# Key Concept #2 – Polynomials

A. Perform arithmetic operations, including adding, subtracting, and multiplying with complex numbers. [N.CN.2, N.CN.1]

B. Graph, transform and identify the key features of a polynomial function and make connections between algebraic and graphical forms. [F.IF.4, F.IF.7c, A.APR.3, F.BF.3]
   - Key features: minimum and maximum, domain and range, end behavior, intercepts
   - Identify the degree of the polynomial
   - Multiplicity of factors
   - Create a rough sketch of a graph using zeros.
   - Odd and even functions – Technology Lab 4-1

C. Add, subtract, and multiply polynomials and explain why solutions are equivalent. [A.SSE.2, A.APR.1]

D. Create polynomials functions given factors and zeros. [A.CED.2]

E. Apply the Remainder Theorem to determine the factors and zeros of a polynomial function. [A.APR.2, A.APR.6]
   - Multiplicity of factors
   - Rewrite a rational expression into its quotient and remainder/ divisor.
   - Long division and synthetic division.

F. Solve polynomials algebraically and graphically by using technology. [A.SSE.2, A.REI.1, A.REI.11, A.APR.3, A.APR.4]
   - Prove polynomial identities*** (revisit) (Technology Lab 4-6)
   - $(x^2-1)(x^2+1)$

G. Analyze a polynomial function in multiple representations (equation, table or graph) within a context and make conclusions on the features. [N.Q.2, F.IF.4, F.IF.5, F.IF.6, F.IF.7c]
   - Increasing and decreasing intervals
   - Interpret x and/or y intercepts.
   - Interpret relative min and max.
   - Estimate rate of change.
   - Math III book, section 4.3 problem #6 **Very good question!

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**Target 2A: Perform arithmetic operations, including adding, subtracting, and multiplying with complex numbers. [N.CN.2, N.CN.1]**

You know that the real numbers consist of the rational numbers and the irrational numbers. The set of real numbers is itself a subset of a larger set of numbers, the *complex numbers*. Curiously, the complex numbers include a number like $a$ in the Solve It. Its fifth power is itself.

**Essential Understanding** The complex numbers are based on a number whose square is $-1$.

The **imaginary unit** $i$ is the complex number whose square is $-1$. So $i^2 = -1$, and $i = \sqrt{-1}$.

take note

**Key Concept** Square Root of a Negative Real Number

**Algebra**

For any positive number $a$,

$\sqrt{-a} = \sqrt{-1 \cdot a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$

Note that $(\sqrt{-5})^2 = (i\sqrt{5})^2 = i^2(\sqrt{5})^2 = -1 \cdot 5 = -5$ (not 5).

**Example**

$\sqrt{-5} = i\sqrt{5}$

## Problem 1 Simplifying a Number Using $i$

**How do you write $\sqrt{-18}$ by using the imaginary unit $i$?**

$$\begin{aligned}
\sqrt{-18} &= \sqrt{-1 \cdot 18} \\
&= \sqrt{-1} \cdot \sqrt{18} && \text{Multiplication Property of Square Roots} \\
&= i \cdot \sqrt{18} && \text{Definition of } i \\
&= i \cdot 3\sqrt{2} && \text{Simplify.} \\
&= 3i\sqrt{2}
\end{aligned}$$

An **imaginary number** is any number of the form $a + bi$, where $a$ and $b$ are real numbers and $b \neq 0$. Imaginary numbers and real numbers together make up the set of *complex numbers*.

take note

### Key Concept Complex Numbers

You can write a **complex number** in the form $a + bi$, where $a$ and $b$ are real numbers.

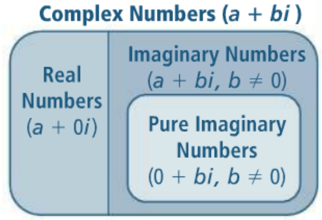
If $b = 0$, the number $a + bi$ is a real number.

If $a = 0$ and $b \neq 0$, the number $a + bi$ is a **pure imaginary number**.

$a \quad + \quad bi$

$\uparrow$ Real part $\qquad \uparrow$ Imaginary part

**Complex Numbers ($a + bi$)**




## Problem 3 Adding and Subtracting Complex Numbers

**What is each sum or difference?**

**A** $(4 - 3i) + (-4 + 3i)$

$4 + (-4) + (-3i) + 3i$ Use the commutative and associative properties.

$0 + 0 = 0$ $\quad 4 - 3i$ and $-4 + 3i$ are additive inverses.

**B** $(5 - 3i) - (-2 + 4i)$

$5 - 3i + 2 - 4i$ To subtract, add the opposite.

$5 + 2 - 3i - 4i$ Use the commutative and associative properties

$7 - 7i$ Simplify.

You multiply complex numbers $a + bi$ and $c + di$ as you would multiply binomials. For imaginary parts $bi$ and $di$, $(bi)(di) = bd(i)^2 = bd(-1) = -bd$.

## Problem 4 Multiplying Complex Numbers

**What is each product?**

**A** $(3i)(-5 + 2i)$

$-15i + 6i^2$ Distributive Property

$-15i + 6(-1)$ Substitute $-1$ for $i^2$.

$-6 - 15i$ Simplify.

**B** $(4 + 3i)(-1 - 2i)$

$-4 - 8i - 3i - 6i^2$

$-4 - 8i - 3i - 6(-1)$

$2 - 11i$

Substitute $-1$ for $i^2$.

**C** $(-6 + i)(-6 - i)$

$36 + 6i - 6i - i^2$

$36 + 6i - 6i - (-1)$

$37$

## Problem 5 Dividing Complex Numbers

**What is each quotient?**

**A** $\dfrac{9 + 12i}{3i}$

$\dfrac{9 + 12i}{3i} \cdot \dfrac{-3i}{-3i}$

$\dfrac{-27i - 36i^2}{-9i^2}$

$\dfrac{-27i - 36(-1)}{-9(-1)}$

$\dfrac{36 - 27i}{9}$

$4 - 3i$

Multiply numerator and denominator by the complex conjugate of the denominator.

Substitute $-1$ for $i^2$.

**B** $\dfrac{2 + 3i}{1 - 4i}$

$\dfrac{2 + 3i}{1 - 4i} \cdot \dfrac{1 + 4i}{1 + 4i}$

$\dfrac{2 + 8i + 3i + 12i^2}{1 + 4i - 4i - 16i^2}$

$\dfrac{2 + 8i + 3i + 12(-1)}{1 + 4i - 4i - 16(-1)}$

$\dfrac{-10 + 11i}{17}$

$-\dfrac{10}{17} + \dfrac{11}{17}i$

**Target 2B:** Graph, transform and identify the key features of a polynomial function and make connections between algebraic and graphical forms. [F.IF.4, F.IF.7c, A.APR.3, F.BF.3]

The sequence of numbers in the first column of the Solve It are values of a particular *polynomial function*. For such a sequence, you can use patterns of 1st differences, 2nd differences, 3rd differences, and so on, to learn more about the polynomial function.

**Essential Understanding** A polynomial function has distinguishing "behaviors." You can look at its algebraic form and know something about its graph. You can look at its graph and know something about its algebraic form.

A **monomial** is a real number, a variable, or a product of a real number and one or more variables with whole-number exponents. The **degree of a monomial** in one variable is the exponent of the variable. A **polynomial** is a monomial or a sum of monomials. The **degree of a polynomial** in one variable is the greatest degree among its monomial terms.

A polynomial with the variable $x$ defines a **polynomial function** of $x$. The degree of the polynomial function is the same as the degree of the polynomial.

take note

## Key Concepts Standard Form of a Polynomial Function

The **standard form of a polynomial function** arranges the terms by degree in descending numerical order.

A polynomial function $P(x)$ in standard form is

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

where $n$ is a nonnegative integer and $a_n, \ldots, a_0$ are real numbers.

$$P(x) = 4x^3 + 3x^2 + 5x - 2$$

Cubic term | Quadratic term | Linear term | Constant term

You can classify a polynomial by its degree or by its number of terms. Polynomials of degrees zero through five have specific names, as shown in this table.

| Degree | Name Using Degree | Polynomial Example | Number of Terms | Name Using Number of Terms |
|---|---|---|---|---|
| 0 | constant | $5$ | 1 | monomial |
| 1 | linear | $x + 4$ | 2 | binomial |
| 2 | quadratic | $4x^2$ | 1 | monomial |
| 3 | cubic | $4x^3 - 2x^2 + x$ | 3 | trinomial |
| 4 | quartic | $2x^4 + 5x^2$ | 2 | binomial |
| 5 | quintic | $-x^5 + 4x^2 + 2x + 1$ | 4 | polynomial of 4 terms |

## Problem 1 Classifying Polynomials

**Write each polynomial in standard form. What is the classification of each polynomial by degree? by number of terms?**

**A** $3x + 9x^2 + 5$

$9x^2 + 3x + 5$

The polynomial has degree 2 and 3 terms. It is a quadratic trinomial.

**B** $4x - 6x^2 + x^4 + 10x^2 - 12$

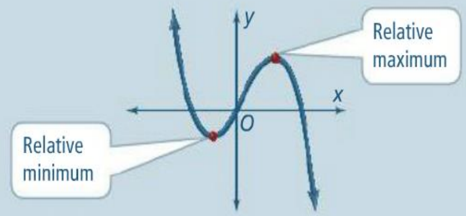
$x^4 + 4x^2 + 4x - 12$

The polynomial has degree 4 and 4 terms. It is a quartic polynomial of 4 terms.

The degree of a polynomial function affects the shape of its graph and determines the maximum number of **turning points**, or places where the graph changes direction. It also affects the **end behavior**, or the directions of the graph to the far left and to the far right.

The table below shows you examples of polynomial functions and the four types of end behavior. The table also shows intervals where the functions are increasing and decreasing. A function is *increasing* when the $y$-values increase as $x$-values increase. A function is *decreasing* when the $y$-values decrease as $x$-values increase.

If the graph of a polynomial function has several turning points, the function can have a **relative maximum** and a **relative minimum**. A relative maximum is the value of the function at an up-to-down turning point. A relative minimum is the value of the function at a down-to-up turning point.




take note

## Key Concepts Polynomial Functions

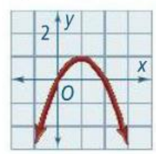
$y = 4x^4 + 6x^3 - x$


End behavior: up and up

Turning points: $(-1.07, -1.04)$, $(-0.27, 0.17)$, and $(0.22, -0.15)$

The function is decreasing when $x < -1.07$ and $-0.27 < x < 0.22$. The function is increasing when $-1.07 < x < -0.27$ and $x > 0.22$.

$y = -x^2 + 2x$




End behavior: down and down

Turning point: $(1, 1)$

The function is increasing when $x < 1$ and is decreasing when $x > 1$.

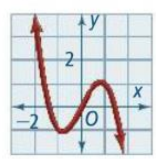
$y = x^3$


End behavior: down and up

No turning points

The function is increasing for all $x$.

$y = -x^3 + 2x$




End behavior: up and down

Turning Points: $(-0.82, -1.09)$ and $(0.82, 1.09)$

The function is decreasing when $x < -0.82$ and when $x > 0.82$. The function is increasing when $-0.82 < x < 0.82$.

You can determine the end behavior of a polynomial function of degree $n$ from the leading term $ax^n$ of the standard form.

**End Behavior of a Polynomial Function With Leading Term $ax^n$**

| | $n$ Even ($n \neq 0$) | $n$ Odd |
|---|---|---|
| $a$ Positive | Up and up | Down and up |
| $a$ Negative | Down and down | Up and down |

In general, the graph of a polynomial function of degree $n$ ($n \geq 1$) has at most $n - 1$ turning points. The graph of a polynomial function of odd degree has an even number of turning points. The graph of a polynomial function of even degree has an odd number of turning points.

## Problem 2 Describing End Behavior of Polynomial Functions

**Consider the leading term of each polynomial function. What is the end behavior of the graph? Check your answer with a graphing calculator.**

**A** $y = 4x^3 - 3x$

The leading term is $4x^3$. Since $n$ is odd and $a$ is positive, the end behavior is down and up.

**B** $y = -2x^4 + 8x^3 - 8x^2 + 2$

The leading term is $-2x^4$. Since $n$ is even and $a$ is negative, the end behavior is down and down.

## Problem 3 Graphing Cubic Functions

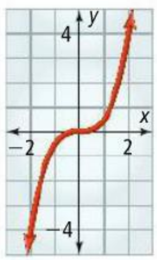
**What is the graph of each cubic function? Describe the graph, including end behavior, turning points, and increasing/decreasing intervals.**

**A** $y = \frac{1}{2}x^3$

**Step 1**

| $x$ | $y$ |
|---|---|
| −2 | −4 |
| −1 | −0.5 |
| 0 | 0 |
| 1 | 0.5 |
| 2 | 4 |

**Step 2**




**Step 3**

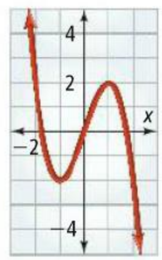
The end behavior is down and up. There are no turning points. The function increases from $-\infty$ to $\infty$.

**B** $y = 3x - x^3$

**Step 1**

| $x$ | $y$ |
|---|---|
| −2 | 2 |
| −1 | −2 |
| 0 | 0 |
| 1 | 2 |
| 2 | −2 |

**Step 2**




**Step 3**

The end behavior is up and down. There are turning points at $(-1, -2)$ and $(1, 2)$. The function decreases from $-\infty$ to $-1$, increases from $-1$ to $1$, and decreases from $1$ to $\infty$.

In this activity you will learn how to sketch the graph of a polynomial function by using the zeros, turning points, and end behavior.

## Example

**Sketch the graph of the function $f(x) = (x - 3)(x + 1)(x - 2)$.**

**Step 1** Identify the zeros and plot them on a coordinate grid.

$f(x) = (x - 3)(x + 1)(x - 2)$

$0 = (x - 3)(x + 1)(x - 2)$

$0 = x - 3$ or $0 = x + 1$ or $0 = x - 2$

$3 = x$ $\quad -1 = x$ $\quad 2 = x$

The function has zeros at $(3, 0)$, $(-1, 0)$, and $(2, 0)$.

**Step 2** Determine whether the polynomial is positive or negative over each interval.

To determine whether $f(x)$ is positive or negative over the interval $x < -1$, choose an $x$-value within the interval. Let $x = -2$. Then evaluate $f(-2)$.

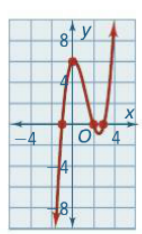
$f(-2) = (-2 - 3)(-2 + 1)(-2 - 2) = (-5)(-1)(-4) = -20$

$f(x)$ is negative over the interval $x < -1$.

Repeat the process for the intervals $-1 < x < 2$, $2 < x < 3$, and $x > 3$.

| Interval | $x$ | $f(x)$ |
|---|---|---|
| $x < -1$ | −2 | −20 |
| $-1 < x < 2$ | 0 | 6 |
| $2 < x < 3$ | 2.5 | −0.875 |
| $x > 3$ | 4 | 10 |

**Step 3** Sketch the graph.



**Target 2C: Add, subtract, and multiply polynomials and explain why solutions are equivalent. [A.SSE.2, A.APR.1]**

**Essential Understanding** You can add, subtract, multiply, and divide functions based on how you perform these operations for real numbers. One difference, however, is that you must consider the domain of each function.

take note

**Key Concepts** Function Operations

| | |
|---|---|
| Addition | $(f+g)(x) = f(x) + g(x)$ |
| Subtraction | $(f-g)(x) = f(x) - g(x)$ |
| Multiplication | $(f \cdot g)(x) = f(x) \cdot g(x)$ |
| Division | $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} g(x) \neq 0$ |

The domains of the sum, difference, product, and quotient functions consist of the $x$-values that are in the domains of *both* $f$ and $g$. Also, the domain of the quotient function does not contain any $x$-value for which $g(x) = 0$.

## Problem 1 Adding and Subtracting Functions

**Let $f(x) = 4x + 7$ and $g(x) = \sqrt{x} + x$. What are $f + g$ and $f - g$? What are their domains?**

$$(f+g)(x) = f(x) + g(x) = (4x+7) + (\sqrt{x} + x) = 5x + \sqrt{x} + 7$$

$$(f-g)(x) = f(x) - g(x) = (4x+7) - (\sqrt{x} + x) = 3x - \sqrt{x} + 7$$

The domain of $f$ is the set of all real numbers. The domain of $g$ is all $x \geq 0$. The domain of both $f + g$ and $f - g$ is the set of numbers common to the domains of both $f$ and $g$, which is all $x \geq 0$.

## Problem 2 Multiplying and Dividing Functions

**Let $f(x) = x^2 - 9$ and $g(x) = x + 3$. What are $f \cdot g$ and $\frac{f}{g}$ and their domains?**

$$(f \cdot g)(x) = f(x) \cdot g(x) = (x^2 - 9)(x + 3)$$
$$= x^3 + 3x^2 - 9x - 27$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 9}{x + 3} = \frac{(x-3)(x+3)}{x+3} = x - 3, x \neq -3$$

The domain of both $f$ and $g$ is the set of real numbers, so the domain of $f \cdot g$ is also the set of real numbers.
The domain of $\frac{f}{g}$ is the set of all real numbers except $x \neq -3$, because $g(-3) = 0$.
The definition of $\frac{f}{g}$ requires that you consider the zero denominator in the *original* expression for $\frac{f(x)}{g(x)}$ despite the fact that the simplified form has the domain all real numbers.

## Problem 2 Subtracting Polynomials

**What is the difference $(6x^4 - 2x^2 + 3x) - (4x^4 - 9x^2 + 7)$?**

**Method 1** Subtract vertically.

$$\begin{array}{r} 6x^4 - 2x^2 + 3x \\ \underline{-(4x^4 - 9x^2 \qquad + 7)} \end{array}$$

Line up like terms.

$$\begin{array}{r} 6x^4 - 2x^2 + 3x \\ \underline{-4x^4 + 9x^2 \qquad - 7} \\ 2x^4 + 7x^2 + 3x - 7 \end{array}$$

Combine like terms.

**Method 2** Subtract horizontally.

$(6x^4 - 2x^2 + 3x) - (4x^4 - 9x^2 + 7)$

$= (6x^4 - 2x^2 + 3x) - 4x^4 + 9x^2 - 7$ — Write the opposite of each term of the second polynomial.

$= (6x^4 - 4x^4) + (-2x^2 + 9x^2) + (3x) - 7$ — Combine like terms.

$= 2x^4 + 7x^2 + 3x - 7$ — Combine like terms.

## Problem 3 Multiplying Polynomials

**A What is the product $(2x + 3y)(7x - 4y)$?**

$(2x + 3y)(7x - 4y) = 14x^2 - 8xy + 21xy - 12y^2$ — Use the FOIL method.

$= 14x^2 + 13xy - 12y^2$ — Combine like terms.

**B What is the product $(x - 5)(4x^2 + 9x - 1)$?**

$(x - 5)(4x^2 + 9x - 1)$

$= x(4x^2 + 9x - 1) - 5(4x^2 + 9x - 1)$ — Distributive Property

$= 4x^3 + 9x^2 - x - 20x^2 - 45x + 5$

$= 4x^3 - 11x^2 - 46x + 5$ — Combine like terms.

**C What is the product $(c^2 + 2)(c + 3)(2c + 1)$?**

$(c^2 + 2)(c - 3)(2c + 1)$

$= (c^2 + 2)(2c^2 - 5c - 3)$ — Multiply $(c - 3)$ and $(2c + 1)$.

$= c^2(2c^2 - 5c - 3) + 2(2c^2 - 5c - 3)$ — Distributive Property

$= 2c^4 - 5c^3 - 3c^2 + 4c^2 - 10c - 6$

$= 2c^4 - 5c^3 + c^2 - 10c - 6$ — Combine like terms.

**Target 2D: Create polynomials functions given factors and zeros. [A.CED.2]**

## take note — Key Concepts: Roots, Zeros, and $x$-intercepts

The following are equivalent statements about a real number $b$ and a polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$.

- $x - b$ is a linear factor of the polynomial $P(x)$.
- $b$ is a zero of the polynomial function $y = P(x)$.
- $b$ is a root (or solution) of the polynomial equation $P(x) = 0$.
- $b$ is an $x$-intercept of the graph of $y = P(x)$.

## take note — Theorem 102: Factor Theorem

The expression $x - a$ is a factor of a polynomial if and only if the value $a$ is a zero of the related polynomial function.

## Problem 3 Writing a Polynomial Function From Its Zeros

**A What is a cubic polynomial function in standard form with zeros −2, 2, and 3?**

$$\begin{array}{ll} \quad\;\; -2 \quad\;\; 2 \quad\;\; 3 & -2, 2, \text{ and } 3 \text{ are zeros.} \\ \quad\;\;\; \downarrow \quad\;\; \downarrow \quad\;\; \downarrow & \\ f(x) = (x+2)(x-2)(x-3) & \text{Write a linear factor for each zero.} \\ \quad\;\; = (x+2)(x^2-5x+6) & \text{Multiply } (x-2) \text{ and } (x-3). \\ \quad\;\; = x(x^2-5x+6) + 2(x^2-5x+6) & \text{Distributive Property} \\ \quad\;\; = x^3 - 5x^2 + 6x + 2x^2 - 10x + 12 & \text{Distributive Property} \\ \quad\;\; = x^3 - 3x^2 - 4x + 12 & \text{Simplify.} \end{array}$$

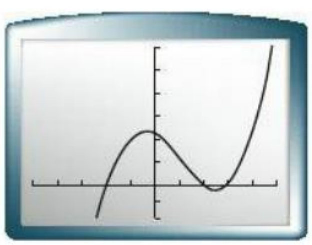
The cubic polynomial $f(x) = x^3 - 3x^2 - 4x + 12$ has zeros −2, 2, and 3.

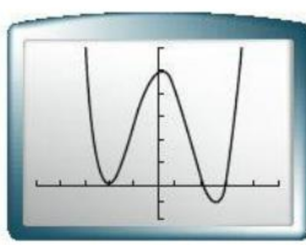
**B What is a quartic polynomial function in standard form with zeros −2, −2, 2, and 3?**

$$\begin{array}{ll} \quad\;\; -2 \quad\; -2 \quad\;\; 2 \quad\;\; 3 & -2, -2, 2, \text{ and } 3 \text{ are zeros.} \\ \quad\;\;\; \downarrow \quad\;\;\; \downarrow \quad\;\; \downarrow \quad\;\; \downarrow & \\ g(x) = (x+2)(x+2)(x-2)(x-3) & \text{Write a linear factor for each zero.} \\ \quad\;\; = x^4 - x^3 - 10x^2 + 4x + 24 & \text{Simplify.} \end{array}$$

The quartic polynomial $g(x) = x^4 - x^3 - 10x^2 + 4x + 24$ has zeros −2, −2, 2, and 3.

**C Graph both functions. How do the graphs differ? How are they similar?**





Both screens:
$x$-scale: 1
$y$-scale: 5

Both graphs have $x$-intercepts at −2, 2, and 3. The cubic has down-and-up end behavior. The quartic has up-and-up end behavior.

The cubic function has two turning points, and it crosses the $x$-axis at −2. The quartic function touches the $x$-axis at −2 but does not cross it. The quartic function has three turning points.

Recall that $i^2 = -1$, or $i = \sqrt{-1}$. An imaginary number is any number of the form $a + bi$, where $a$ and $b$ are real numbers and $b \neq 0$. Imaginary numbers and real numbers together make up the set of *complex numbers*.

The complex numbers $a + bi$ and $a - bi$ are conjugates. Similarly, the irrational numbers $a + \sqrt{b}$ and $a - \sqrt{b}$ are conjugates. If a complex number or an irrational number is a root of a polynomial equation with rational coefficients, so is its conjugate.

take note

### Theorem 105 Conjugate Root Theorem

If $P(x)$ is a polynomial with *rational* coefficients, then irrational roots of $P(x) = 0$ that have the form $a + \sqrt{b}$ occur in conjugate pairs. That is, if $a + \sqrt{b}$ is an irrational root with $a$ and $b$ rational, then $a - \sqrt{b}$ is also a root.

If $P(x)$ is a polynomial with *real* coefficients, then the complex roots of $P(x) = 0$ occur in conjugate pairs. That is, if $a + bi$ is a complex root with $a$ and $b$ real, then $a - bi$ is also a root.

## Problem 3 Using the Conjugate Root Theorem to Identify Roots

**A quartic polynomial $P(x)$ has rational coefficients. If $\sqrt{2}$ and $1 + i$ are roots of $P(x) = 0$, what are the two other roots?**

Since $P(x)$ has rational coefficients and $0 + \sqrt{2}$ is a root of $P(x) = 0$, it follows from the Conjugate Root Theorem that $0 - \sqrt{2}$ is also a root.

Since $P(x)$ has real coefficients and $1 + i$ is a root of $P(x) = 0$, it follows that $1 - i$ is also a root.

The two other roots are $-\sqrt{2}$ and $1 - i$.

## Problem 4 Using Conjugates to Construct a Polynomial

**Multiple Choice What is a third-degree polynomial function $y = P(x)$ with rational coefficients so that $P(x) = 0$ has roots $-4$ and $2i$?**

A $P(x) = x^3 - 2x^2 - 16x + 32$

B $P(x) = x^3 - 4x^2 + 4x - 16$

C $P(x) = x^3 + 4x^2 + 4x + 16$

D $P(x) = x^3 + 4x^2 - 4x - 16$

Since $2i$ is a root, then $-2i$ is also a root.

| | |
|---|---|
| $P(x) = (x + 2i)(x - 2i)(x + 4)$ | Write the polynomial function. |
| $= (x^2 + 4)(x + 4)$ | Multiply the complex conjugates. |
| $= x^3 + 4x^2 + 4x + 16$ | Write the polynomial function in standard form. |

The equation $x^3 + 4x^2 + 4x + 16 = 0$ has rational coefficients and has roots $-4$ and $2i$. The correct answer is C.

**Target 2E: Apply the Remainder Theorem to determine the factors and zeros of a polynomial function. [A.APR.2, A.APR.6]**

take note

**Theorem 103 The Remainder Theorem**

If you divide a polynomial $P(x)$ of degree $n \geq 1$ by $x - a$, then the remainder is $P(a)$.

**Synthetic division** simplifies the long-division process for dividing by a linear expression $x - a$. To use synthetic division, write the coefficients (including zeros) of the polynomial in standard form. Omit all variables and exponents. For the divisor, reverse the sign (use $a$). This allows you to add instead of subtract throughout the process.

## Problem 3 Using Synthetic Division

**Use synthetic division to divide $x^3 - 14x^2 + 51x - 54$ by $x + 2$. What is the quotient and remainder?**

**Step 1** Reverse the sign of $+2$. Write the coefficients of the polynomial.

$$\begin{array}{r|rrrr} -2 & 1 & -14 & 51 & -54 \end{array}$$

**Step 2** Bring down the first coefficient.

$$\begin{array}{r|rrrr} -2 & 1 & -14 & 51 & -54 \\ & & & & \\ \hline & 1 & & & \end{array}$$

**Step 3** Multiply the coefficient by the divisor. Add to the next coefficient.

$$\begin{array}{r|rrrr} -2 & 1 & -14 & 51 & -54 \\ & & -2 & & \\ \hline & 1 & -16 & & \end{array}$$

**Step 4** Continue multiplying and adding through the last coefficient.

$$\begin{array}{r|rrrr} -2 & 1 & -14 & 51 & -54 \\ & & -2 & 32 & -166 \\ \hline & 1 & -16 & 83 & -220 \end{array}$$

The quotient is $x^2 - 16x + 83$, R $-220$.

## Problem 5 Evaluating a Polynomial

**Given that $P(x) = x^5 - 2x^3 - x^2 + 2$, what is $P(3)$?**

By the Remainder Theorem, $P(3)$ is the remainder when you divide $P(x)$ by $x - 3$.

$$\begin{array}{r|rrrrrr} 3 & 1 & 0 & -2 & -1 & 0 & 2 \\ & & 3 & 9 & 21 & 60 & 180 \\ \hline & 1 & 3 & 7 & 20 & 60 & 182 \end{array}$$

$P(3) = 182$.

**Target 2F: Solve polynomials algebraically and graphically by using technology. [A.SSE.2, A.REI.1, A.REI.11, A.APR.3, A.APR.4]**

Factoring a polynomial like $ax^2 + bx + c$ can help you solve a polynomial equation like $ax^2 + bx + c = 0$.

**Essential Understanding** If $(x - a)$ is a factor of a polynomial, then the polynomial has value 0 when $x = a$. If $a$ is a real number, then the graph of the polynomial has $(a, 0)$ as an $x$-intercept.

To solve a polynomial equation by factoring:

1. Write the equation in the form $P(x) = 0$ for some polynomial function $P$.
2. Factor $P(x)$. Use the Zero-Product Property to find the roots.

## Problem 1 Solving Polynomial Equations Using Factors

**What are the real or imaginary solutions of each polynomial equation?**

**A** $2x^3 - 5x^2 = 3x$

| | |
|---|---|
| $2x^3 - 5x^2 - 3x = 0$ | Rewrite in the form $P(x) = 0$. |
| $x(2x^2 - 5x - 3) = 0$ | Factor out the GCF, $x$. |
| $x(2x + 1)(x - 3) = 0$ | Factor $2x^2 - 5x - 3$. |
| $x = 0$ or $2x + 1 = 0$ or $x - 3 = 0$ | Zero Product Property |
| $x = 0 \quad x = -\frac{1}{2} \quad x = 3$ | Solve each equation for $x$. |

The solutions are 0, $-\frac{1}{2}$, and 3.

**B** $3x^4 + 12x^2 = 6x^3$

| | |
|---|---|
| $3x^4 - 6x^3 + 12x^2 = 0$ | Rewrite in the form $P(x) = 0$. |
| $x^4 - 2x^3 + 4x^2 = 0$ | Multiply by $\frac{1}{3}$ to simplify. |
| $x^2(x^2 - 2x + 4) = 0$ | Factor out the GCF, $x^2$. |
| $x^2 = 0$ or $x^2 - 2x + 4 = 0$ | Zero Product Property |

$x = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

Use the Quadratic Formula to solve $x^2 - 2x + 4 = 0$. Substitute $a = 1$, $b = -2$, and $c = 4$.

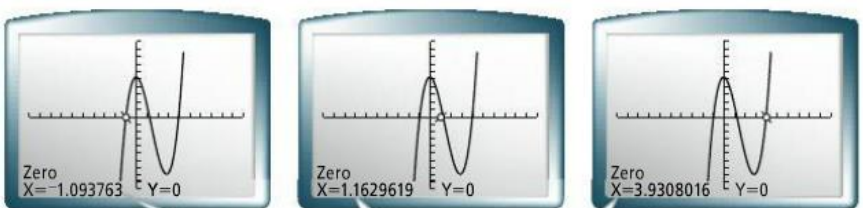
$$x = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

The solutions are 0, $1 + i\sqrt{3}$, and $1 - i\sqrt{3}$.

## Problem 3 Finding Real Roots by Graphing

**What are the real solutions of the equation $x^3 + 5 = 4x^2 + x$?**

Rewrite the equation as $x^3 - 4x^2 - x + 5 = 0$. Graph the related function $y = x^3 - 4x^2 - x + 5$. Use the **ZERO** feature.




The solutions are the same as identified using Method 1.

Approximate solutions are $x = -1.09$, $x = 1.16$, and $x = 3.93$.

**Target 2G: Analyze a polynomial function in multiple representations (equation, table or graph) within a context and make conclusions on the features. [N.Q.2, F.IF.4, F.IF.5, F.IF.6, F.IF.7c]**

Complete the table for $P(x) = x^3 - 2x^2 - 5x + 6$.

| $x$ | −5 | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(x)$ | -144 | -70 | -24 | 0 | 8 | 6 | 0 | -4 | 0 | 18 | 56 |

Y

Determine an interval where the polynomial is increasing.

(-2,-1) (3,4) ...

Determine an interval where the polynomial is decreasing.

(1,2) (0,1)...

Determine the interval in which the relative minimum occurs.

(1,3)

Determine the interval in which the relative maximum occurs.

(-2,0)

Complete the table for $P(x) = 3x^3 - 6x^2 - 16x + 32$.

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(x)$ | -413 | -192 | -55 | 16 | 39 | 32 | 13 | 0 | 11 | 64 | 177 |

Determine the interval in which the relative minimum occurs.

(1, 3)

Determine the interval in which the relative maximum occurs.

(-2, 0)

## Problem 6 Using a Polynomial Function to Maximize Volume

**Technology The design of a digital box camera maximizes the volume while keeping the sum of the dimensions at 6 inches. If the length must be 1.5 times the height, what should each dimension be?**

**Step 1** Define a variable $x$.

Let $x =$ the height of the camera.

**Step 2** Determine length and width.

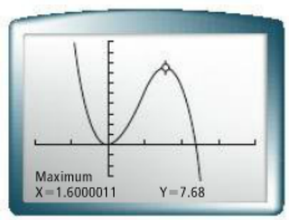
length $= 1.5x$; width $= 6 - (x + 1.5x) = 6 - 2.5x$

**Step 3** Model the volume.

$V = (\text{length})(\text{width})(\text{height}) = (1.5x)(6 - 2.5x)(x)$

$= -3.75x^3 + 9x^2$

**Step 4** Graph the polynomial function. Use the **MAXIMUM** feature to find that the maximum volume is 7.68 in.$^3$ for a height of 1.6 in.




height $= x = 1.6$

length $= 1.5x = 1.5(1.6) = 2.4$

width $= 6 - 2.5x = 6 - 2.5(1.6) = 2$

The dimensions of the camera should be 2.4 in. long by 2 in. wide by 1.6 in. high.