in the third.) **Sample answer: 1. The function that is written closest to the variable is applied first. 2. Work from the inside to the outside. 3. Work from right to left.**

Enrichment, p. 416

Relative Maximum Values

The graph of $f(x) = x^3 - 6x - 9$ shows a relative maximum value somewhere between $f(-2)$ and $f(-1)$. You can obtain a closer approximation by comparing values such as those shown in the table.

To the nearest tenth a relative maximum value for $f(x)$ is -3.3.



x	$f(x)$
−2	−5
−1.5	−3.375
−1.4	−3.344
−1.3	−3.397
−1	−4

Using a calculator to find points, graph each function. To the nearest tenth, find a relative maximum value of the function.

1. $f(x) = x(x^2 - 3)$ **rel. max. of 2.0**
2. $f(x) = x^3 - 3x - 3$ **rel. max. of −1.0**

56. **CRITICAL THINKING** If $f(0) = 4$ and $f(x + 1) = 3f(x) - 2$, find $f(4)$. **244**

57. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

Why is it important to combine functions in business?

Include the following in your answer:

- a description of how to write a new function that represents the profit, using the revenue and cost functions, and
- an explanation of the benefits of combining two functions into one function.

Standardized Test Practice

58. If $h(x) = 7x - 5$ and $g[h(x)] = 2x + 3$, then $g(x) =$ **A**

Ⓐ $\frac{2x + 31}{7}$. Ⓑ $-5x + 8$.
Ⓒ $5x - 8$. Ⓓ $\frac{2x + 26}{7}$.

59. If $f(x) = 4x^4 + 5x^3 - 3x^2 - 14x + 31$ and $g(x) = 7x^3 - 4x^2 + 5x - 42$, then $(f - g)(x) =$ **C**

Ⓐ $4x^4 + 12x^3 - 7x^2 - 9x - 11$. Ⓑ $4x^4 - 2x^3 - 7x^2 - 19x - 11$.
Ⓒ $4x^4 - 2x^3 + x^2 - 19x + 73$. Ⓓ $-3x^4 - 2x^3 - 7x^2 - 19x + 73$.

Maintain Your Skills

Mixed Review

61. $\pm1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm2, \pm3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm6$

63. $x^3 - 4x^2 - 17x + 60$

64. $x^3 - 3x^2 - 34x - 48$

65. $6x^3 - 13x^2 + 9x - 2$

68. $x^4 + x^3 - 14x^2 + 26x - 20$

List all of the possible rational zeros of each function. *(Lesson 7-6)*

60. $r(x) = x^2 - 6x + 8$ **$\pm1, \pm2, \pm4, \pm8$**
61. $f(x) = 4x^3 - 2x^2 + 6$
62. $g(x) = 9x^2 - 1$ **$\pm1, \pm\frac{1}{3}, \pm\frac{1}{9}$**

Write a polynomial function of least degree with integral coefficients that has the given zeros. *(Lesson 7-5)*

63. 5, 3, −4
64. −3, −2, 8
65. $1, \frac{1}{2}, \frac{2}{3}$
66. 6, $2i$ **$x^3 - 6x^2 + 4x - 24$**
67. 3, $3 - 2i$ **$x^3 - 9x^2 + 31x - 39$**
68. −5, 2, $1 - i$

69. **ELECTRONICS** There are three basic things to be considered in an electrical circuit: the flow of the electrical current I, the resistance to the flow Z called impedance, and electromotive force E called voltage. These quantities are related in the formula $E = I \cdot Z$. The current of a circuit is to be $35 - 40j$ amperes. Electrical engineers use the letter j to represent the imaginary unit. Find the impedance of the circuit if the voltage is to be $430 - 330j$ volts. *(Lesson 5-9)* **$10 + 2j$**

Find the inverse of each matrix, if it exists. *(Lesson 4-7)*

70. $\begin{bmatrix} 8 & 6 \\ 7 & 5 \end{bmatrix}$ **$-\frac{1}{2}\begin{bmatrix} 5 & -6 \\ -7 & 8 \end{bmatrix}$**

71. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ **$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$**

72. $\begin{bmatrix} 8 & 4 \\ 6 & 3 \end{bmatrix}$ **does not exist**

73. $\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$ **$\frac{1}{2}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$**

74. $\begin{bmatrix} 6 & -2 \\ 9 & -3 \end{bmatrix}$ **does not exist**

75. $\begin{bmatrix} 2 & 2 \\ 3 & -5 \end{bmatrix}$ **$\frac{1}{16}\begin{bmatrix} 5 & 2 \\ 3 & -2 \end{bmatrix}$**

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation or formula for the specified variable.**
*(To review **solving equations for a variable**, see Lesson 1-3.)*

76. $2x - 3y = 6$, for x **$x = \frac{6 + 3y}{2}$**
77. $4x^2 - 5xy + 2 = 3$, for y **$y = \frac{1 - 4x^2}{-5x}$**
78. $3x + 7xy = -2$, for x **$x = \frac{-2}{3 + 7y}$**
79. $I = prt$, for t **$t = \frac{I}{pr}$**
80. $C = \frac{5}{9}(F - 32)$, for F **$F = \frac{9}{5}C + 32$**
81. $F = G\frac{Mm}{r^2}$, for m **$m = \frac{Fr^2}{GM}$**

4 Assess

Open-Ended Assessment

Modeling In some courses, a function $f(x)$ is modeled by a "machine" that accepts values for x as inputs and then outputs values for $f(x)$. Using this model, ask students to explain how the composition of two functions could be modeled by two such "machines" linked together. Also ask them to use the model to explain how the composition of two functions could be undefined for some initial input values.

Getting Ready for Lesson 7-8

PREREQUISITE SKILL In Lesson 7-8, students will find the inverse of a function. One method used will involve students solving an equation for a variable. Use Exercises 76–81 to determine your students' familiarity with solving equations for a variable.

Assessment Options

Quiz (Lessons 7-6 and 7-7) is available on p. 444 of the *Chapter 7 Resource Masters.*

Answer

57. Answers should include the following.

- **Using the revenue and cost functions, a new function that represents the profit is $p(x) = r(c(x))$.**
- **The benefit of combining two functions into one function is that there are fewer steps to compute and it is less confusing to the general population of people reading the formulas.**

Lesson 7-7 Operations on Functions 389

7-8 Lesson Notes

1 Focus

5-Minute Check Transparency 7-8 Use as a quiz or review of Lesson 7-7.

Mathematical Background notes are available for this lesson on p. 344D.

How are inverse functions related to measurement conversions?

Ask students:

- When would you need to convert from SI units to customary units? **Sample answer: reading a recipe printed in a cookbook that was published in Canada**
- When would you need to convert between units within the customary system? **Sample answer: You might change pounds to ounces in order to estimate the cost per ounce of different grocery items.**
- When you are driving a car at a speed of 55 miles per hour, how many meters would you guess you are traveling each second? **Sample answer: 20 m/s**
- What is the calculated value for $f(55)$? **about 24.4 m/s**

7-8 Inverse Functions and Relations

What You'll Learn

- Find the inverse of a function or relation.
- Determine whether two functions or relations are inverses.

Vocabulary

- inverse relation
- inverse function
- one-to-one

How are inverse functions related to measurement conversions?

Most scientific formulas involve measurements given in SI (International System) units. The SI units for speed are meters per second. However, the United States uses customary measurements such as miles per hour. To convert x miles per hour to an approximate equivalent in meters per second, you can evaluate

$f(x) = \frac{x \cancel{\text{ miles}}}{1 \cancel{\text{ hour}}} \cdot \frac{1600 \text{ meters}}{1 \cancel{\text{ mile}}} \cdot \frac{1 \cancel{\text{ hour}}}{3600 \text{ seconds}}$ or $f(x) = \frac{4}{9}x$. To convert x meters per second to an approximate equivalent in miles per hour, you can evaluate

$g(x) = \frac{x \cancel{\text{ meters}}}{1 \cancel{\text{ second}}} \cdot \frac{3600 \cancel{\text{ seconds}}}{1 \text{ hour}} \cdot \frac{1 \text{ mile}}{1600 \cancel{\text{ meters}}}$ or $g(x) = \frac{9}{4}x$.

Notice that $f(x)$ multiplies a number by 4 and divides it by 9. The function $g(x)$ does the inverse operation of $f(x)$. It divides a number by 4 and multiplies it by 9. The functions $f(x) = \frac{4}{9}x$ and $g(x) = \frac{9}{4}x$ are inverses.

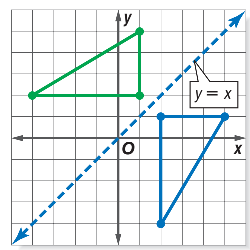
FIND INVERSES Recall that a relation is a set of ordered pairs. The **inverse relation** is the set of ordered pairs obtained by reversing the coordinates of each original ordered pair. The domain of a relation becomes the range of the inverse, and the range of a relation becomes the domain of the inverse.

Key Concept — *Inverse Relations*

- **Words** Two relations are inverse relations if and only if whenever one relation contains the element (a, b), the other relation contains the element (b, a).
- **Example** $Q = \{(1, 2), (3, 4), (5, 6)\}$ $S = \{(2, 1), (4, 3), (6, 5)\}$ Q and S are inverse relations.

Example 1 *Find an Inverse Relation*

GEOMETRY **The ordered pairs of the relation $\{(2, 1), (5, 1), (2, -4)\}$ are the coordinates of the vertices of a right triangle. Find the inverse of this relation and determine whether the resulting ordered pairs are also the vertices of a right triangle.**



To find the inverse of this relation, reverse the coordinates of the ordered pairs.

The inverse of the relation is $\{(1, 2), (1, 5), (-4, 2)\}$.

Plotting the points shows that the ordered pairs also describe the vertices of a right triangle. Notice that the graphs of the relation and the inverse relation are reflections over the graph of $y = x$.

Resource Manager

Workbook and Reproducible Masters

Chapter 7 Resource Masters

- Study Guide and Intervention, pp. 417–418
- Skills Practice, p. 419
- Practice, p. 420
- Reading to Learn Mathematics, p. 421
- Enrichment, p. 422

School-to-Career Masters, p. 14

Teaching Algebra With Manipulatives Masters, pp. 256–257, 258

Transparencies

5-Minute Check Transparency 7-8

Answer Key Transparencies

Technology

Interactive Chalkboard

Study Tip

Reading Math
f^{-1} is read *f inverse* or *the inverse of f.* Note that -1 is not an exponent.

The ordered pairs of **inverse functions** are also related. We can write the inverse of function $f(x)$ as $f^{-1}(x)$.

Key Concept — *Property of Inverse Functions*

Suppose f and f^{-1} are inverse functions. Then, $f(a) = b$ if and only if $f^{-1}(b) = a$.

Let's look at the inverse functions $f(x) = x + 2$ and $f^{-1}(x) = x - 2$.

Evaluate $f(5)$.	Now, evaluate $f^{-1}(7)$.
$f(x) = x + 2$	$f^{-1}(x) = x - 2$
$f(5) = 5 + 2$ or 7	$f^{-1}(7) = 7 - 2$ or 5

Since $f(x)$ and $f^{-1}(x)$ are inverses, $f(5) = 7$ and $f^{-1}(7) = 5$. The inverse function can be found by exchanging the domain and range of the function.

Example 2 Find an Inverse Function

a. Find the inverse of $f(x) = \frac{x+6}{2}$.

Step 1 Replace $f(x)$ with y in the original equation.

$f(x) = \frac{x+6}{2} \rightarrow y = \frac{x+6}{2}$

Step 2 Interchange x and y.

$x = \frac{y+6}{2}$

Step 3 Solve for y.

$x = \frac{y+6}{2}$ Inverse

$2x = y + 6$ Multiply each side by 2.

$2x - 6 = y$ Subtract 6 from each side.

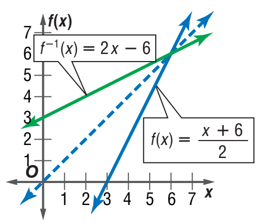
Step 4 Replace y with $f^{-1}(x)$.

$y = 2x - 6 \rightarrow f^{-1}(x) = 2x - 6$

The inverse of $f(x) = \frac{x+6}{2}$ is $f^{-1}(x) = 2x - 6$.

b. Graph the function and its inverse.

Graph both functions on the coordinate plane. The graph of $f^{-1}(x) = 2x - 6$ is the reflection of the graph of $f(x) = \frac{x+6}{2}$ over the line $y = x$.



INVERSES OF RELATIONS AND FUNCTIONS

You can determine whether two functions are inverses by finding both of their compositions. If both equal the **identity function** $I(x) = x$, then the functions are inverse functions.

Key Concept — *Inverse Functions*

- **Words** Two functions f and g are inverse functions if and only if both of their compositions are the identity function.
- **Symbols** $[f \circ g](x) = x$ and $[g \circ f](x) = x$

www.algebra2.com/extra_examples

2 Teach

FIND INVERSES

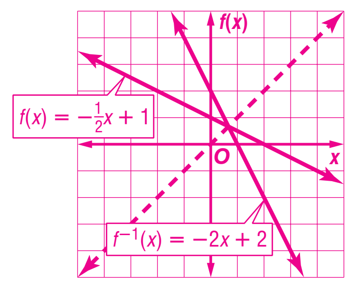
In-Class Examples Power Point®

1 **GEOMETRY** The ordered pairs of the relation {(1, 3), (6, 3), (6, 0), (1, 0)} are the coordinates of the vertices of a rectangle. Find the inverse of this relation and determine whether the resulting ordered pairs are also the coordinates of the vertices of a rectangle. **The inverse of the relation is {(3, 1), (3, 6), (0, 6), (0, 1)}. These ordered pairs also describe the vertices of a rectangle.**

2

a. Find the inverse of $f(x) = -\frac{1}{2}x + 1$. $f^{-1}(x) = -2x + 2$

b. Graph the function and its inverse.



DAILY INTERVENTION

Differentiated Instruction

Logical After completing Example 3 on p. 392 and discussing how to determine if two functions are inverses, challenge students to find two functions, f and g, such that $f(g(x)) \neq g(f(x))$, with one of the two compositions having a value of x. You may wish to have students work in groups to brainstorm as they attempt this puzzler.

INVERSES OF RELATIONS AND FUNCTIONS

In-Class Example

Power Point®

3 Determine whether $f(x) = \frac{3}{4}x - 6$ and $g(x) = \frac{4}{3}x + 8$ are inverse functions. **The functions are inverses since both $[f \circ g](x)$ and $[g \circ f](x)$ equal x.**

Teaching Tip Have students verify the results of Example 3 by graphing the two functions on their graphing calculators and checking that the graphs are reflections over the line $y = x$.

3 Practice/Apply

Study Notebook

Have students—
- *add the definitions/examples of the vocabulary terms to their Vocabulary Builder worksheets for Chapter 7.*
- *write the definitions of inverse relations and inverse functions.*
- *include any other item(s) that they find helpful in mastering the skills in this lesson.*

About the Exercises...

Organization by Objective
- **Find Inverses:** 14–31
- **Inverses of Relations and Functions:** 32–37

Odd/Even Assignments
Exercises 14–37 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide

Basic: 15–37 odd, 41, 44–61

Average: 15–37 odd, 41–61

Advanced: 14–36 even, 38–40, 42–55 (optional: 56–61)

Study Tip

Inverse Functions
Both compositions of $f(x)$ and $g(x)$ must be the identity function for $f(x)$ and $g(x)$ to be inverses. It is necessary to check them both.

Example 3 Verify Two Functions are Inverses

Determine whether $f(x) = 5x + 10$ and $g(x) = \frac{1}{5}x - 2$ are inverse functions.

Check to see if the compositions of $f(x)$ and $g(x)$ are identity functions.

$$\begin{aligned}[f \circ g](x) &= f[g(x)] \\ &= f\left(\frac{1}{5}x - 2\right) \\ &= 5\left(\frac{1}{5}x - 2\right) + 10 \\ &= x - 10 + 10 \\ &= x\end{aligned} \qquad \begin{aligned}[g \circ f](x) &= g[f(x)] \\ &= g(5x + 10) \\ &= \frac{1}{5}(5x + 10) - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

The functions are inverses since both $[f \circ g](x)$ and $[g \circ f](x)$ equal x.

You can also determine whether two functions are inverse functions by graphing. The graphs of a function and its inverse are mirror images with respect to the graph of the identity function $I(x) = x$.

Algebra Activity

Inverses of Functions

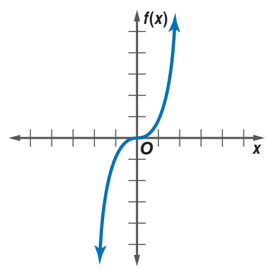
- Use a full sheet of grid paper. Draw and label the x- and y-axes.
- Graph $y = 2x - 3$.
- On the same coordinate plane, graph $y = x$ as a dashed line.
- Place a geomirror so that the drawing edge is on the line $y = x$. Carefully plot the points that are part of the reflection of the original line. Draw a line through the points.

Analyze

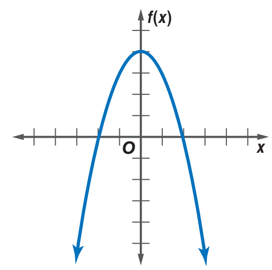
1. What is the equation of the drawn line? $y = \frac{x + 3}{2}$
2. What is the relationship between the line $y = 2x - 3$ and the line that you drew? Justify your answer.
3. Try this activity with the function $y = |x|$. Is the inverse also a function? Explain. **No; the graph does not pass the vertical line test.**

2. They are inverses. $f(x) = 2x - 3$, $g(x) = \frac{x + 3}{2}$; $f[g(x)] = 2\left(\frac{x + 3}{2}\right) - 3 = x$; $g[f(x)] = \frac{(2x - 3) + 3}{2} = x$

When the inverse of a function is a function, then the original function is said to be **one-to-one**. To determine if the inverse of a function is a function, you can use the *horizontal line test*.



No horizontal line can be drawn so that it passes through more than one point. The inverse of this function is a function.



A horizontal line can be drawn that passes through more than one point. The inverse of this function is not a function.

Algebra Activity

Materials: grid paper, geomirror

- Watch for students who mistakenly place the geomirror on the graph of the function instead of the line $y = x$.
- Before students attempt Exercise 3, you may want to provide a brief review of the graphs of absolute value functions.

Check for Understanding

Concept Check

2. Switch x and y in the equation and solve for y.
3. Sample answer: $f(x) = 2x$, $f^{-1}(x) = 0.5x$; $f[f^{-1}(x)] = f^{-1}[f(x)] = x$

1. **Determine** whether $f(x) = 3x + 6$ and $g(x) = x - 2$ are inverses. **no**
2. **Explain** the steps you would take to find an inverse function.
3. **OPEN ENDED** Give an example of a function and its inverse. Verify that the two functions are inverses.
4. **Determine** the values of n for which $f(x) = x^n$ has an inverse that is a function. Assume that n is a whole number. ***n* is an odd whole number.**

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
5, 6	1
7–9, 12, 13	2
10, 11	3

Find the inverse of each relation.

5. {(2, 4), (−3, 1), (2, 8)} **{(4, 2), (1, −3), (8, 2)}**
6. {(1, 3), (1, −1), (1, −3), (1, 1)} **{(3, 1), (−1, 1), (−3, 1), (1, 1)}**

Find the inverse of each function. Then graph the function and its inverse.

7. $f(x) = -x$ $\;f^{-1}(x) = -x$
8. $g(x) = 3x + 1$ $\;g^{-1}(x) = \frac{1}{3}x - \frac{1}{3}$
9. $y = \frac{1}{2}x + 5$ $\;y = 2x - 10$

7–9. See pp. 407A–407H for graphs.

Determine whether each pair of functions are inverse functions.

10. $f(x) = x + 7$
    $g(x) = x - 7$ **yes**
11. $g(x) = 3x - 2$
    $f(x) = \frac{x-2}{3}$ **no**

Application

PHYSICS For Exercises 12 and 13, use the following information.

The acceleration due to gravity is 9.8 meters per second squared (m/s²). To convert to feet per second squared, you can use the following chain of operations:

$$\frac{9.8 \text{ m}}{\text{s}^2} \times \frac{100 \text{ cm}}{1 \text{ m}} \times \frac{1 \text{ in.}}{2.54 \text{ cm}} \times \frac{1 \text{ ft}}{12 \text{ in.}}$$

12. 32.2 ft/s²

12. Find the value of the acceleration due to gravity in feet per second squared.
13. An object is accelerating at 50 feet per second squared. How fast is it accelerating in meters per second squared? **15.24 m/s²**

Practice and Apply

Homework Help

For Exercises	See Examples
14–19	1
20–31, 38–43	2
32–37	3

Extra Practice

See page 844.

20–28. See pp. 407A–407H for graphs.

21. $g^{-1}(x) = -\frac{1}{2}x$
22. $f^{-1}(x) = x + 5$
23. $g^{-1}(x) = x - 4$
24. $f^{-1}(x) = \frac{1}{3}x - 1$
25. $y = -\frac{1}{2}x - \frac{1}{2}$
27. $f^{-1}(x) = \frac{8}{5}x$
28. $f^{-1}(x) = 3x - 12$

29–31. See pp. 407A–407H.

Find the inverse of each relation. 14–19. See margin.

14. {(2, 6), (4, 5), (−3, −1)}
15. {(3, 8), (4, −2), (5, −3)}
16. {(7, −4), (3, 5), (−1, 4), (7, 5)}
17. {(−1, −2), (−3, −2), (−1, −4), (0, 6)}
18. {(6, 11), (−2, 7), (0, 3), (−5, 3)}
19. {(2, 8), (−6, 5), (8, 2), (5, −6)}

Find the inverse of each function. Then graph the function and its inverse.

20. $y = -3$ $\;x = -3$
21. $g(x) = -2x$
22. $f(x) = x - 5$
23. $g(x) = x + 4$
24. $f(x) = 3x + 3$
25. $y = -2x - 1$
26. $y = \frac{1}{3}x$ $\;y = 3x$
27. $f(x) = \frac{5}{8}x$
28. $f(x) = \frac{1}{3}x + 4$
29. $f(x) = \frac{4}{5}x - 7$
30. $g(x) = \frac{2x+3}{6}$
31. $f(x) = \frac{7x-4}{8}$

Determine whether each pair of functions are inverse functions.

32. $f(x) = x - 5$
    $g(x) = x + 5$ **yes**
33. $f(x) = 3x + 4$
    $g(x) = 3x - 4$ **no**
34. $f(x) = 6x + 2$
    $g(x) = x - \frac{1}{3}$ **no**
35. $g(x) = 2x + 8$
    $f(x) = \frac{1}{2}x - 4$ **yes**
36. $h(x) = 5x - 7$
    $g(x) = \frac{1}{5}(x + 7)$ **yes**
37. $g(x) = 2x + 1$
    $f(x) = \frac{x-1}{2}$ **yes**

www.algebra2.com/self_check_quiz

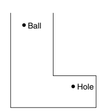
Answers

14. {(6, 2), (5, 4), (−1, −3)}
15. {(8, 3), (−2, 4), (−3, 5)}
16. {(−4, 7), (5, 3), (4, −1), (5, 7)}
17. {(−2, −1), (−2, −3), (−4, −1), (6, 0)}
18. {(11, 6), (7, −2), (3, 0), (3, −5)}
19. {(8, 2), (5, −6), (2, 8), (−6, 5)}

Enrichment, p. 422

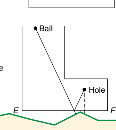
Miniature Golf

In miniature golf, the object of the game is to roll the golf ball into the hole in as few shots as possible. As in the diagram at the right, the hole is often placed so that a direct shot is impossible. Reflections can be used to help determine the direction that the ball should be rolled in order to score a hole-in-one.



Example 1 **Using wall $\overline{EF}$, find the path to use to score a hole-in-one.**

Find the reflection image of the "hole" with respect to $\overline{EF}$ and label it H'. The intersection of $\overline{BH'}$ with wall $\overline{EF}$ is the point at which the shot should be directed.



Study Guide and Intervention, p. 417 (shown) and p. 418

Find Inverses

Inverse Relations	Two relations are inverse relations if and only if whenever one relation contains the element (*a*, *b*), the other relation contains the element (*b*, *a*).
Property of Inverse Functions	Suppose f and f^{-1} are inverse functions. Then $f(a) = b$ if and only if $f^{-1}(b) = a$.

Example **Find the inverse of the function $f(x) = \frac{2}{5}x - \frac{1}{5}$. Then graph the function and its inverse.**

Step 1 Replace $f(x)$ with y in the original equation.

$f(x) = \frac{2}{5}x - \frac{1}{5} \rightarrow y = \frac{2}{5}x - \frac{1}{5}$

Step 2 Interchange x and y.

$x = \frac{2}{5}y - \frac{1}{5}$

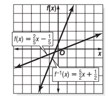
Step 3 Solve for y.

$x = \frac{2}{5}y - \frac{1}{5}$ Inverse

$5x = 2y - 1$ Multiply each side by 5.

$5x + 1 = 2y$ Add 1 to each side.

$\frac{1}{2}(5x + 1) = y$ Divide each side by 2.

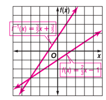


The inverse of $f(x) = \frac{2}{5}x - \frac{1}{5}$ is $f^{-1}(x) = \frac{1}{2}(5x + 1)$.

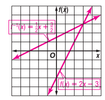
Exercises

Find the inverse of each function. Then graph the function and its inverse.

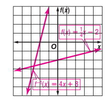
1. $f(x) = \frac{2}{3}x - 1$ $\;f^{-1}(x) = \frac{3}{2}x + \frac{3}{2}$



2. $f(x) = 2x - 3$ $\;f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}$



3. $f(x) = \frac{1}{4}x - 2$ $\;f^{-1}(x) = 4x + 8$



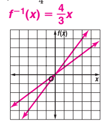
Skills Practice, p. 419 and Practice, p. 420 (shown)

Find the inverse of each relation.

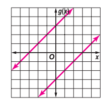
1. {(0, 3), (4, 2), (5, −6)} **{(3, 0), (2, 4), (−6, 5)}**
2. {(−5, 1), (−5, −1), (−5, 8)} **{(1, −5), (−1, −5), (8, −5)}**
3. {(−3, −7), (0, −1), (5, 9), (7, 13)} **{(−7, −3), (−1, 0), (9, 5), (13, 7)}**
4. {(8, −2), (10, 5), (12, 6), (14, 7)} **{(−2, 8), (5, 10), (6, 12), (7, 14)}**
5. {(−5, −4), (1, 2), (3, 4), (7, 8)} **{(−4, −5), (2, 1), (4, 3), (8, 7)}**
6. {(−3, 9), (−2, 4), (0, 0), (1, 1)} **{(9, −3), (4, −2), (0, 0), (1, 1)}**

Find the inverse of each function. Then graph the function and its inverse.

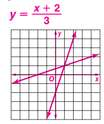
7. $f(x) = \frac{3}{4}x$ $\;f^{-1}(x) = \frac{4}{3}x$



8. $g(x) = 3 + x$ $\;g^{-1}(x) = x - 3$



9. $y = 3x - 2$ $\;y = \frac{x+2}{3}$



Determine whether each pair of functions are inverse functions.

10. $f(x) = x + 6$, $g(x) = x - 6$ **yes**
11. $f(x) = -4x + 1$, $g(x) = \frac{1}{4}(1 - x)$ **yes**
12. $g(x) = 13x - 13$, $h(x) = \frac{1}{13}x - 1$ **no**
13. $f(x) = 2x$, $g(x) = -2x$ **no**
14. $f(x) = \frac{6}{7}x$, $g(x) = \frac{7}{6}x$ **yes**
15. $g(x) = 2x - 8$, $h(x) = \frac{1}{2}x + 4$ **yes**

16. **MEASUREMENT** The points (63, 121), (71, 180), (67, 140), (65, 108), and (72, 165) give the weight in pounds as a function of height in inches for 5 students in a class. Give the points for these students that represent height as a function of weight. **(121, 63), (180, 71), (140, 67), (108, 65), (165, 72)**

REMODELING For Exercises 17 and 18, use the following information.

The Clearys are replacing the flooring in their 15 foot by 18 foot kitchen. The new flooring costs $17.99 per square yard. The formula $f(x) = 9x$ converts square yards to square feet.

17. Find the inverse $f^{-1}(x)$. What is the significance of $f^{-1}(x)$ for the Clearys? $f^{-1}(x) = \frac{x}{9}$; **It will allow them to convert the square footage of their kitchen floor to square yards, so they can then calculate the cost of the new flooring.**
18. What will the new flooring cost the Cleary's? **$539.70**

Reading to Learn Mathematics, p. 421 ELL

Pre-Activity **How are inverse functions related to measurement conversions?**

Read the introduction to Lesson 7-8 at the top of page 390 in your textbook.

A function multiplies a number by 3 and then adds 5 to the result. What does the inverse function do, and in what order? **Sample answer: It first subtracts 5 from the number and then divides the result by 3.**

Reading the Lesson

1. Complete each statement.
   a. If two relations are inverses, the domain of one relation is the **range** of the other.
   b. Suppose that $g(x)$ is a relation and that the point (4, −2) is on its graph. Then a point on the graph of $g^{-1}(x)$ is **(−2, 4)**.
   c. The **horizontal line** test can be used on the graph of a function to determine whether the function has an inverse function.
   d. If you are given the graph of a function, you can find the graph of its inverse by reflecting the original graph over the line with equation **$y = x$**.
   e. If f and g are inverse functions, then $(f \circ g)(x) =$ **x** and $(g \circ f)(x) =$ **x**.
   f. A function has an inverse that is also a function only if the given function is **one-to-one**.
   g. Suppose that $h(x)$ is a function whose inverse is also a function. If $h(5) = 12$, then $h^{-1}(12) =$ **5**.
2. Assume that $f(x)$ is a one-to-one function defined by an algebraic equation. Write the four steps you would follow in order to find the equation for $f^{-1}(x)$.
   1. **Replace $f(x)$ with y in the original equation.**
   2. **Interchange x and y.**
   3. **Solve for y.**
   4. **Replace y with $f^{-1}(x)$.**

Helping You Remember

3. A good way to remember something new is to relate it to something you already know. How are the vertical and horizontal line tests related? **Sample answer: The vertical line test determines whether a relation is a function because the ordered pairs in a function can have no repeated *x*-values. The horizontal line test determines whether a function is one-to-one because a one-to-one function cannot have any repeated *y*-values.**

4 Assess

Open-Ended Assessment

Modeling On a large coordinate grid, have students model the graph of the identity function $f(x) = x$ using a length of string, a piece of raw spaghetti, or something similar. Then place a second length of string or spaghetti to model the graph of a function. Have students model the graph of the inverse of this function.

Getting Ready for Lesson 7-9

PREREQUISITE SKILL Students will graph square root functions and inequalities in Lesson 7-9. Students will need to solve radical equations. Use Exercises 56–61 to determine your students' familiarity with solving radical equations.

Answers

42. $C^{-1}(x) = \frac{9}{5}x + 32$;
$C[C^{-1}(x)] = C^{-1}[C(x)] = x$

43. It can be used to convert Celsius to Fahrenheit.

44. Sample answer: $f(x) = x$ and $f^{-1}(x) = x$ or $f(x) = -x$ and $f^{-1}(x) = -x$

45. Inverses are used to convert between two units of measurement. Answers should include the following.

- Even if it is not necessary, it is helpful to know the imperial units when given the metric units because most measurements in the U.S. are given in imperial units so it is easier to understand the quantities using our system.
- To convert the speed of light from meters per second to miles per hour,

$$f(x) \approx \frac{3.0 \times 10^8 \text{ meters}}{1 \text{ second}} \cdot \frac{3600 \text{ seconds}}{1 \text{ hour}} \cdot \frac{1 \text{ mile}}{1600 \text{ meters}}$$
$$\approx 675{,}000{,}000 \text{ mi/hr}$$

NUMBER GAMES **For Exercises 38–40, use the following information.**
Damaso asked Sophia to choose a number between 1 and 20. He told her to add 7 to that number, multiply by 4, subtract 6, and divide by 2.

38. Write an equation that models this problem. $y = \frac{4(x + 7) - 6}{2}$

39. Find the inverse. $y = \frac{1}{2}x - \frac{11}{2}$

40. Sophia's final number was 35. What was her original number? **12**

41. **SALES** Sales associates at Electronics Unlimited earn \$8 an hour plus a 4% commission on the merchandise they sell. Write a function to describe their income, and find how much merchandise they must sell in order to earn \$500 in a 40-hour week. $I(m) = 320 + 0.04m$; **\$4500**

More About...

Temperature
The Fahrenheit temperature scale was established in 1724 by a physicist named Gabriel Daniel Fahrenheit. The Celsius temperature scale was established in the same year by an astronomer named Anders Celsius.
Source: www.infoplease.com

TEMPERATURE **For Exercises 42 and 43, use the following information.**
A formula for converting degrees Fahrenheit to Celsius is $C(x) = \frac{5}{9}(x - 32)$.

42. Find the inverse $C^{-1}(x)$. Show that $C(x)$ and $C^{-1}(x)$ are inverses.

43. Explain what purpose $C^{-1}(x)$ serves. **42–44. See margin.**

44. **CRITICAL THINKING** Give an example of a function that is its own inverse.

45. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. **See margin.**

How are inverse functions related to measurement conversions?

Include the following items in your answer:
- an explanation of why you might want to know the customary units if you are given metric units even if it is not necessary for you to perform additional calculations, and
- a demonstration of how to convert the speed of light $c = 3.0 \times 10^8$ meters per second to miles per hour.

Standardized Test Practice

46. Which of the following is the inverse of the function $f(x) = \frac{3x - 5}{2}$? **A**

Ⓐ $g(x) = \frac{2x + 5}{3}$ Ⓑ $g(x) = \frac{3x + 5}{2}$ Ⓒ $g(x) = 2x + 5$ Ⓓ $g(x) = \frac{2x - 5}{3}$

47. For which of the following functions is the inverse also a function? **B**

I. $f(x) = x^3$ **II.** $f(x) = x^4$ **III.** $f(x) = -|x|$

Ⓐ I and II only Ⓑ I only Ⓒ I, II, and III Ⓓ III only

Maintain Your Skills

Mixed Review

Find $[g \circ h](x)$ and $[h \circ g](x)$. *(Lesson 7-7)*

48. $g(x) = 4x$, $h(x) = x + 5$

49. $g(x) = 3x + 2$, $h(x) = 2x - 4$

50. $g(x) = x + 4$, $h(x) = x^2 - 3x - 28$

48. $g[h(x)] = 4x + 20$, $h[g(x)] = 4x + 5$
49. $g[h(x)] = 6x - 10$, $h[g(x)] = 6x$
50. $g[h(x)] = x^2 - 3x - 24$, $h[g(x)] = x^2 + 5x - 14$

Find all of the rational zeros of each function. *(Lesson 7-6)*

51. $f(x) = x^3 + 6x^2 - 13x - 42$ **−7, −2, 3**

52. $h(x) = 24x^3 - 86x^2 + 57x + 20$ $-\frac{1}{4}, \frac{4}{3}, \frac{5}{2}$

Evaluate each expression. *(Lesson 5-7)*

53. $16^{\frac{3}{2}}$ **64**

54. $64^{\frac{1}{3}} \cdot 64^{\frac{1}{2}}$ **32**

55. $\frac{3^{\frac{4}{3}}}{81^{\frac{1}{12}}}$ **3**

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation.**
(To review ***solving radical equations****, see Lesson 5-8.)*

56. $\sqrt{x} - 5 = -3$ **4**

57. $\sqrt{x + 4} = 11$ **117**

58. $12 - \sqrt{x} = -2$ **196**

59. $\sqrt{x - 5} = \sqrt{2x + 2}$ **−7**

60. $\sqrt{x - 3} = \sqrt{2} - \sqrt{x}$ $\frac{25}{8}$

61. $3 - \sqrt{x} = \sqrt{x - 6}$ $\frac{25}{4}$

394 Chapter 7 Polynomial Functions

7-9 Square Root Functions and Inequalities

What You'll Learn

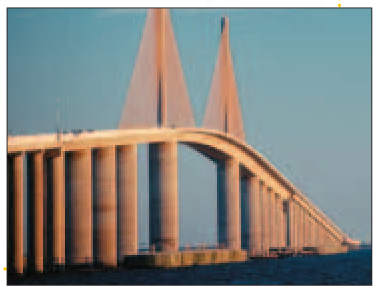
- Graph and analyze square root functions.
- Graph square root inequalities.

Vocabulary

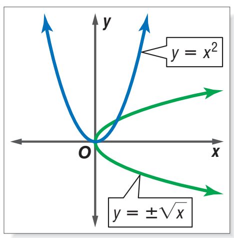
- square root function
- square root inequality

How are square root functions used in bridge design?

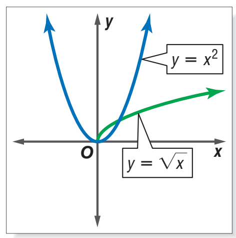
The Sunshine Skyway Bridge across Tampa Bay, Florida, is supported by 21 steel cables, each 9 inches in diameter. The amount of weight that a steel cable can support is given by $w = 8d^2$, where d is the diameter of the cable in inches and w is the weight in tons. If you need to know what diameter a steel cable should have to support a given weight, you can use the equation $d = \sqrt{\frac{w}{8}}$.



SQUARE ROOT FUNCTIONS If a function contains a square root of a variable, it is called a **square root function**. The inverse of a quadratic function is a square root function only if the range is restricted to nonnegative numbers.



$y = \pm\sqrt{x}$ is not a function.



$y = \sqrt{x}$ is a function.

In order for a square root to be a real number, the radicand cannot be negative. When graphing a square root function, determine when the radicand would be negative and exclude those values from the domain.

Example 1 Graph a Square Root Function

Graph $y = \sqrt{3x + 4}$. State the domain, range, x- and y-intercepts.

Since the radicand cannot be negative, identify the domain.

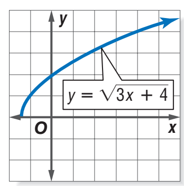
$3x + 4 \geq 0$ — Write the expression inside the radicand as ≥ 0.

$x \geq -\frac{4}{3}$ — Solve for x.

The x-intercept is $-\frac{4}{3}$.

Make a table of values and graph the function. From the graph, you can see that the domain is $x \geq -\frac{4}{3}$, and the range is $y \geq 0$. The y-intercept is 2.

x	y
$-\frac{4}{3}$	0
−1	1
0	2
2	3.2
4	4



7-9 Lesson Notes

1 Focus

5-Minute Check Transparency 7-9 Use as a quiz or review of Lesson 7-8.

Mathematical Background notes are available for this lesson on p. 344D.

How are square root functions used in bridge design?

Ask students:

- Using the given data and formula, how much total weight can be supported by the bridge cables on the Sunshine Skyway Bridge? **13,608 tons**
- If the bridge had been designed to support 20,000 tons, how many 9-inch cables would have been used in the bridge design? **31 cables**

2 Teach

SQUARE ROOT FUNCTIONS

Teaching Tip In the graph on the right, point out the two graphs and lead students to see how the domain and range of the graph of $y = \sqrt{x}$ have been restricted.

Resource Manager

Workbook and Reproducible Masters

Chapter 7 Resource Masters

- Study Guide and Intervention, pp. 423–424
- Skills Practice, p. 425
- Practice, p. 426
- Reading to Learn Mathematics, p. 427
- Enrichment, p. 428
- Assessment, p. 444

Transparencies

5-Minute Check Transparency 7-9

Answer Key Transparencies

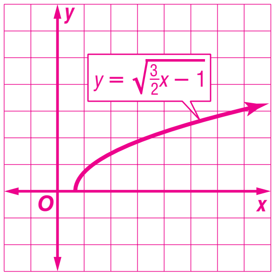
Technology

Interactive Chalkboard

In-Class Examples

Power Point®

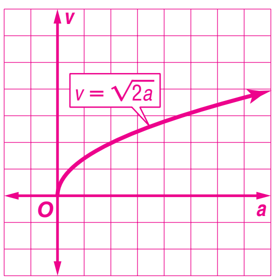
1 Graph $y = \sqrt{\frac{3}{2}x - 1}$. State the domain, range, x- and y-intercepts.



The domain is $x \geq \frac{2}{3}$, and the range is $y \geq 0$. The x-intercept is $\frac{2}{3}$. There is no y-intercept.

2 **PHYSICS** When an object is spinning in a circular path of radius 2 meters with velocity v, in meters per second, the centripetal acceleration a, in meters per second squared, is directed toward the center of the circle. The velocity v and acceleration a of the object are related by the function $v = \sqrt{2a}$.

a. Graph the function. State the domain and range. **The domain is $a \geq 0$, and the range is $v \geq 0$.**



b. What would be the centripetal acceleration of an object spinning along the circular path with a velocity of 4 meters per second? **8 m/s²**

More About...

Submarines

Submarines were first used by The United States in 1776 during the Revolutionary War.

Source: www.infoplease.com

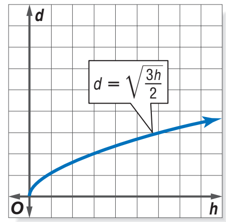
Example 2 Solve a Square Root Problem

SUBMARINES **A lookout on a submarine is h feet above the surface of the water. The greatest distance d in miles that the lookout can see on a clear day is given by the square root of the quantity h multiplied by $\frac{3}{2}$.**

a. Graph the function. State the domain and range.

The function is $d = \sqrt{\frac{3h}{2}}$. Make a table of values and graph the function.

h	d
0	0
2	$\sqrt{3}$ or 1.73
4	$\sqrt{6}$ or 2.45
6	$\sqrt{9}$ or 3.00
8	$\sqrt{12}$ or 3.46
10	$\sqrt{15}$ or 3.87



The domain is $h \geq 0$, and the range is $d \geq 0$.

b. A ship is 3 miles from a submarine. How high would the submarine have to raise its periscope in order to see the ship?

$d = \sqrt{\frac{3h}{2}}$ Original equation

$3 = \sqrt{\frac{3h}{2}}$ Replace d with 3.

$9 = \frac{3h}{2}$ Square each side.

$18 = 3h$ Multiply each side by 2.

$6 = h$ Divide each side by 3.

The periscope would have to be 6 feet above the water. Check this result on the graph.

Graphs of square root functions can be transformed just like quadratic functions.

Graphing Calculator Investigation

Square Root Functions

You can use a TI-83 Plus graphing calculator to graph square root functions. Use [2nd] [$\sqrt{\ }$] to enter the functions in the Y= list.

Think and Discuss **1–3. See pp. 407A–407H for graphs.**

1. Graph $y = \sqrt{x}$, $y = \sqrt{x} + 1$, and $y = \sqrt{x} - 2$ in the viewing window [−2, 8] by [−4, 6]. State the domain and range of each function and describe the similarities and differences among the graphs.
2. Graph $y = \sqrt{x}$, $y = \sqrt{2x}$, and $y = \sqrt{8x}$ in the viewing window [0, 10] by [0, 10]. State the domain and range of each function and describe the similarities and differences among the graphs.
3. Make a conjecture on how you could write an equation that translates the parent graph $y = \sqrt{x}$ to the left three units. Test your conjecture with the graphing calculator. **$y = \sqrt{x + 3}$**

1. D: $x \geq 0$, R: $y \geq 0$; D: $x \geq 0$, R: $y \geq 1$; D: $x \geq 0$, R: $y \geq -2$; Graphs are the same except they are translated vertically.

2. D: $x \geq 0$, R: $y \geq 0$; D: $x \geq 0$, R: $y \geq 0$; D: $x \geq 0$, R: $y \geq 0$; Graphs are the same except they get increasingly less steep.

Graphing Calculator Investigation

Square Roots Students who have worked with non-graphing calculators will likely be used to finding square roots by typing a value first and then pressing the square root key. On a graphing calculator, the square root key is pressed first, followed by the expression whose square root is to be found. Point out that the calculator treats the radical symbol like an open set of parentheses. If students want to graph $y = \sqrt{\frac{3}{2}x - 1}$, they need to enter a right parenthesis after the 1.

396 Chapter 7 Polynomial Functions

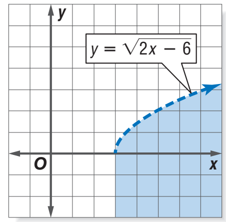
SQUARE ROOT INEQUALITIES

A **square root inequality** is an inequality involving square roots. You can use what you know about square root functions to graph square root inequalities.

Example 3 Graph a Square Root Inequality

a. Graph $y < \sqrt{2x - 6}$.

Graph the related equation $y = \sqrt{2x - 6}$. Since the boundary should not be included, the graph should be dashed.



The domain includes values for $x \geq 3$, so the graph is to the right of $x = 3$. Select a point and test its ordered pair.

Test (4, 1).

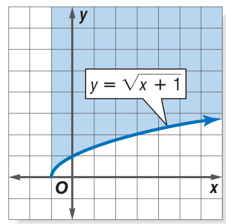
$1 < \sqrt{2(4) - 6}$

$1 < \sqrt{2}$ true

Shade the region that includes the point (4, 1).

b. Graph $y \geq \sqrt{x + 1}$.

Graph the related equation $y = \sqrt{x + 1}$.



The domain includes values for $x \geq -1$, so the graph includes $x = -1$ and the values of x to the right of $x = -1$. Select a point and test its ordered pair.

Test (2, 1).

$y \geq \sqrt{x + 1}$.

$1 \geq \sqrt{2 + 1}$

$1 \geq \sqrt{3}$ false

Shade the region that does not include (2, 1).

Check for Understanding

Concept Check 1–2. See margin.

1. **Explain** why the inverse of $y = 3x^2$ is not a square root function.
2. **Describe** the difference between the graphs of $y = \sqrt{x} - 4$ and $y = \sqrt{x - 4}$.
3. **OPEN ENDED** **Write** a square root function with a domain of $\{x \mid x \geq 2\}$. Sample answer: $y = \sqrt{2x - 4}$

Guided Practice

GUIDED PRACTICE KEY	
Exercises	Examples
4–7	1
8–11	3
12, 13	2

Graph each function. State the domain and range of the function.

4. $y = \sqrt{x} + 2$ D: $x \geq 0$, R: $y \geq 2$
5. $y = \sqrt{4x}$ D: $x \geq 0$; R: $y \geq 0$
6. $y = 3 - \sqrt{x}$ D: $x \geq 0$; R: $y \leq 3$
7. $y = \sqrt{x - 1} + 3$ D: $x \geq 1$; R: $y \geq 3$

4–7. See pp. 407A–407H for graphs.

Graph each inequality. 8–11. See pp. 407A–407H.

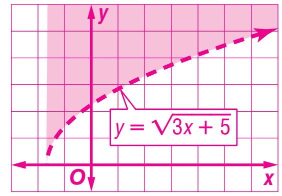
8. $y \leq \sqrt{x - 4} + 1$
9. $y > \sqrt{2x + 4}$
10. $y < 3 - \sqrt{5x + 1}$
11. $y \geq \sqrt{x + 2} - 1$

www.algebra2.com/extra_examples

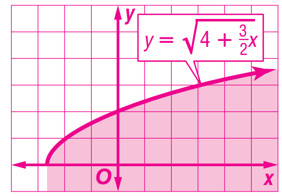
SQUARE ROOT INEQUALITIES

In-Class Example Power Point®

3 **a.** Graph $y > \sqrt{3x + 5}$.



b. Graph $y \leq \sqrt{4 + \frac{3}{2}x}$.



3 Practice/Apply

Study Notebook

Have students—

- complete the definitions/examples for the remaining terms on their Vocabulary Builder worksheets for Chapter 7.
- include any other item(s) that they find helpful in mastering the skills in this lesson.

Answers

1. In order for it to be a square root function, only the nonnegative range can be considered.
2. Both have the shape of the graph of $y = \sqrt{x}$, but $y = \sqrt{x} - 4$ is shifted down 4 units, and $y = \sqrt{x - 4}$ is shifted to the right 4 units.

DAILY INTERVENTION

Differentiated Instruction

Auditory/Musical Divide the class into groups of 4 to 6 students. Challenge each group to give themselves a rock or rap group name based on the vocabulary in this lesson, such as "The Intercepts." Have students write a musical verse about some of the key facts in the lesson, such as the domain, range, and intercepts of a graph.

Application

FIREFIGHTING **For Exercises 12 and 13, use the following information.**
When fighting a fire, the velocity v of water being pumped into the air is the square root of twice the product of the maximum height h and g, the acceleration due to gravity (32 ft/s^2).

12. Determine an equation that will give the maximum height of the water as a function of its velocity. $v = \sqrt{2gh}$

13. The Coolville Fire Department must purchase a pump that is powerful enough to propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 ft/s meet the fire department's need? Explain. **Yes; sample answer: the advertised pump will reach a maximum height of 87.9 ft.**

Practice and Apply

Homework Help

For Exercises	See Examples
14–25	1
26–31	3
32–34	2

Extra Practice

See page 844.

14. D: $x \geq 0$, R: $y \geq 0$
15. D: $x \geq 0$, R: $y \leq 0$
16. D: $x \geq 0$, R: $y \leq 0$
17. D: $x \geq 0$, R: $y \geq 0$
18. D: $x \geq -2$, R: $y \geq 0$
19. D: $x \geq 7$, R: $y \geq 0$
20. D: $x \geq -0.5$, R: $y \leq 0$
21. D: $x \geq 0.6$, R: $y \geq 0$
22. D: $x \geq -6$, R: $y \geq -3$
23. D: $x \geq -4$, R: $y \leq 5$
24. D: $x \geq 2$, R: $y \geq 4$
25. D: $x \leq 0.75$, R: $y \geq 3$

Graph each function. State the domain and range of each function.

14. $y = \sqrt{3x}$ **15.** $y = -\sqrt{5x}$ **16.** $y = -4\sqrt{x}$

17. $y = \frac{1}{2}\sqrt{x}$ **18.** $y = \sqrt{x + 2}$ **19.** $y = \sqrt{x - 7}$

20. $y = -\sqrt{2x + 1}$ **21.** $y = \sqrt{5x - 3}$ **22.** $y = \sqrt{x + 6} - 3$

23. $y = 5 - \sqrt{x + 4}$ **24.** $y = \sqrt{3x - 6} + 4$ **25.** $y = 2\sqrt{3 - 4x} + 3$

14–25. See pp. 407A–407H for graphs.

Graph each inequality. **26–31. See pp. 407A–407H.**

26. $y \leq -6\sqrt{x}$ **27.** $y < \sqrt{x + 5}$ **28.** $y > \sqrt{2x + 8}$

29. $y \geq \sqrt{5x - 8}$ **30.** $y \geq \sqrt{x - 3} + 4$ **31.** $y < \sqrt{6x - 2} + 1$

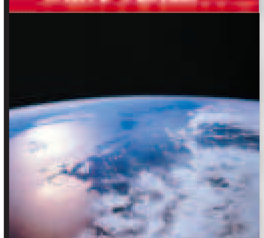
32. ROLLER COASTERS The velocity of a roller coaster as it moves down a hill is $v = \sqrt{v_0^2 + 64h}$, where v_0 is the initial velocity and h is the vertical drop in feet. An engineer wants a new coaster to have a velocity of 90 feet per second when it reaches the bottom of the hill. If the initial velocity of the coaster at the top of the hill is 10 feet per second, how high should the engineer make the hill? **125 ft**

AEROSPACE **For Exercises 33 and 34, use the following information.**
The force due to gravity decreases with the square of the distance from the center of Earth. So, as an object moves further from Earth, its weight decreases. The radius of Earth is approximately 3960 miles. The formula relating weight and distance is $r = \sqrt{\frac{3960^2 W_E}{W_S}} - 3960$, where W_E represents the weight of a body on Earth, W_S represents the weight of a body a certain distance from the center of Earth, and r represents the distance of an object above Earth's surface.

33. An astronaut weighs 140 pounds on Earth and 120 pounds in space. How far is he above Earth's surface? **317.29 mi**

34. An astronaut weighs 125 pounds on Earth. What is her weight in space if she is 99 miles above the surface of Earth? **119 lb**

More About . . .



Aerospace

The weight of a person is equal to the product of the person's mass and the acceleration due to Earth's gravity. Thus, as a person moves away from Earth, the person's weight decreases. However, mass remains constant.

35. RESEARCH Use the Internet or another resource to find the weights, on Earth, of several space shuttle astronauts and the average distance they were from Earth during their missions. Use this information to calculate their weights while in orbit. **See students' work.**

36. CRITICAL THINKING Recall how values of a, h, and k can affect the graph of a quadratic function of the form $y = a(x - h)^2 + k$. Describe how values of a, h, and k can affect the graph of a square root function of the form $y = a\sqrt{x - h} + k$. **See margin.**

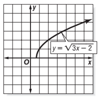
Study Guide and Intervention, p. 423 (shown) and p. 424

Square Root Functions A function that contains the square root of a variable expression is a **square root function**.

Example **Graph $y = \sqrt{3x - 2}$. State its domain and range.**

Since the radicand cannot be negative, $3x - 2 \geq 0$ or $x \geq \frac{2}{3}$.
The x-intercept is $\frac{2}{3}$. The range is $y \geq 0$.
Make a table of values and graph the function.

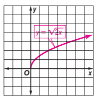
x	y
$\frac{2}{3}$	0
1	1
2	2
3	$\sqrt{7}$



Exercises

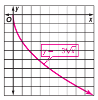
Graph each function. State the domain and range of the function.

1. $y = \sqrt{2x}$



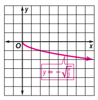
D: $x \geq 0$; R: $y \geq 0$

2. $y = -3\sqrt{x}$



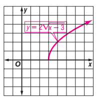
D: $x \geq 0$; R: $y \leq 0$

3. $y = -\sqrt{\frac{x}{2}}$



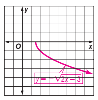
D: $x \geq 0$; R: $y \leq 0$

4. $y = 2\sqrt{x - 3}$



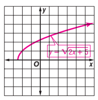
D: $x \geq 3$; R: $y \geq 0$

5. $y = -\sqrt{2x - 3}$



D: $x \geq \frac{3}{2}$; R: $y \leq 0$

6. $y = \sqrt{2x + 5}$

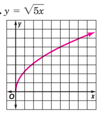


D: $x \geq -\frac{5}{2}$; R: $y \geq 0$

Skills Practice, p. 425 and Practice, p. 426 (shown)

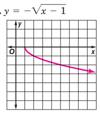
Graph each function. State the domain and range of each function.

1. $y = \sqrt{5x}$



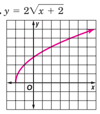
D: $x \geq 0$, R: $y \geq 0$

2. $y = -\sqrt{x - 1}$



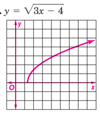
D: $x \geq 1$, R: $y \leq 0$

3. $y = 2\sqrt{x + 2}$



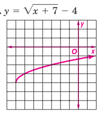
D: $x \geq -2$, R: $y \geq 0$

4. $y = \sqrt{3x - 4}$



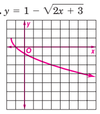
D: $x \geq \frac{4}{3}$, R: $y \geq 0$

5. $y = \sqrt{x + 7} - 4$



D: $x \geq -7$, R: $y \geq -4$

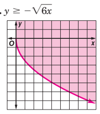
6. $y = 1 - \sqrt{2x + 3}$



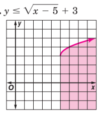
D: $x \geq -\frac{3}{2}$, R: $y \leq 1$

Graph each inequality.

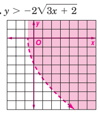
7. $y \geq -\sqrt{6x}$



8. $y \leq \sqrt{x - 5} + 3$



9. $y > -2\sqrt{3x + 2}$



10. ROLLER COASTERS The velocity of a roller coaster as it moves down a hill is $v = \sqrt{v_0^2 + 64h}$, where v_0 is the initial velocity and h is the vertical drop in feet. If $v = 70$ feet per second and $v_0 = 8$ feet per second, find h. **about 75.6 ft**

11. WEIGHT Use the formula $d = \sqrt{\frac{3960^2 W_E}{W_s}} - 3960$, which relates distance from Earth d in miles to weight. If an astronaut's weight on Earth W_E is 148 pounds and in space W_s is 115 pounds, how far from Earth is the astronaut? **about 532 mi**

Reading to Learn Mathematics, p. 427 ELL

Pre-Activity **How are square root functions used in bridge design?**

Read the introduction to Lesson 7-9 at the top of page 395 in your textbook.

If the weight to be supported by a steel cable is doubled, should the diameter of the support cable also be doubled? If not, by what number should the diameter be multiplied?

no; $\sqrt{2}$

Reading the Lesson

1. Match each square root function from the list on the left with its domain and range from the list on the right.

a. $y = \sqrt{x}$ **iv**	**i.** domain: $x \geq 0$; range: $y \geq 3$
b. $y = \sqrt{x + 3}$ **viii**	**ii.** domain: $x \geq 0$; range: $y \leq 0$
c. $y = \sqrt{x} + 3$ **i**	**iii.** domain: $x \geq 0$; range: $y \leq -3$
d. $y = \sqrt{x - 3}$ **v**	**iv.** domain: $x \geq 0$; range: $y \geq 0$
e. $y = -\sqrt{x}$ **ii**	**v.** domain: $x \geq 3$; range: $y \geq 0$
f. $y = -\sqrt{x - 3}$ **vii**	**vi.** domain: $x \leq 3$; range: $y \geq 3$
g. $y = \sqrt{3 - x} + 3$ **vi**	**vii.** domain: $x \geq 3$; range: $y \leq 0$
h. $y = -\sqrt{x} - 3$ **iii**	**viii.** domain: $x \geq -3$; range: $y \geq 0$

2. The graph of the inequality $y \leq \sqrt{3x + 6}$ is a shaded region. Which of the following points lie inside this region?

(3, 0) (2, 4) (5, 2) (4, −2) (6, 6)

(3, 0), (5, 2), (4, −2)

Helping You Remember

3. A good way to remember something is to explain it to someone else. Suppose you are studying this lesson with a classmate who thinks that you cannot have square root functions because every positive real number has two square roots. How would you explain the idea of square root functions to your classmate?

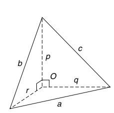
Sample answer: To form a square root function, choose *either* the positive or negative square root. For example, $y = \sqrt{x}$ and $y = -\sqrt{x}$ are two separate functions.

Enrichment, p. 428

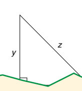
Reading Algebra

If two mathematical problems have basic structural similarities, they are said to be **analogous.** Using analogies is one way of discovering and proving new theorems.

The following numbered sentences discuss a three-dimensional analogy to the Pythagorean theorem.



01 Consider a tetrahedron with three perpendicular faces that meet at vertex O.
02 Suppose you want to know how the areas A, B, and C of the three faces that meet at vertex O are related to the area D of the face opposite vertex O.
03 It is natural to expect a formula analogous to the Pythagorean theorem $z^2 = x^2 + y^2$, which is true for a similar situation in two dimensions.
04 To explore the three-dimensional case, you might guess a formula and then try to prove it.
05 Two re[illegible]able guesses are $D^3 = A^3 + B^3 + C^3$ and [illegible]



37. WRITING IN MATH Answer the question that was posed at the beginning of the lesson. See margin.

How are square root functions used in bridge design?

Include the following in your answer:

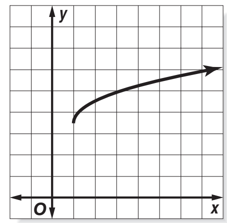
- the weights for which a diameter less than 1 is reasonable, and
- the weight that the Sunshine Skyway Bridge can support.

Standardized Test Practice

38. What is the domain of $f(x) \geq \sqrt{5x - 3}$? C

(A) $\left\{x \mid x > \frac{3}{5}\right\}$ (B) $\left\{x \mid x > -\frac{3}{5}\right\}$ (C) $\left\{x \mid x \geq \frac{3}{5}\right\}$ (D) $\left\{x \mid x \geq -\frac{3}{5}\right\}$

39. Given the graph of the square root function at the right, which of the following must be true? D



I. The domain is all real numbers.

II. The function is $y = \sqrt{x} + 3.5$.

III. The range is $\{y \mid y \geq 3.5\}$.

(A) I only (B) I, II, and III

(C) II and III (D) III only

Maintain Your Skills

Mixed Review

Determine whether each pair of functions are inverse functions. *(Lesson 7-8)*

40. $f(x) = 3x$ yes
$g(x) = \frac{1}{3}x$

41. $f(x) = 4x - 5$ no
$g(x) = \frac{1}{4}x - \frac{5}{16}$

42. $f(x) = \frac{3x + 2}{7}$ yes
$g(x) = \frac{7x - 2}{3}$

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. *(Lesson 7-7)*

43. $f(x) = x + 5$
$g(x) = x - 3$

44. $f(x) = 10x - 20$
$g(x) = x - 2$

45. $f(x) = 4x^2 - 9$
$g(x) = \frac{1}{2x + 3}$

43–45. See margin.

46. ENTERTAINMENT A magician asked a member of his audience to choose any number. He said, "Multiply your number by 3. Add the sum of your number and 8 to that result. Now divide by the sum of your number and 2." The magician announced the final answer without asking the original number. What was the final answer? How did he know what it was? *(Lesson 5-4)*

46. 4; If x is your number, you can write the expression $\frac{3x + x + 8}{x + 2}$, which equals 4 after dividing the numerator and denominator by the GCF, $x + 2$.

Simplify. *(Lesson 5-2)*

47. $(x + 2)(2x - 8)$
$2x^2 - 4x - 16$

48. $(3p + 5)(2p - 4)$
$6p^2 - 2p - 20$

49. $(a^2 + a + 1)(a - 1)$
$a^3 - 1$

WebQuest Internet Project

Population Explosion

It is time to complete your project. Use the information and data you have gathered about the population to prepare a Web page. Be sure to include graphs, tables, and equations in the presentation.

www.algebra2.com/webquest

www.algebra2.self_check_quiz

About the Exercises...

Organization by Objective

- **Square Root Functions:** 14–25
- **Square Root Inequalities:** 26–31

Odd/Even Assignments

Exercises 14–31 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Alert! Exercise 35 involves research on the Internet or other reference materials.

Assignment Guide

Basic: 15–31 odd, 33–49

Average: 15–31 odd, 33–49

Advanced: 14–32 even, 33–34, 36–49

4 Assess

Open-Ended Assessment

Writing Have students write a paragraph explaining why the domain and range of square root functions and square root inequalities must be restricted.

Assessment Options

Quiz (Lesson 7-9) is available on p. 444 of the *Chapter 7 Resource Masters*.

Answers

43. $2x + 2$; 8; $x^2 + 2x - 15$; $\frac{x + 5}{x - 3}$, $x \neq 3$

44. $11x - 22$; $9x - 18$; $10x^2 - 40x + 40$; 10, $x \neq 2$

45. $\frac{8x^3 + 12x^2 - 18x - 26}{2x + 3}$, $x \neq -\frac{3}{2}$; $\frac{8x^3 + 12x^2 - 18x - 28}{2x + 3}$, $x \neq -\frac{3}{2}$; $2x - 3$, $x \neq -\frac{3}{2}$; $8x^3 + 12x^2 - 18x - 27$, $x \neq -\frac{3}{2}$

Answers

36. If a is negative, the graph is reflected over the x-axis. The larger the value of a, the less steep the graph. If h is positive, the origin is translated to the right, and if h is negative, the origin is translated to the left. When k is positive, the origin is translated up, and when k is negative, the origin is translated down.

37. Square root functions are used in bridge design because the engineers must determine what diameter of steel cable needs to be used to support a bridge based on its weight. Answers should include the following.

- Sample answer: When the weight to be supported is less than 8 tons.
- 13,608 tons

Chapter 7 Study Guide and Review

Vocabulary and Concept Check

- This alphabetical list of vocabulary terms in Chapter 7 includes a page reference where each term was introduced.
- **Assessment** A vocabulary test/review for Chapter 7 is available on p. 442 of the *Chapter 7 Resource Masters*.

Lesson-by-Lesson Review

For each lesson,

- the main ideas are summarized,
- additional examples review concepts, and
- practice exercises are provided.

Vocabulary PuzzleMaker

ELL The Vocabulary PuzzleMaker software improves students' mathematics vocabulary using four puzzle formats—crossword, scramble, word search using a word list, and word search using clues. Students can work on a computer screen or from a printed handout.

MindJogger Videoquizzes

ELL MindJogger Videoquizzes provide an alternative review of concepts presented in this chapter. Students work in teams in a game show format to gain points for correct answers. The questions are presented in three rounds.

Round 1 Concepts (5 questions)
Round 2 Skills (4 questions)
Round 3 Problem Solving (4 questions)

Chapter 7 Study Guide and Review

Vocabulary and Concept Check

Complex Conjugates Theorem (p. 374)
composition of functions (p. 384)
degree of a polynomial (p. 346)
depressed polynomial (p. 366)
Descartes' Rule of Signs (p. 372)
end behavior (p. 348)
Factor Theorem (p. 366)
Fundamental Theorem of Algebra (p. 371)
identity function (p. 391)
Integral Zero Theorem (p. 378)
inverse function (p. 391)
inverse relation (p. 390)
leading coefficients (p. 346)
Location Principle (p. 353)
one-to-one (p. 392)
polynomial function (p. 347)
polynomial in one variable (p. 346)
quadratic form (p. 360)
Rational Zero Theorem (p. 378)
relative maximum (p. 354)
relative minimum (p. 354)
Remainder Theorem (p. 365)
square root function (p. 395)
square root inequality (p. 397)
synthetic substitution (p. 365)

Choose the letter that best matches each statement or phrase.

1. A point on the graph of a polynomial function that has no other nearby points with lesser y-coordinates is a ______. **f**
2. The ______ is the coefficient of the term in a polynomial function with the highest degree. **d**
3. The ______ says that in any polynomial function, if an imaginary number is a zero of that function, then its conjugate is also a zero. **a**
4. When a polynomial is divided by one of its binomial factors, the quotient is called a(n) ______. **b**
5. $(x^2)^2 - 17(x^2) + 16 = 0$ is written in ______. **e**
6. $f(x) = 6x - 2$ and $g(x) = \frac{x+2}{6}$ are ______ since $[f \circ g](x)$ and $[g \circ f](x) = x$. **c**

a. Complex Conjugates Theorem
b. depressed polynomial
c. inverse functions
d. leading coefficient
e. quadratic form
f. relative minimum

Lesson-by-Lesson Review

7-1 Polynomial Functions

See pages 346–352.

Concept Summary

- The degree of a polynomial function in one variable is determined by the greatest exponent of its variable.

Example Find $p(a + 1)$ if $p(x) = 5x - x^2 + 3x^3$.

$p(a + 1) = 5(a + 1) - (a + 1)^2 + 3(a + 1)^3$ Replace x with $a + 1$.

$= 5a + 5 - (a^2 + 2a + 1) + 3(a^3 + 3a^2 + 3a + 1)$ Evaluate $5(a + 1)$, $(a + 1)^2$, and $3(a + 1)^3$.

$= 5a + 5 - a^2 - 2a - 1 + 3a^3 + 9a^2 + 9a + 3$

$= 3a^3 + 8a^2 + 12a + 7$ Simplify.

Exercises Find $p(-4)$ and $p(x + h)$ for each function.

See Examples 2 and 3 on pages 347 and 348.

7. $p(x) = x - 2$ **-6; $x + h - 2$**
8. $p(x) = -x + 4$ **8; $-x - h + 4$**
9. $p(x) = 6x + 3$ **-21; $6x + 6h + 3$**
10. $p(x) = x^2 + 5$
11. $p(x) = x^2 - x$
12. $p(x) = 2x^3 - 1$

10. 21; $x^2 + 2xh + h^2 + 5$ 11. 20; $x^2 + 2xh + h^2 - x - h$ 12. -129; $2x^3 + 6x^2h + 6xh^2 + 2h^3 - 1$

www.algebra2.com/vocabulary_review

FOLDABLES™ Study Organizer

For more information about Foldables, see *Teaching Mathematics with Foldables.*

Have students look through the chapter to make sure they have included notes and examples for each lesson in this chapter in their Foldable.

Encourage students to refer to their Foldables while completing the Study Guide and Review and to use them in preparing for the Chapter Test.

7-2 Graphing Polynomial Functions

See pages 353–358.

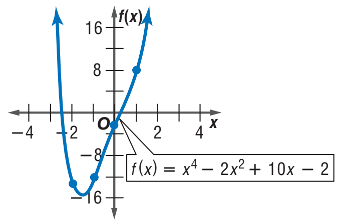
Concept Summary

- The Location Principle: Since zeros of a function are located at x-intercepts, there is also a zero between each pair of these zeros.
- Turning points of a function are called relative maxima and relative minima.

Example **Graph $f(x) = x^4 - 2x^2 + 10x - 2$ by making a table of values.**

Make a table of values for several values of x and plot the points. Connect the points with a smooth curve.

x	$f(x)$
−3	31
−2	−14
−1	−13
0	−2
1	7
2	26



Exercises **For Exercises 13–18, complete each of the following.**

a. Graph each function by making a table of values.

b. Determine consecutive values of x between which each real zero is located.

c. Estimate the x-coordinates at which the relative maxima and relative minima occur. *See Example 1 on page 353.* **13–18. See margin.**

13. $h(x) = x^3 - 6x - 9$ **14.** $f(x) = x^4 + 7x + 1$

15. $p(x) = x^5 + x^4 - 2x^3 + 1$ **16.** $g(x) = x^3 - x^2 + 1$

17. $r(x) = 4x^3 + x^2 - 11x + 3$ **18.** $f(x) = x^3 + 4x^2 + x - 2$

7-3 Solving Equations Using Quadratic Techniques

See pages 360–364.

Concept Summary

- Solve polynomial equations by using quadratic techniques.

Example **Solve $x^3 - 3x^2 - 54x = 0$.**

$x^3 - 3x^2 - 54x = 0$ Original equation

$x(x^2 - 3x - 54) = 0$ Factor out the GCF.

$x(x - 9)(x + 6) = 0$ Factor the trinomial.

$x = 0$ or $x - 9 = 0$ or $x + 6 = 0$ Zero Product Property

$x = 0$ $x = 9$ $x = -6$

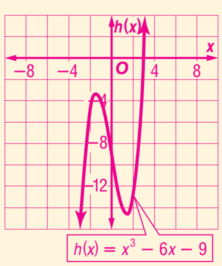
Exercises **Solve each equation.** *See Example 2 on page 361.*

19. $3x^3 + 4x^2 - 15x = 0$ **20.** $m^4 + 3m^3 = 40m^2$ **21.** $a^3 - 64 = 0$

22. $r + 9\sqrt{r} = -8$ $\varnothing$ **23.** $x^4 - 8x^2 + 16 = 0$ **24.** $x^{\frac{2}{3}} - 9x^{\frac{1}{3}} + 20 = 0$ **64, 125**

19. $\frac{5}{3}$, −3, 0
20. −8, 0, 5
21. 4, $-2 \pm 2i\sqrt{3}$
23. 2, −2

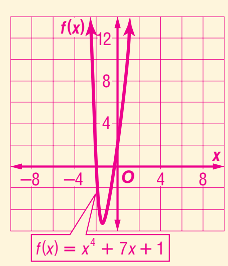
13a.



13b. at $x = 3$

13c. Sample answer: rel. max. at $x = -1.4$, rel. min. at $x = 1.4$

14a.

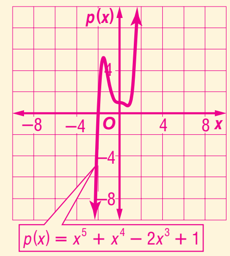


Answers

14b. between −2 and −1, and between −1 and 0

14c. Sample answer: no rel. max., rel. min. at $x = -1.2$

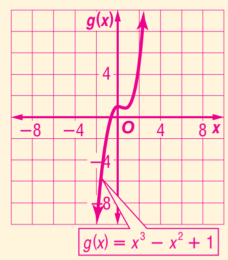
15a.



15b. between −2 and −3

15c. Sample answer: rel. max. at $x = 0$ $x = -1.6$, rel. min. at $x = 0.8$

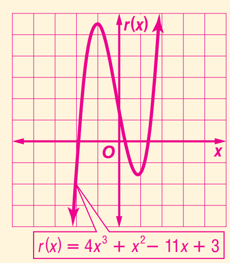
16a.



16b. between −1 and 0

16c. Sample answer: rel. max. at $x =$ 0, rel. min. at $x = 0.7$

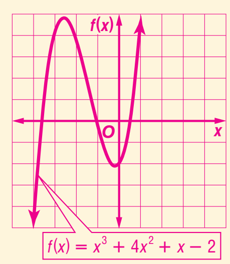
17a.



17b. between −2 and −1, between 0 and 1, and between 1 and 2

17c. Sample answer: rel. max. at $x = -1$, rel. min. at $x = 0.9$

18a.



18b. between −4 and −3, at $x = -1$, and between 0 and 1

18c. Sample answer: rel. max. at $x = -2.5$, rel. min. at $x = -0.1$

7-4 The Remainder and Factor Theorems

See pages 365–370.

Concept Summary

- Remainder Theorem: If a polynomial $f(x)$ is divided by $x - a$, the remainder is the constant $f(a)$ and $f(x) = q(x) \cdot (x - a) + f(a)$ where $q(x)$ is a polynomial with degree one less than the degree of $f(x)$.
- Factor Theorem: $x - a$ is a factor of polynomial $f(x)$ if and only if $f(a) = 0$.

Example **Show that $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$. Then find any remaining factors of the polynomial.**

-2	1	-2	-5	6
		-2	8	-6
	1	-4	3	0

The remainder is 0, so $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$. Since $x^3 - 2x^2 - 5x + 6 = (x + 2)(x^2 - 4x + 3)$, the factors are of $x^3 - 2x^2 - 5x + 6$ are $(x + 2)(x - 3)(x - 1)$.

Exercises **Use synthetic substitution to find $f(3)$ and $f(-2)$ for each function.**
See Example 2 on page 367. **26. 1, 16 27. 20, −20**

25. $f(x) = x^2 - 5$ **4, −1** **26.** $f(x) = x^2 - 4x + 4$ **27.** $f(x) = x^3 - 3x^2 + 4x + 8$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. *See Example 3 on page 367.*

28. $x^3 + 5x^2 + 8x + 4$; $x + 1$ **29.** $x^3 + 4x^2 + 7x + 6$; $x + 2$ **$x^2 + 2x + 3$**

28. $x + 2$, $x + 2$

7-5 Roots and Zeros

See pages 371–377.

Concept Summary

- Fundamental Theorem of Algebra: Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.
- Use Descartes' Rule of Signs to determine types of zeros of polynomial functions.
- Complex Conjugates Theorem: If $a + bi$ is a zero of a polynomial function, then $a - bi$ is also a zero of the function.

Example **State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = 5x^4 + 6x^3 - 8x + 12$.**

Since $f(x)$ has two sign changes, there are 2 or 0 real positive zeros.

$f(-x) = 5x^4 - 6x^3 + 8x + 12$ Two sign changes → 0 or 2 negative real zeros

There are 0, 2, or 4 imaginary zeros. **30. 3 or 1; 1; 2 or 0**
33. 3 or 1; 1; 0 or 2 34. 2 or 0; 2 or 0; 4, 2, or 0 35. 2 or 0; 2 or 0; 4, 2, or 0

Exercises **State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.** *See Example 2 on page 373.*

30. $f(x) = 2x^4 - x^3 + 5x^2 + 3x - 9$ **31.** $f(x) = 7x^3 + 5x - 1$ **1; 0; 2**
32. $f(x) = -4x^4 - x^2 - x + 1$ **1; 1; 2** **33.** $f(x) = 3x^4 - x^3 + 8x^2 + x - 7$
34. $f(x) = x^4 + x^3 - 7x + 1$ **35.** $f(x) = 2x^4 - 3x^3 - 2x^2 + 3$

402 Chapter 7 Polynomial Functions

7-6 Rational Zero Theorem

See pages 378–382.

Concept Summary

- Use the Rational Zero Theorem to find possible zeros of a polynomial function.
- Integral Zero Theorem: If the coefficients of a polynomial function are integers such that $a_0 = 1$ and $a_n \neq 0$, any rational zeros of the function must be factors of a_n.

Examples **Find all of the zeros of $f(x) = x^3 + 7x^2 - 36$.**

There are exactly three complex zeros.

There are either one or three positive real zeros and two or zero negative real zeros.

The possible rational zeros are ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18, ±36.

2	1	7	0	−36
		2	18	36
	1	9	18	0

$x^3 + 7x^2 - 36 = (x - 2)(x^2 + 9x + 18)$
$= (x - 2)(x + 3)(x + 6)$

Therefore, the zeros are 2, −3, and −6.

Exercises **Find all of the rational zeros of each function.** *See Example 3 on page 379.*

36. $f(x) = 2x^3 - 13x^2 + 17x + 12$ $-\frac{1}{2}, 3, 4$
37. $f(x) = x^4 + 5x^3 + 15x^2 + 19x + 8$
38. $f(x) = x^3 - 3x^2 - 10x + 24$ $-3, 2, 4$
39. $f(x) = x^4 - 4x^3 - 7x^2 + 34x - 24$
40. $f(x) = 2x^3 - 5x^2 - 28x + 15$
41. $f(x) = 2x^4 - 9x^3 + 2x^2 + 21x - 10$

37. −1, −1 39. 1, 2, 4, −3 40. −3, 5, $\frac{1}{2}$ 41. $\frac{1}{2}$, 2

7-7 Operations of Functions

See pages 383–389.

Concept Summary

Operation	Definition	Operation	Definition
Sum	$(f + g)(x) = f(x) + g(x)$	Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$
Difference	$(f - g)(x) = f(x) - g(x)$	Composition	$[f \circ g](x) = f[g(x)]$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	—	—

Example **If $f(x) = x^2 - 2$ and $g(x) = 8x - 1$. Find $g[f(x)]$ and $f[g(x)]$.**

$g[f(x)] = 8(x^2 - 2) - 1$ Replace $f(x)$ with $x^2 - 2$.
$= 8x^2 - 16 - 1$ Multiply.
$= 8x^2 - 17$ Simplify.
$f[g(x)] = (8x - 1)^2 - 2$ Replace $g(x)$ with $8x - 1$.
$= 64x^2 - 16x + 1 - 2$ Expand the binomial.
$= 64x^2 - 16x - 1$ Simplify.

Exercises **Find $[g \circ h](x)$ and $[h \circ g](x)$.** *See Example 4 on page 385.*

42. $h(x) = 2x - 1$, $g(x) = 3x + 4$ $6x + 1$; $6x + 7$
43. $h(x) = x^2 + 2$, $g(x) = x - 3$
44. $h(x) = x^2 + 1$, $g(x) = -2x + 1$
45. $h(x) = -5x$, $g(x) = 3x - 5$
46. $h(x) = x^3$, $g(x) = x - 2$
47. $h(x) = x + 4$, $g(x) = |x|$ $|x + 4|$; $|x| + 4$

43. $x^2 - 1$; $x^2 - 6x + 11$
44. $-2x^2 - 1$; $4x^2 - 4x + 2$
45. $-15x - 5$; $-15x + 25$
46. $x^3 - 2$; $x^3 - 6x^2 + 12x - 8$

Chapter 7 Study Guide and Review 403

Chapter 7 For More ...
- Extra Practice, see pages 842–844.
- Mixed Problem Solving, see page 868.

7-8 Inverse Functions and Relations

See pages 390–394.

Concept Summary

- Reverse the coordinates of ordered pairs to find the inverse of a relation.
- Two functions are inverses if and only if both of their compositions are the identity function. $[f \circ g](x) = x$ and $[g \circ f](x) = x$
- A function is one-to-one when the inverse of the function is a function.

Example **Find the inverse of $f(x) = -3x + 1$.**

Rewrite $f(x)$ as $y = -3x + 1$. Then interchange the variables and solve for y.

$x = -3y + 1$ Interchange the variables.

$3y = -x + 1$ Solve for y.

$y = \frac{-x + 1}{3}$ Divide each side by 3.

$f^{-1}(x) = \frac{-x + 1}{3}$ Rewrite in function notation.

Exercises **Find the inverse of each function. Then graph the function and its inverse.** *See Example 2 on page 391.* **48–53. See pp. 407A–407H.**

48. $f(x) = 3x - 4$
49. $f(x) = -2x - 3$
50. $g(x) = \frac{1}{3}x + 2$
51. $f(x) = \frac{-3x + 1}{2}$
52. $y = x^2$
53. $y = (2x + 3)^2$

7-9 Square Root Functions and Inequalities

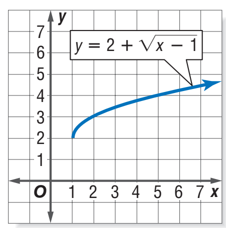
See pages 395–399.

Concept Summary

- Graph square root inequalities in a similar manner as graphing square root equations.

Example **Graph $y = 2 + \sqrt{x - 1}$.**

x	y
1	2
2	3
3	$2 + \sqrt{2}$ or 3.4
4	$2 + \sqrt{3}$ or 3.7
5	4



Exercises **Graph each function. State the domain and range of each function.** *See Examples 1 and 2 on pages 395 and 396.* **54–56. See margin.**

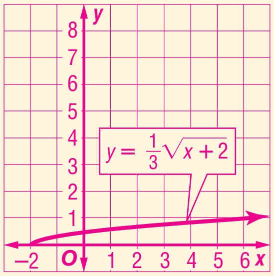
54. $y = \frac{1}{3}\sqrt{x + 2}$
55. $y = \sqrt{5x - 3}$
56. $y = 4 + 2\sqrt{x - 3}$

Graph each inequality. *See Example 3 on page 397.* **57–58. See margin.**

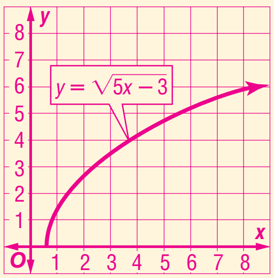
57. $y \geq \sqrt{x} - 2$
58. $y < \sqrt{4x - 5}$

Answers

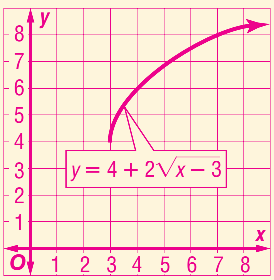
54. D: $x \geq -2$, R: $y \geq 0$



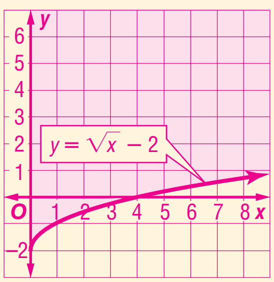
55. D: $x \geq \frac{3}{5}$, R: $y \geq 0$



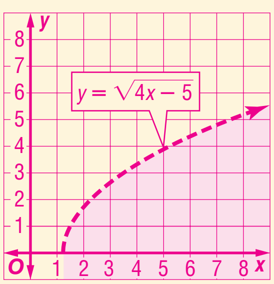
56. D: $x \geq 3$, R: $y \geq 4$



57.



58.



Answers (p. 405)

20. $x^2 + 2x - 1$

21. $-x^2 + 2x - 7$

22. $2x^3 - 4x^2 + 6x - 12$

23. $\frac{2x - 4}{x^2 + 3}$

24a. $A = 1000(1 + r)^6 + 1000(1 + r)^5 + 1000(1 + r)^4 + 1200(1 + r)^3 + 1200(1 + r)^2 + 2000(1 + r)$

Chapter 7 Practice Test

Vocabulary and Concepts

Match each statement with the term that it best describes.

a. quadratic form
b. composition of functions
c. inverse functions

1. $[f \circ g](x) = f[g(x)]$ **b**
2. $[f \circ g](x) = x$ and $[g \circ f](x) = x$ **c**
3. $(\sqrt{x})^2 - 2(\sqrt{x}) + 4 = 0$ **a**

Skills and Applications

For Exercises 4–7, complete each of the following. **4–7. See pp. 407A–407H.**

a. Graph each function by making a table of values.
b. Determine consecutive values of x between which each real zero is located.
c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

4. $g(x) = x^3 + 6x^2 + 6x - 4$
5. $h(x) = x^4 + 6x^3 + 8x^2 - x$
6. $f(x) = x^3 + 3x^2 - 2x + 1$
7. $g(x) = x^4 - 2x^3 - 6x^2 + 8x + 5$

Solve each equation. **8. $0, -4 \pm \sqrt{34}$ 10. $\pm\sqrt{6}, \pm\sqrt{3}$**

8. $p^3 + 8p^2 = 18p$
9. $16x^4 - x^2 = 0$ **$0, \pm\frac{1}{4}$**
10. $r^4 - 9r^2 + 18 = 0$
11. $p^{\frac{3}{2}} - 8 = 0$ **4**

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

12. $x^3 - x^2 - 5x - 3; x + 1$ **$x - 3, x + 1$**
13. $x^3 + 8x + 24; x + 2$ **$x^2 - 2x + 12$**

State the possible number of positive real zeros, negative real zeros, and imaginary zeros for each function.

14. $f(x) = x^3 - x^2 - 14x + 24$ **2 or 0; 1; 0 or 2**
15. $f(x) = 2x^3 - x^2 + 16x - 5$ **3 or 1; 0; 0 or 2**

Find all of the rational zeros of each function.

16. $g(x) = x^3 - 3x^2 - 53x - 9$ **9**
17. $h(x) = x^4 + 2x^3 - 23x^2 + 2x - 24$ **−6, 4**

Determine whether each pair of functions are inverse functions.

18. $f(x) = 4x - 9, g(x) = \frac{x - 9}{4}$ **no**
19. $f(x) = \frac{1}{x + 2}, g(x) = \frac{1}{x} - 2$ **yes**

If $f(x) = 2x - 4$ and $g(x) = x^2 + 3$, find each value. **20–23. See margin.**

20. $(f + g)(x)$
21. $(f - g)(x)$
22. $(f \cdot g)(x)$
23. $\left(\frac{f}{g}\right)(x)$

24. **FINANCIAL PLANNING** Toshi will start college in six years. According to their plan, Toshi's parents will save $1000 each year for the next three years. During the fourth and fifth years, they will save $1200 each year. During the last year before he starts college, they will save $2000.
    a. In the formula $A = P(1 + r)^t$, A = the balance, P = the amount invested, r = the interest rate, and t = the number of years the money has been invested. Use this formula to write a polynomial equation to describe the balance of the account when Toshi starts college. **See margin.**
    b. Find the balance of the account if the interest rate is 6%. **$8916.76**

25. **STANDARDIZED TEST PRACTICE** Which value is included in the graph of $y < \sqrt{2x}$? **D**
    (A) (−2, −2) (B) (−1, −1) (C) (0, 0) (D) None of these

www.algebra2.com/chapter_test

Portfolio Suggestion

Introduction In mathematics, polynomial equations can be used to model many real-world problems. The solution to the polynomial equation provides a solution to the real-world problem.

Ask Students From your work in this chapter, select a real-world problem modeled by a polynomial equation and show how you solved it. Explain how the solution to the polynomial equation relates to the solution of the real-world problem. Place your work in your portfolio.

Chapter 7 Practice Test

Assessment Options

Vocabulary Test A vocabulary test/review for Chapter 7 can be found on p. 442 of the *Chapter 7 Resource Masters*.

Chapter Tests There are six Chapter 7 Tests and an Open-Ended Assessment task available in the *Chapter 7 Resource Masters*.

Chapter 7 Tests

Form	Type	Level	Pages
1	MC	basic	429–430
2A	MC	average	431–432
2B	MC	average	433–434
2C	FR	average	435–436
2D	FR	average	437–438
3	FR	advanced	439–440

MC = multiple-choice questions
FR = free-response questions

Open-Ended Assessment Performance tasks for Chapter 7 can be found on p. 441 of the *Chapter 7 Resource Masters*. A sample scoring rubric for these tasks appears on p. A34.

Unit 2 Test A unit test/review can be found on pp. 449–450 of the *Chapter 7 Resource Masters*.

First Semester Test A test for Chapters 1–7 can be found on pp. 451–454 of the *Chapter 7 Resource Masters*.

TestCheck and Worksheet Builder

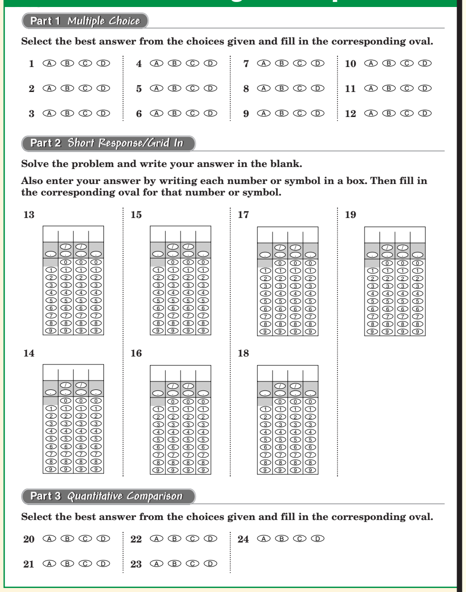
This **networkable software** has three modules for assessment.

- **Worksheet Builder** to make worksheets and tests.
- **Student Module** to take tests on-screen.
- **Management System** to keep student records.

These two pages contain practice questions in the various formats that can be found on the most frequently given standardized tests.

A practice answer sheet for these two pages can be found on p. A1 of the *Chapter 7 Resource Masters*.

Standardized Test Practice Student Recording Sheet, p. A1


Part 1 Multiple Choice
Select the best answer from the choices given and fill in the corresponding oval.
1 A B C D 4 A B C D 7 A B C D 10 A B C D
2 A B C D 5 A B C D 8 A B C D 11 A B C D
3 A B C D 6 A B C D 9 A B C D 12 A B C D
Part 2 Short Response/Grid In
Solve the problem and write your answer in the blank.
Also enter your answer by writing each number or symbol in a box. Then fill in the corresponding oval for that number or symbol.
13 15 17 19
14 16 18
Part 3 Quantitative Comparison
Select the best answer from the choices given and fill in the corresponding oval.
20 A B C D 22 A B C D 24 A B C D
21 A B C D 23 A B C D

Additional Practice

See pp. 447–448 in the *Chapter 7 Resource Masters* for additional standardized test practice.

Chapter 7 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. If $\frac{2}{p} - \frac{4}{p^2} = -\frac{2}{p^3}$, then what is the value of p? **B**

(A) -1 (B) 1 (C) $-\frac{1}{2}$ (D) $\frac{1}{2}$

2. There are n gallons of liquid available to fill a tank. After k gallons of the liquid have filled the tank, how do you represent in terms of n and k the percent of liquid that has filled the tank? **A**

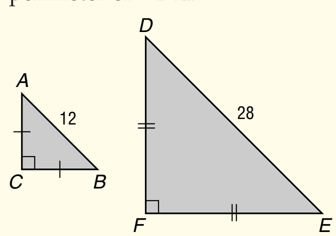
(A) $\frac{100k}{n}\%$ (B) $\frac{n}{100k}\%$

(C) $\frac{100n}{k}\%$ (D) $\frac{n}{100(n-k)}\%$

3. How many different triangles have sides of lengths 4, 9 and s, where s is an integer and $4 < s < 9$? **D**

(A) 0 (B) 1 (C) 2 (D) 3

4. Triangles ABC and DEF are similar. The area of $\triangle ABC$ is 72 square units. What is the perimeter of $\triangle DEF$? **B**



(A) 56 units (B) $28 + 28\sqrt{2}$ units

(C) $56\sqrt{2}$ units (D) $28 + 14\sqrt{2}$ units

5. If $2 - 3x > -1$ and $x + 5 > 0$, then x could equal each of the following *except* **A**

(A) -5. (B) -4. (C) -2. (D) 0.

6. What is the midpoint of the line segment whose endpoints are represented on the coordinate grid by the points $(-5, -3)$ and $(-1, 4)$? **B**

(A) $\left(-3, -\frac{1}{2}\right)$ (B) $\left(-3, \frac{1}{2}\right)$

(C) $\left(-2, -\frac{7}{2}\right)$ (D) $\left(-2, \frac{1}{2}\right)$

7. For all $n \neq 0$, what is the slope of the line passing through (n, k) and $(-n, -k)$? **D**

(A) 0 (B) 1 (C) $\frac{n}{k}$ (D) $\frac{k}{n}$

8. Which of the following is a quadratic equation in one variable? **B**

(A) $3(x + 4) + 1 = 4x - 9$

(B) $3x(x + 4) + 1 = 4x - 9$

(C) $3x(x^2 + 4) + 1 = 4x - 9$

(D) $y = 3x^2 + 8x + 10$

9. Simplify $\sqrt[4]{t^3} \cdot \sqrt[8]{t^2}$. **D**

(A) $t^{\frac{3}{16}}$ (B) $t^{\frac{1}{2}}$ (C) $t^{\frac{3}{4}}$ (D) t

10. Which of the following is a quadratic equation that has roots of $2\frac{1}{2}$ and $\frac{2}{3}$? **C**

(A) $5x^2 + 11x - 7 = 0$

(B) $5x^2 - 11x + 10 = 0$

(C) $6x^2 - 19x + 10 = 0$

(D) $6x^2 + 11x + 10 = 0$

11. If $f(x) = 3x - 5$ and $g(x) = 2 + x^2$, then what is equal to $f[g(2)]$? **D**

(A) 3 (B) 6 (C) 12 (D) 13

12. Which of the following is a root of $f(x) = x^3 - 7x + 6$? **B**

(A) -1 (B) 2 (C) 3 (D) 6

The Princeton Review *Log On* for Test Practice

The Princeton Review offers additional test-taking tips and practice problems at their web site. Visit www.princetonreview.com or www.review.com

TestCheck and Worksheet Builder

Special banks of standardized test questions similar to those on the SAT, ACT, TIMSS 8, NAEP 8, and Algebra 1 End-of-Course tests can be found on this CD-ROM.

406 Chapter 7 Polynomial Functions

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

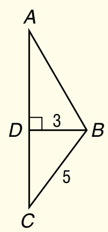
13. A group of 34 people is to be divided into committees so that each person serves on exactly one committee. Each committee must have at least 3 members and not more than 5 members. If N represents the maximum number of committees that can be formed and n represents the minimum number of committees that can be formed, what is the value of $N - n$? **4**

14. Raisins selling for $2.00 per pound are to be mixed with peanuts selling for $3.00 per pound. How many pounds of peanuts are needed to produce a 20-pound mixture that sells for $2.75 per pound? **15**

15. The mean of 15 scores is 82. If the mean of 7 of these scores is 78, what is the mean of the remaining 8 scores? **85.5**

16. Jars X, Y, and Z each contain 10 marbles. What is the minimum number of marbles that must be transferred among the jars so that the ratio of the number of marbles in jar X to the number of marbles in jar Y to the number of marbles in jar Z is 1 : 2 : 3? **5**

17. If the area of $\triangle BCD$ is 40% of the area of $\triangle ABC$, what is the measure of $\overline{AD}$? **6**



Test-Taking Tip

Questions 13, 16, and 18 Words such as *maximum*, *minimum*, *least*, and *greatest* indicate that a problem may involve an inequality. Take special care when simplifying inequalities that involve negative numbers.

www.algebra2.com/standardized_test

18. If the measures of the sides of a triangle are 3, 8, and x and x is an integer, then what is the least possible perimeter of the triangle? **17**

19. If the operation ❖ is defined by the equation $x ❖ y = 3x - y$, what is the value of w in the equation $w ❖ 6 = 2 ❖ w$? **3**

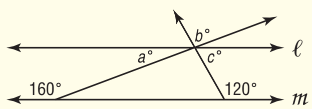
Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- **Ⓐ the quantity in Column A is greater,**
- **Ⓑ the quantity in Column B is greater,**
- **Ⓒ the two quantities are equal, or**
- **Ⓓ the relationship cannot be determined from the information given.**

	Column A	Column B
20. D	2.5% of 10x	0.025x

21. B

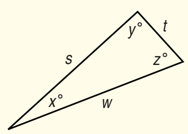


Column A	Column B
$a + c$	b

22. B $x > 0$

Column A	Column B
$\frac{x}{0.4}$	$3x$

23. C



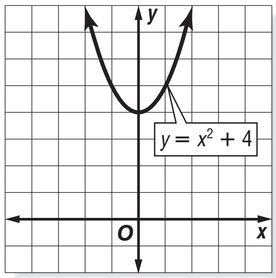
$y = x + z$

Column A	Column B
w	$\sqrt{s^2 + t^2}$

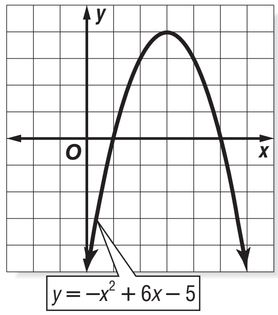
	Column A	Column B
24. C	2^8	$2^7 + 2^7$

Pages 350–352, Lesson 7-1

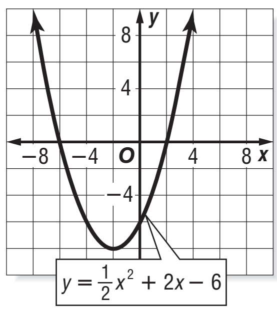
68.



69.

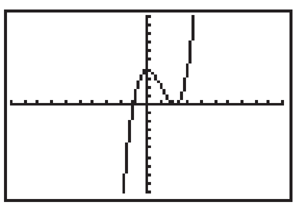


70.



Page 356, Lesson 7-2
Graphing Calculator Investigation

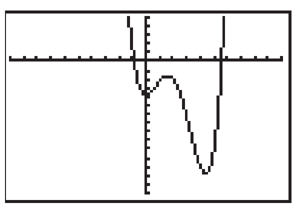
1.



rel. max. at $x = 0$, rel. min. at $x = 2$

2. rel. max. at (0, 4) rel. min. at (2, 0)

3.

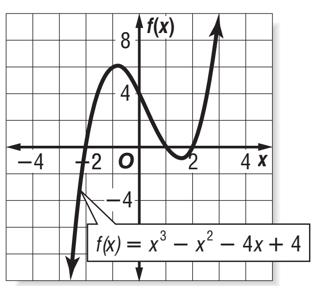


Sample answer: two rel. min. points at (0, −8) and (4.4, −25.8) and one rel. max. point at (1.6, −3.2)

Pages 356–358, Lesson 7-2

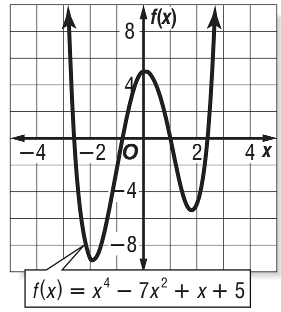
4.

x	f(x)
−3	−20
−2	0
−1	6
0	4
1	0
2	0
3	10

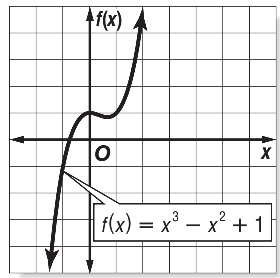


5.

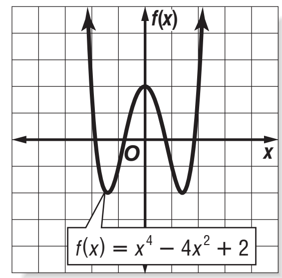
x	f(x)
−3	20
−2	−9
−1	−2
0	5
1	0
2	−5
3	26



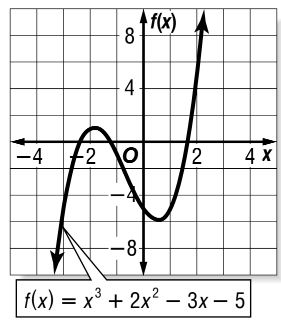
6.



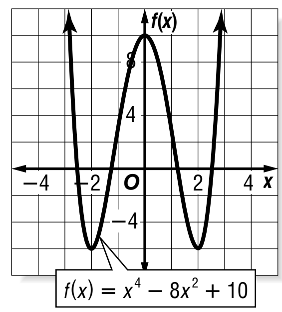
7.



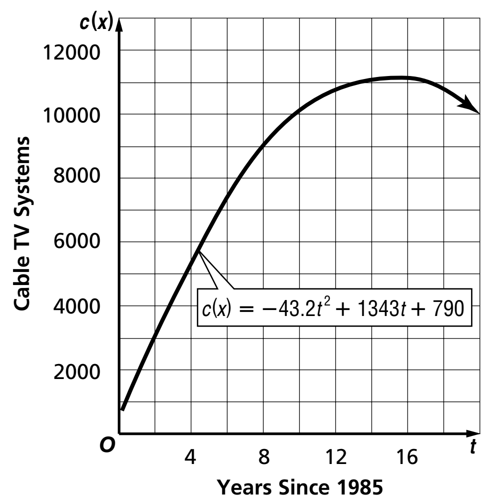
8.



9.



10.

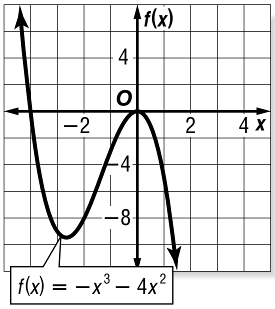


11. rel. max. between $x = 15$ and $x = 16$, and no rel. min.; $f(x) \to -\infty$ as $x \to -\infty$, $f(x) \to -\infty$ as $x \to +\infty$.

12. The number of cable TV systems rose steadily from 1985 to 2000. Then the number began to decline.

13a.

x	$f(x)$
−5	25
−4	0
−3	−9
−2	−8
−1	−3
0	0
1	−5
2	−24

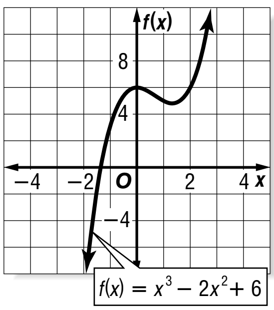


13b. at $x = -4$ and $x = 0$

13c. Sample answer: rel. max. at $x = 0$, rel. min. at $x = -3$

14a.

x	$f(x)$
−2	−10
−1	3
0	6
1	5
2	6
3	15
4	38

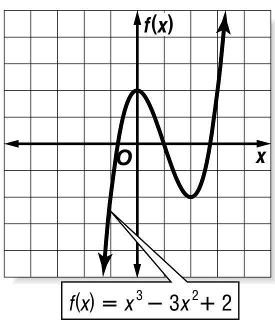


14b. between −2 and −1

14c. Sample answer: rel. max. at $x = 0$, rel. min. at $x = \frac{3}{2}$

15a.

x	$f(x)$
−2	−18
−1	−2
0	2
1	0
2	−2
3	2
4	18

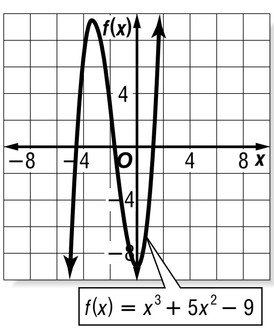


15b. at $x = 1$, between −1 and 0, and between 2 and 3

15c. Sample answer: rel. max. at $x = 0$, rel. min. at $x = 2$

16a.

x	$f(x)$
−5	−9
−4	7
−3	9
−2	3
−1	−5
0	−9
1	−3
2	19

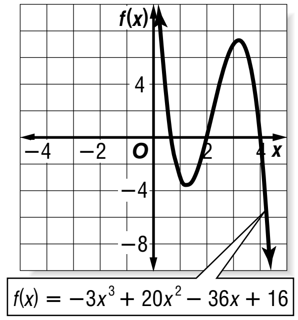


16b. between −5 and −4, between −2 and −1, and between 1 and 2

16c. Sample answer: rel. max. at $x = -3$, rel. min. at $x = 0$

17a.

x	$f(x)$
−1	75
0	16
1	−3
2	0
3	7
4	0
5	−39

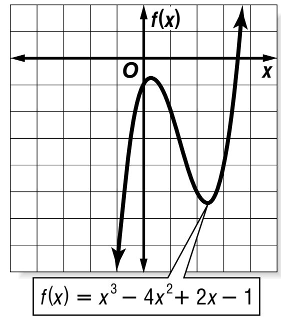


17b. between 0 and 1, at $x = 2$, and at $x = 4$

17c. Sample answer: rel. max. at $x = 3$, rel. min. at $x = 1$

18a.

x	$f(x)$
−2	−29
−1	−8
0	−1
1	−2
2	−5
3	−4
4	7
5	34

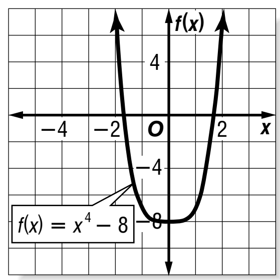


18b. between 3 and 4

18c. Sample answer: rel. max. at $x = 0.5$, rel. min. at $x = 2.5$

19a.

x	$f(x)$
−3	73
−2	8
−1	−7
0	−8
1	−7
2	8
3	73

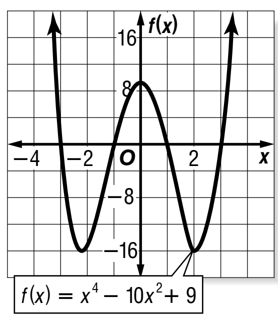


19b. between −2 and −1, and between 1 and 2

19c. Sample answer: no rel. max., rel. min. at $x = 0$

20a.

x	$f(x)$
−3	0
−2	−15
−1	0
0	9
1	0
2	−15
3	0
4	105

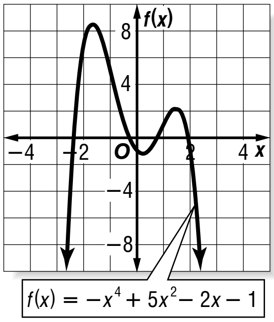


20b. at $x = -3$, $x = -1$, $x = 1$, and $x = 3$

20c. Sample answer: rel. max. at $x = 0$, rel. min. at $x = -2$ and $x = 2$

21a.

x	$f(x)$
−4	−169
−3	−31
−2	7
−1	5
0	−1
1	1
2	−1
3	−43

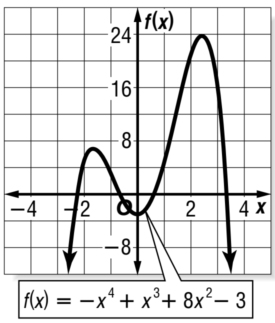


21b. between −3 and −2, between −1 and 0, between 0 and 1, and between 1 and 2

21c. Sample answer: rel. max. at $x = -2$ and at $x = 1.5$, rel. min. at $x = 0$

22a.

x	$f(x)$
−3	−39
−2	5
−1	3
0	−3
1	5
2	21
3	15
4	−67

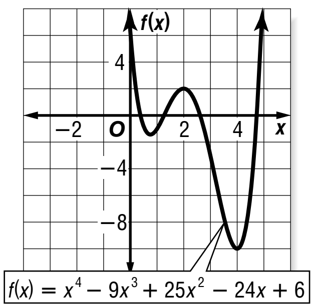


22b. between −3 and −2, between −1 and 0, between 0 and 1, and between 3 and 4

22c. Sample answer: rel. max. at $x = -1.5$ and at $x = 2.5$, rel. min. at $x = 0$

23a.

x	$f(x)$
−1	65
0	6
1	−1
2	2
3	−3
4	−10
5	11

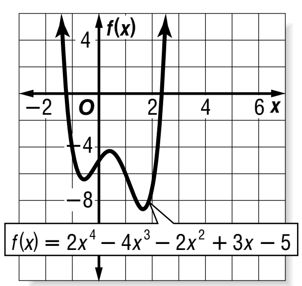


23b. between 0 and 1, between 1 and 2, between 2 and 3, and between 4 and 5

23c. Sample answer: rel. max. at $x = 2$, rel. min. at $x = 0.5$ and at $x = 4$

24a.

x	$f(x)$
−2	45
−1	−4
0	−5
1	−6
2	−7
3	40

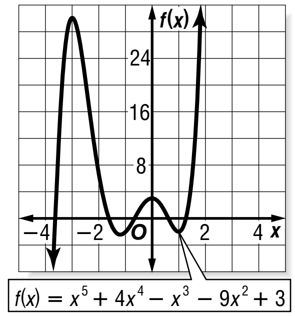


24b. between −2 and −1, and between 2 and 3

24c. Sample answer: rel. max. at $x = 0.5$, rel. min. at $x = -0.5$ and at $x = 1.5$

25a.

x	$f(x)$
−4	−77
−3	30
−2	7
−1	−2
0	3
1	−2
2	55

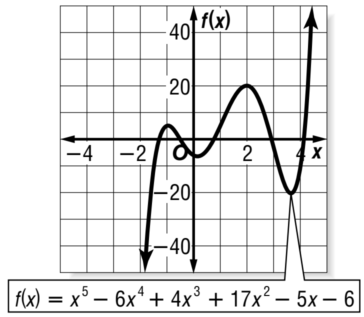


25b. between −4 and −3, between −2 and −1, between −1 and 0, between 0 and 1, and between 1 and 2

25c. Sample answer: rel. max. at $x = -3$ and at $x = 0$, rel. min. at $x = -1$ and at $x = 1$

26a.

x	$f(x)$
−2	−88
−1	5
0	−6
1	5
2	20
3	−3
4	−10
5	269

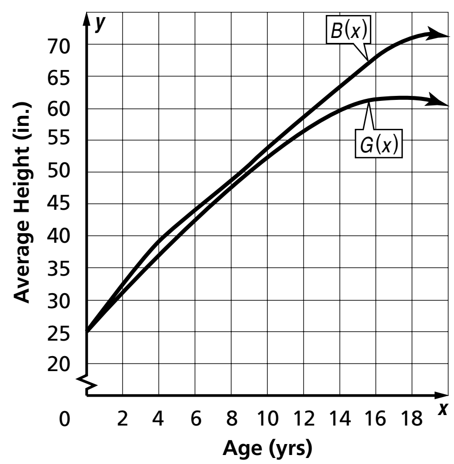


26b. between −2 and −1, between −1 and 0, between 0 and 1, between 2 and 3, and between 4 and 5

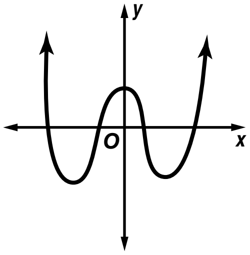
26c. Sample answer: rel. max. at $x = -1$ and at $x = 2$, rel. min. at $x = 0$ and at $x = 3.5$

31.

x	0	2	4	6	8	10	12	14	16	18	20
$B(x)$	25	34	40	45	50	54	59	64	68	71	71
$G(x)$	26	33	39	44	49	53	56	59	61	61	60



36.



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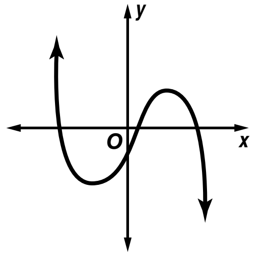

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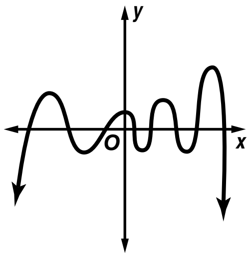

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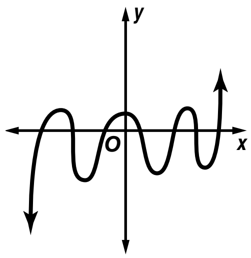
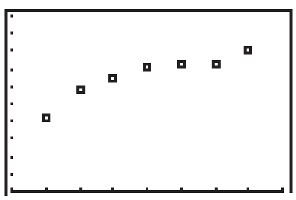


40. The turning points of a polynomial function that models a set of data can indicate fluctuations that may repeat. Answers should include the following.

- Polynomial equations best model data that contain turning points, rather than a constant increase or decrease like linear equations.
- To determine when the percentage of foreign-born citizens was at its highest, look for rel. max. of the graph, which is at $t = 5$. The lowest percentage is found at $t = 75$, the rel. min. of the graph.

Page 359, Follow-Up of Lesson 7-2 Graphing Calculator Investigation

1.



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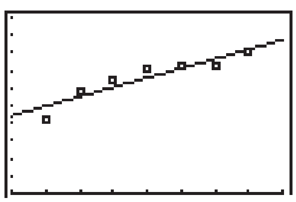

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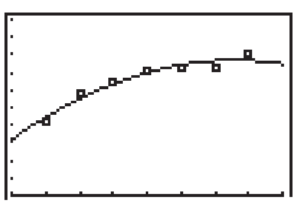

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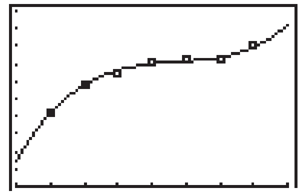


[1930, 2010] scl: 10 by [0, 200] scl: 20

Pages 362–364, Lesson 7-3

17. 0, −4, −3

18. 0, −1, −5

19. $-\sqrt{3}, \sqrt{3}, -i\sqrt{3}, i\sqrt{3}$

20. $0, -4, 4, -4i, 4i$

21. $2, -2, 2\sqrt{2}, -2\sqrt{2}$

22. $\sqrt{2}, -\sqrt{2}, 3, -3$

23. $-9, \frac{9 + 9i\sqrt{3}}{2}, \frac{9 - 9i\sqrt{3}}{2}$

24. $8, -4 + 4i\sqrt{3}, -4 - 4i\sqrt{3}$

25. 81, 625

26. −343, −64

27. 225, 16

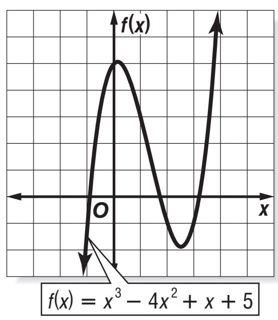
28. 400

38. Answers should include the following.

- Solve the cubic equation $4x^3 + (-164x^2) + 1600x = 3600$ in order to determine the dimensions of the cut square if the desired volume is 3600 in^3. Solutions are 10 in. and $\frac{31 - \sqrt{601}}{2}$ in.
- There can be more than one square cut to produce the same volume because the height of the box is not specified and 3600 has a variety of different factors.

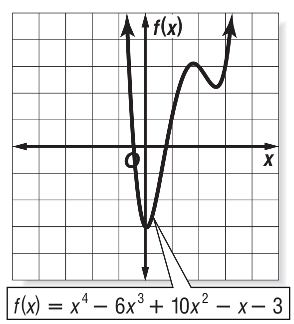
41.

x	f(x)
−2	−21
−1	−1
0	5
1	3
2	−1
3	−1
4	9
5	35



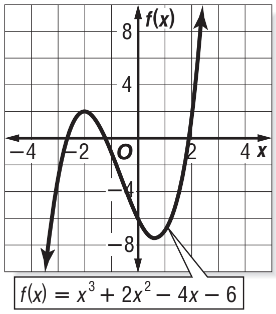
42.

x	f(x)
−1	15
0	−3
1	1
2	3
3	3
4	25



Page 364, Practice Quiz 1

3. Sample answer: maximum at $x = -2$, minimum at $x = 0.5$



Pages 369–370, Lesson 7-4

32.

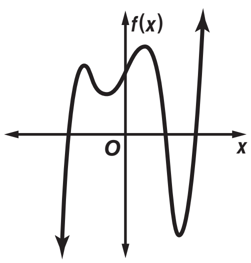
8	1	−4	−29	−24
		8	32	24
	1	4	3	0

37.

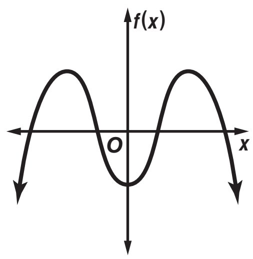
5	1	−14	69	−140	100
		5	−45	120	−100
	1	−9	24	−20	0

Pages 375–377, Lesson 7-5

41a.



41b.



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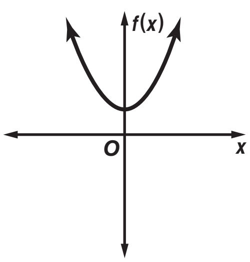
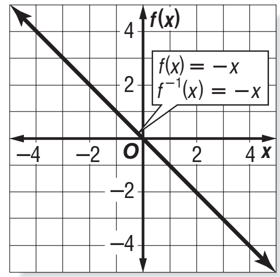


51. If the equation models the level of a medication in a patient's bloodstream, a doctor can use the roots of the equation to determine how often the patient should take the medication to maintain the necessary concentration in the body. Answers should include the following.
    - A graph of this equation reveals that only the first positive real root of the equation, 5, has meaning for this situation, since the next positive real root occurs after the medication level in the bloodstream has dropped below 0 mg. Thus according to this model, after 5 hours there is no significant amount of medicine left in the bloodstream.
    - The patient should not go more than 5 hours before taking their next dose of medication.

Pages 393–394, Lesson 7-8

7.



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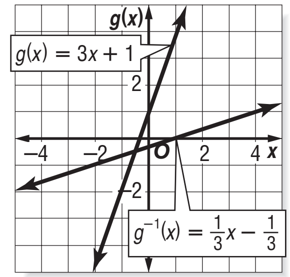

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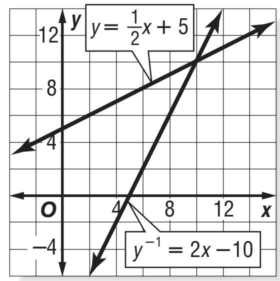

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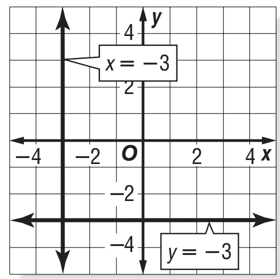

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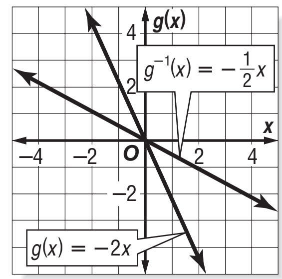

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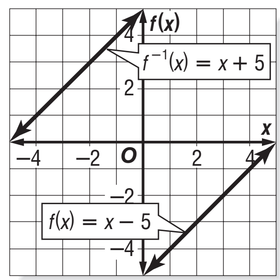

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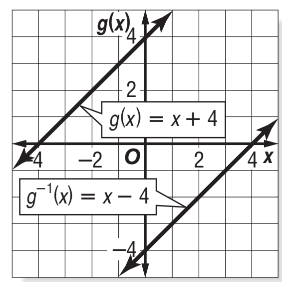

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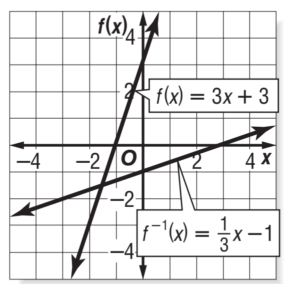

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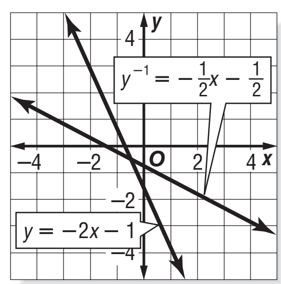

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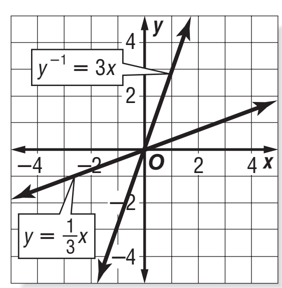

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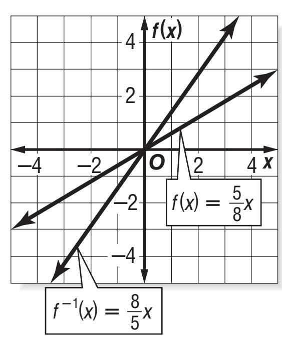

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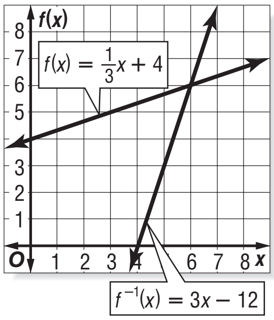
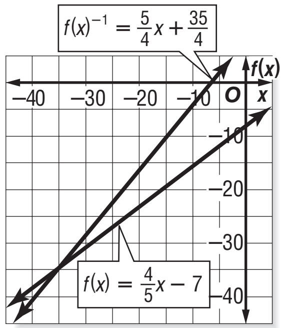
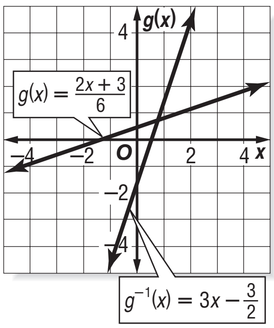


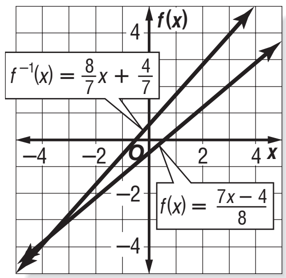
29. $f^{-1}(x) = \frac{5}{4}x + \frac{35}{4}$



30. $g^{-1}(x) = 3x - \frac{3}{2}$

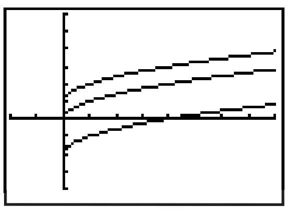


31. $f^{-1}(x) = \frac{8}{7}x + \frac{4}{7}$



Page 396, Lesson 7-9
Graphing Calculator Investigation

1.



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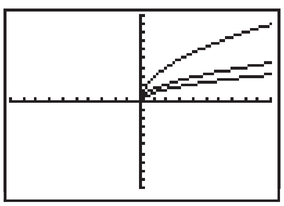

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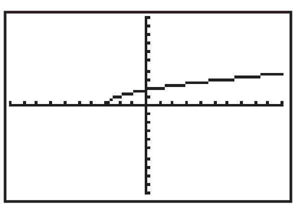

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Pages 397–399, Lesson 7-9

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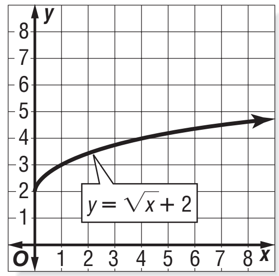

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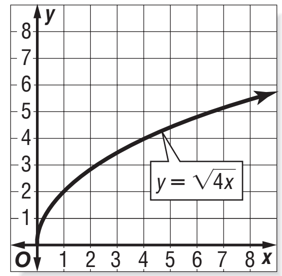

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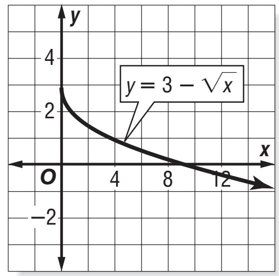

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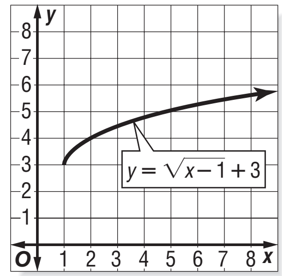

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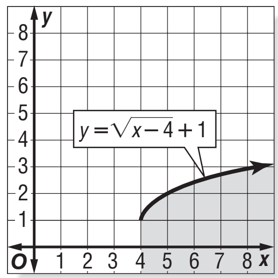

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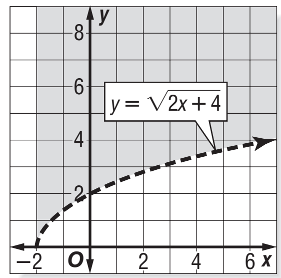

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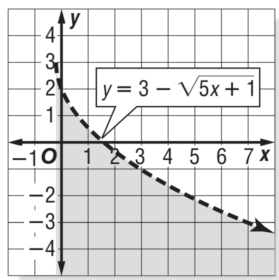

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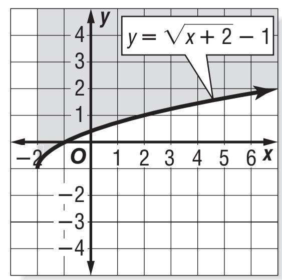

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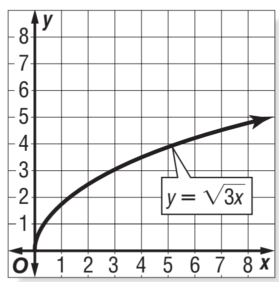

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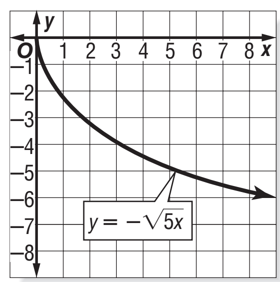

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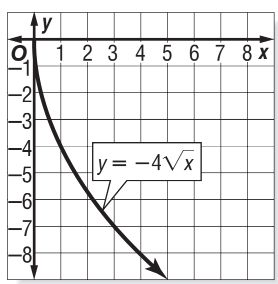

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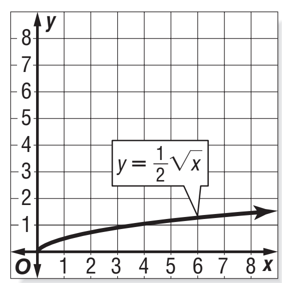

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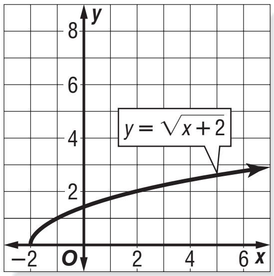

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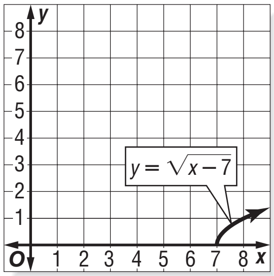

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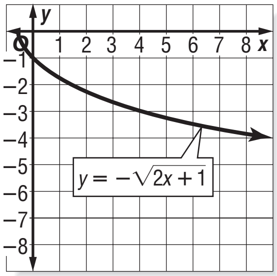

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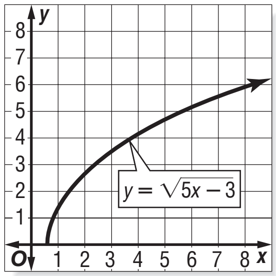

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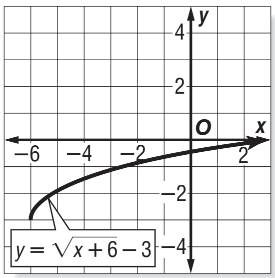

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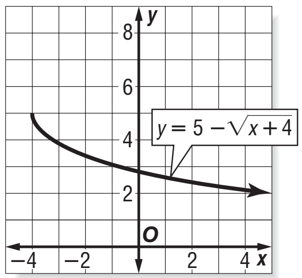

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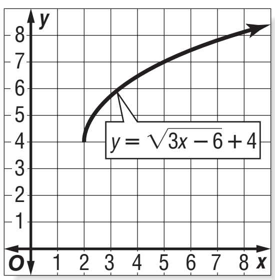

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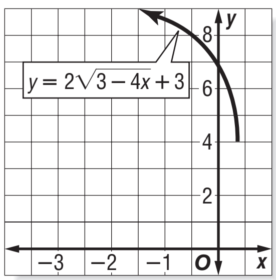

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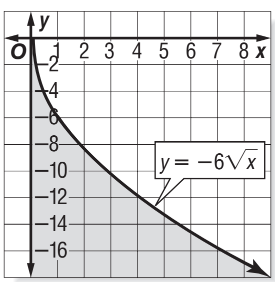

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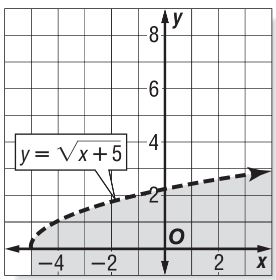

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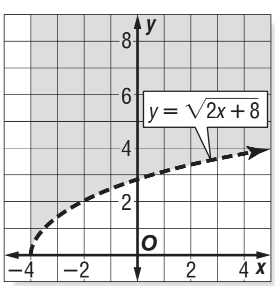

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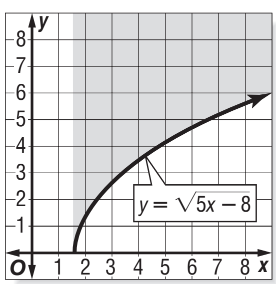

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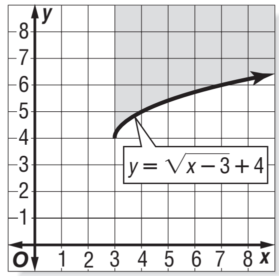

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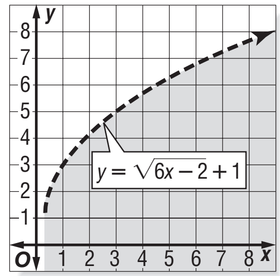
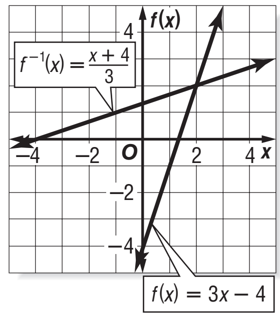
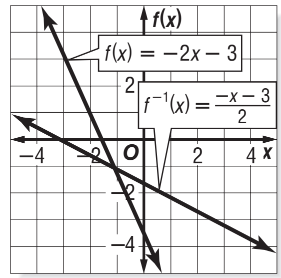


Pages 400–404, Chapter 7 Study Guide and Review

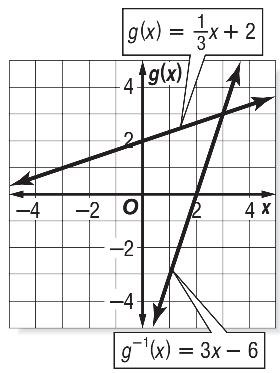
48. $f^{-1}(x) = \frac{x+4}{3}$



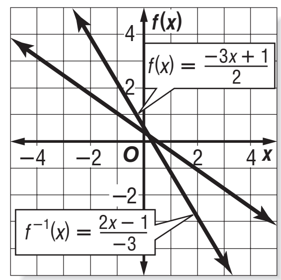
49. $f^{-1}(x) = \frac{-x-3}{2}$



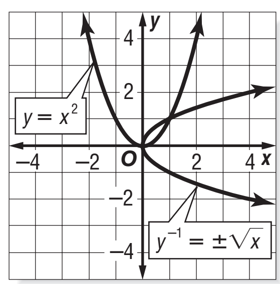
50. $g^{-1}(x) = 3x - 6$



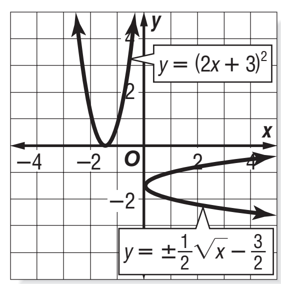
51. $f^{-1}(x) = \frac{2x-1}{-3}$



52. $y^{-1}(x) = \pm\sqrt{x}$

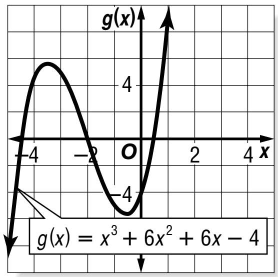


53. $y^{-1}(x) = \pm\frac{1}{2}\sqrt{x} - \frac{3}{2}$



Page 405, Chapter 7 Practice Test

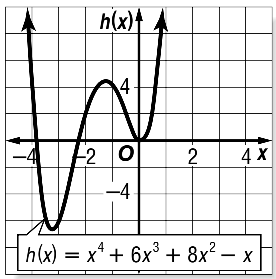
4a.



4b. between −5 and −4, zero at $x = -2$, between 0 and 1

4c. Sample answer: rel. max. at $x = -3.5$, rel. min. at $x = -0.5$

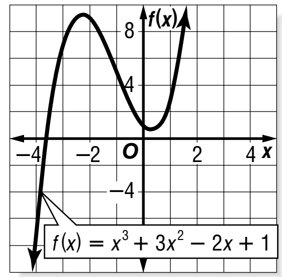
5a.



5b. between −4 and −3, between −3 and −2, zero at $x = 0$

5c. Sample answer: rel. max. at $x = -1$, rel. min. at $x = -3$ and at $x = 0$

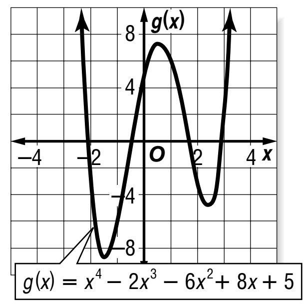
6a.



6b. between −4 and −3

6c. Sample answer: rel. max. at $x = -2.3$, rel. min. at $x = 0.3$

7a.



7b. between −3 and −2, between −1 and 0, between 1 and 2, between 2 and 3

7c. Sample answer: rel. max. at $x = 0.6$, rel. min. at $x = -1.5$ and at $x = 2.4$