Key Concept #3 - Rationals

A. Perform operations with rational expressions to demonstrate the analogy with integers. [A.SSE.1.b, A.SSE.2, A.APR.6, A.APR.7]
   - Simplify rational expressions (polynomial expressions)
   - Perform operations on rational expressions (monomial expressions only)

B. Graph, transform and identify the key features of the graph of a rational function. [F.IF.4, F.IF.7d, F.BF.3]
   - Key features: end behavior, domain and range, intercepts, horizontal and vertical asymptotes, transformations from the parent function.
   - With and without using technology.

C. Solve rational equations and identify, if any, extraneous solutions. [A.REI.1, A.REI.2, A.CED.4]
   - With and without technology

D. Solve systems involving rational equations and identify, if any, extraneous solutions. [A.CED.3, A.REI.11]
   - With and without technology

Target 3A: Perform operations with rational expressions to demonstrate the analogy with integers. [A.SSE.1.b, A.SSE.2, A.APR.6, A.APR.7]

Many problems require finding products and quotients of rational expressions.

Essential Understanding You can multiply and divide rational expressions using the same properties you use to multiply and divide numerical fractions.

If a, b, c, and d represent polynomials (where $b \neq 0$ and $d \neq 0$), then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Problem 1 Multiplying Rational Expressions

What is the product? State any excluded values.

A $\frac{6}{a^2} \cdot \frac{-2}{a^3}$

$\frac{6}{a^2} \cdot \frac{-2}{a^3} = \frac{6(-2)}{a^2(a^3)}$ Multiply numerators and multiply denominators.

$= \frac{-12}{a^5}$ Simplify.

The product is $\frac{-12}{a^5}$, where $a \neq 0$.

B $\frac{x-7}{x} \cdot \frac{x-5}{x+3}$

$\frac{x-7}{x} \cdot \frac{x-5}{x+3} = \frac{(x-7)(x-5)}{x(x+3)}$ Multiply numerators and multiply denominators. Leave the product in factored form.

The product is $\frac{(x-7)(x-5)}{x(x+3)}$, where $x \neq 0$ and $x \neq -3$.

Problem 2 Using Factoring

What is the product $\frac{x+5}{7x-21} \cdot \frac{14x}{x^2+3x-10}$?

$$\frac{x+5}{7x-21} \cdot \frac{14x}{x^2+3x-10} = \frac{x+5}{7(x-3)} \cdot \frac{14x}{(x+5)(x-2)}$$ Factor denominators.

$$= \frac{\cancel{x+5}^{1}}{{}_{1}\cancel{7}(x-3)} \cdot \frac{\cancel{14}^{2}x}{{}_{1}\cancel{(x+5)}(x-2)}$$ Divide out the common factors 7 and $x+5$.

$$= \frac{1}{x-3} \cdot \frac{2x}{x-2}$$ Simplify.

$$= \frac{2x}{(x-3)(x-2)}$$ Multiply numerators and multiply denominators. Leave the product in factored form.

Problem 3 Multiplying a Rational Expression by a Polynomial

What is the product $\frac{2m+5}{3m-6} \cdot (m^2+m-6)$?

$$\frac{2m+5}{3m-6} \cdot (m^2+m-6) = \frac{2m+5}{3(m-2)} \cdot \frac{(m-2)(m+3)}{1}$$ Factor.

$$= \frac{(2m+5)}{3_{1}\cancel{(m-2)}} \cdot \frac{\cancel{(m-2)}^{1}(m+3)}{1}$$ Divide out the common factor $m-2$.

$$= \frac{(2m+5)(m+3)}{3}$$ Multiply. Leave the product in factored form.

Recall that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, where $b \neq 0$, $c \neq 0$, and $d \neq 0$. When you divide rational expressions, first rewrite the quotient as a product using the reciprocal before dividing out common factors.

Problem 4 Dividing Rational Expressions

What is the quotient $\frac{x^2-25}{4x+28} \div \frac{x-5}{x^2+9x+14}$?

Think	Write
To divide by a rational expression, multiply by its reciprocal.	$\frac{x^2-25}{4x+28} \div \frac{x-5}{x^2+9x+14}$
	$= \frac{x^2-25}{4x+28} \cdot \frac{x^2+9x+14}{x-5}$
Before multiplying, factor.	$= \frac{(x+5)(x-5)}{4(x+7)} \cdot \frac{(x+7)(x+2)}{x-5}$
Divide out the common factors $x-5$ and $x+7$.	$= \frac{(x+5)\cancel{(x-5)}^{1}}{4_{1}\cancel{(x+7)}} \cdot \frac{\cancel{(x+7)}^{1}(x+2)}{{}_{1}\cancel{x-5}}$
Multiply numerators and multiply denominators. Leave the quotient in factored form.	$= \frac{(x+5)(x+2)}{4}$

A **complex fraction** is a fraction that contains one or more fractions in its numerator, in its denominator, or in both. You can simplify a complex fraction by dividing its numerator by its denominator.

Any complex fraction of the form $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ (where $b \neq 0$, $c \neq 0$, and $d \neq 0$) can be expressed as $\frac{a}{b} \div \frac{c}{d}$.

Problem 6 Simplifying a Complex Fraction

What is the simplified form of $\dfrac{\frac{1}{x-2}}{\frac{x+3}{x^2-4}}$?

$$\dfrac{\frac{1}{x-2}}{\frac{x+3}{x^2-4}} = \frac{1}{x-2} \div \frac{x+3}{x^2-4}$$ Write as a quotient.

$$= \frac{1}{x-2} \cdot \frac{x^2-4}{x+3}$$ Multiply by the reciprocal.

$$= \frac{1}{x-2} \cdot \frac{(x+2)(x-2)}{x+3}$$ Factor.

$$= \frac{1}{{}_1\cancel{x-2}} \cdot \frac{(x+2)\cancel{(x-2)}^1}{x+3}$$ Divide out the common factor $x - 2$.

$$= \frac{x+2}{x+3}$$ Simplify.

Essential Understanding You can use the same rules to add and subtract rational expressions that you use to add and subtract numerical fractions.

You can add the numerators of rational expressions with like denominators. If a, b, and c represent polynomials (with $c \neq 0$), then $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$.

Problem 1 Adding Expressions With Like Denominators

Problem 2 Subtracting Expressions With Like Denominators

What is the difference $\frac{7x+5}{3x^2-x-2} - \frac{4x+3}{3x^2-x-2}$?

$$\frac{7x+5}{3x^2-x-2} - \frac{4x+3}{3x^2-x-2} = \frac{7x+5-(4x+3)}{3x^2-x-2}$$ Subtract the numerators.

$$= \frac{7x+5-4x-3}{3x^2-x-2}$$ Distributive Property

To add or subtract rational expressions with different denominators, you can write the expressions with the least common denominator (LCD).

$$= \frac{1}{x-1}$$ Simplify.

Problem 3 Adding Expressions With Different Denominators

What is the sum $\frac{5}{6x} + \frac{3}{2x^2}$?

Step 1 Find the LCD of $\frac{5}{6x}$ and $\frac{3}{2x^2}$. First write the denominators $6x$ and $2x^2$ as products of prime factors. To form the LCD, list each factor the greatest number of times it appears in a denominator.

$$6x = 2 \cdot 3 \cdot x \qquad \text{Factor each denominator.}$$

$$2x^2 = 2 \cdot x \cdot x$$

$$\text{LCD} = 2 \cdot 3 \cdot x \cdot x = 6x^2 \qquad \text{The LCD is the LCM of } 6x \text{ and } 2x^2.$$

Step 2 Rewrite each rational expression using the LCD and then add.

$$\frac{5}{6x} + \frac{3}{2x^2} = \frac{5 \cdot x}{6x \cdot x} + \frac{3 \cdot 3}{2x^2 \cdot 3} \qquad \text{Rewrite each fraction using the LCD.}$$

$$= \frac{5x}{6x^2} + \frac{9}{6x^2} \qquad \text{Simplify numerators and denominators.}$$

$$= \frac{5x + 9}{6x^2} \qquad \text{Add the numerators.}$$

Problem 4 Subtracting Expressions With Different Denominators

What is the difference $\frac{3}{d-1} - \frac{2}{d+2}$?

Step 1 Find the LCD of $\frac{3}{d-1}$ and $\frac{2}{d+2}$.

Since there are no common factors, the LCD is $(d-1)(d+2)$.

Step 2 Rewrite each rational expression using the LCD and then subtract.

$$\frac{3}{d-1} - \frac{2}{d+2} = \frac{3(d+2)}{(d-1)(d+2)} - \frac{2(d-1)}{(d-1)(d+2)} \qquad \text{Rewrite each fraction using the LCD.}$$

$$= \frac{3d+6}{(d-1)(d+2)} - \frac{2d-2}{(d-1)(d+2)} \qquad \text{Simplify each numerator.}$$

$$= \frac{3d + 6 - (2d - 2)}{(d-1)(d+2)} \qquad \text{Subtract the numerators.}$$

$$= \frac{d+8}{(d-1)(d+2)} \qquad \text{Simplify the numerator.}$$

Target 3B: Graph, transform and identify the key features of the graph of a rational function. [F.IF.4, F.IF.7d, F.BF.3]

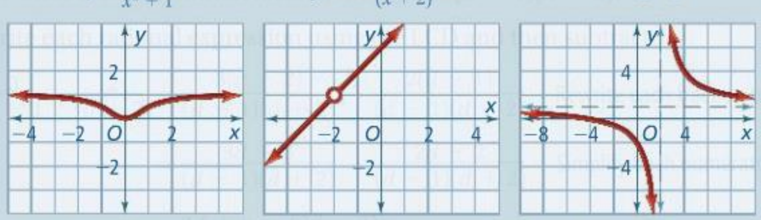
You use a ratio of polynomial functions to form a *rational function*, like $y = \frac{x+3}{x+16}$.

Essential Understanding If a function has a polynomial in its denominator, its graph has a gap at each zero of the polynomial. The gap could be a one-point hole in the graph, or it could be the location of a vertical asymptote for the graph.

A **rational function** is a function that you can write in the form $fx = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions. The domain of $f(x)$ is all real numbers except those values for which $Q(x) = 0$.

Here are graphs of three rational functions:

$y = \frac{x^2}{x^2+1}$ $\qquad y = \frac{(x+3)(x+2)}{(x+2)}$ $\qquad y = \frac{x+4}{x-2}$



For the first rational function, $y = \frac{x^2}{x^2+1}$, there is no value of x that makes the denominator 0. The graph is a **continuous graph** because it has no jumps, breaks, or holes. You can draw the graph and your pencil never leaves the paper.

For the second rational function, $y = \frac{(x+3)(x+2)}{x+2}$, x cannot be -2. For $y = \frac{x+4}{x-2}$, x cannot be 2. The second and third graphs are **discontinuous graphs**.

take note

Key Concept Point of Discontinuity

If a is a real number for which the denominator of a rational function $f(x)$ is zero, then a is not in the domain of $f(x)$. The graph of $f(x)$ is not continuous at $x = a$ and the function has a **point of discontinuity** at $x = a$.

The graph of $y = \frac{(x+3)(x+2)}{x+2}$ has a **removable discontinuity** at $x = -2$. The hole in the graph is called a removable discontinuity because you could make the function continuous by redefining it at $x = -2$ so that $f(-2) = 1$.

The graph of $y = \frac{x+4}{x-2}$ has a **non-removable discontinuity** at $x = 2$. There is no way to redefine the function at 2 to make the function continuous.

When you are looking for discontinuities, it is helpful to factor the numerator and denominator as a first step. The factors of the denominator will reveal the points of discontinuity. The discontinuity caused by $(x-a)^n$ in the denominator is removable if the numerator also has $(x-a)^n$ as a factor.

ONLINE PROBLEMS

Problem 1 Finding Points of Discontinuity

What are the domain and points of discontinuity of each rational function? Are the points of discontinuity removable or non-removable? What are the x- and y-intercepts?

A $y = \frac{x+3}{x^2-4x+3}$

Factor the numerator and denominator to check for common factors.

$$y = \frac{x+3}{x^2-4x+3} = \frac{x+3}{(x-3)(x-1)}$$

The function is undefined where $x - 3 = 0$ and where $x - 1 = 0$, at $x = 3$ and $x = 1$. The domain of the function is the set of all real numbers except $x = 1$ and $x = 3$.

There are non-removable points of discontinuity at $x = 1$ and $x = 3$.

The x-intercept occurs where the numerator equals 0, at $x = -3$.

To find the y-intercept, let $x = 0$ and simplify.

$$y = \frac{0+3}{(0-3)(0-1)} = \frac{3}{(-3)(-1)} = \frac{3}{3} = 1$$

B $y = \frac{x-5}{x^2+1}$

You cannot factor the numerator or the denominator. Also, there are no values of x that make the denominator 0. The domain of the function is all real numbers, and there are no discontinuities.

The x-intercept occurs where the numerator equals 0, at $x = 5$.

To find the y-intercept, let $x = 0$ and simplify: $y = \frac{0-5}{0^2+1} = \frac{-5}{1} = -5$

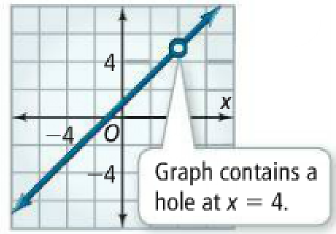
C $y = \frac{x^2 - 3x - 4}{x - 4}$

Factor the numerator and denominator: $y = \frac{x^2 - 3x - 4}{x - 4} = \frac{(x - 4)(x + 1)}{(x - 4)}$

The function is undefined where $x - 4 = 0$, at $x = 4$. The domain of the function is the set of all real numbers except $x = 4$.

Because $y = x + 1$, except at $x = 4$, there is a removable discontinuity at $x = 4$.

At $x = 4, y = x + 1 = 4 + 1 = 5$, so you can redefine the function to remove the discontinuity.



$$y = \begin{cases} \frac{x^2 - 3x - 4}{x - 4}, & \text{if } x \neq 4 \\ 5, & \text{if } x = 4 \end{cases}$$

The x-intercept occurs where the numerator equals 0, at $x = -1$.

To find the y-intercept, let $x = 0$ and simplify.

$$y = \frac{0^2 - 3 \cdot 0 - 4}{0 - 4} = \frac{0 - 0 - 4}{-4} = \frac{-4}{-4} = 1$$

take note

Key Concept Vertical Asymptotes of Rational Functions

The graph of the rational function $f(x) = \frac{P(x)}{Q(x)}$ has a vertical asymptote at each real zero of $Q(x)$ if $P(x)$ and $Q(x)$ have no common zeros. If $P(x)$ and $Q(x)$ have $(x - a)^m$ and $(x - a)^n$ as factors, respectively and $m < n$, then $f(x)$ also has a vertical asymptote at $x = a$.

While the graph of a rational function can have any number of vertical asymptotes, it can have no more than one horizontal asymptote.

take note

Key Concept Horizontal Asymptote of a Rational Function

To find the horizontal asymptote of the graph of a rational function, compare the degree of the numerator m to the degree of the denominator n.

If $m < n$, the graph has horizontal asymptote $y = 0$ (the x-axis).

If $m > n$, the graph has no horizontal asymptote.

If $m = n$, the graph has horizontal asymptote $y = \frac{a}{b}$ where a is the coefficient of the term of greatest degree in the numerator and b is the coefficient of the term of greatest degree in the denominator.

Problem 2 Finding Vertical Asymptotes

What are the vertical asymptotes for the graph of $y = \frac{(x + 1)}{(x - 2)(x - 3)}$?

Since 2 and 3 are zeros of the denominator and neither is a zero of the numerator, the lines $x = 2$ and $x = 3$ are vertical asymptotes.

Problem 3 Finding Horizontal Asymptotes

What is the horizontal asymptote for the rational function?

A $y = \frac{2x}{x - 3}$

The degree of the numerator and denominator are the same.

The horizontal asymptote is $y = \frac{2}{1}$ or $y = 2$.

B $y = \frac{x - 2}{x^2 - 2x - 3}$

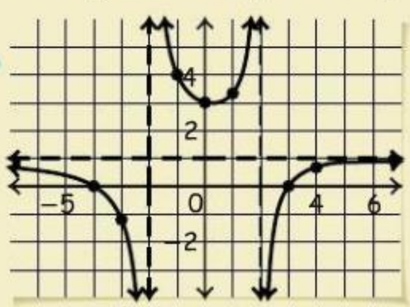
The degree of the numerator is less than the degree of the denominator. The horizontal asymptote is $y = 0$.

C $y = \frac{x^2}{2x - 5}$

The degree of the numerator is greater than the degree of the denominator. There is no horizontal asymptote.

Problem 4 Graphing a Rational Function

What is the graph of the rational function $y = \frac{x^2 + x - 12}{x^2 - 4}$?

Think	Write
The degrees of the numerator and denominator are equal.	$y = \frac{x^2 + x - 12}{x^2 - 4}$ horizontal asymptote: $y = \frac{1}{1} = 1$
Factor the numerator and the denominator. They have no common factor. The graph has no holes. It has two vertical asymptotes at the zeros of the denominator.	$y = \frac{(x + 4)(x - 3)}{(x + 2)(x - 2)}$ vertical asymptotes: $x = -2, x = 2$
Find the *x*- and *y*-intercepts. The *x*-intercepts occur where $y = 0$. The *y*-intercepts occur where $x = 0$.	When the numerator equals zero, $y = 0$. x-intercepts: $(-4, 0)$ and $(3, 0)$ $y = \frac{(0 + 4)(0 - 3)}{(0 + 2)(0 - 2)} = 3$ y-intercept: $(0, 3)$
Find a few more points on the graph.	More points on the graph: $(-3, -\frac{6}{5})$, $(-1, 4)$, $(1, \frac{10}{3})$ and $(4, \frac{2}{3})$
Graph the asymptotes. Then plot the intercepts and additional points. Use the points to sketch the graph.	

Target 3C: Solve rational equations and identify, if any, extraneous solutions. [A.REI.1, A.REI.2, A.CED.4]

Problem 1 Solving a Rational Equation

What are the solutions of the rational equation?

A $\frac{x}{x-3} + \frac{x}{x+3} = \frac{2}{x^2-9}$

Think	Write
Factor the denominators to find the LCD.	$\frac{x}{x-3} + \frac{x}{x+3} = \frac{2}{(x-3)(x+3)}$
Multiply each side by the LCD to clear denominators.	$(x-3)(x+3)\left[\frac{x}{x-3} + \frac{x}{x+3}\right] = (x-3)(x+3)\frac{2}{(x-3)(x+3)}$
Now simplify and solve.	$x^2 + 3x + x^2 - 3x = 2$ $2x^2 = 2$ $x^2 = 1$, so $x = \pm 1$
Check whether $x = 1$ or $x = -1$ is extraneous. Use the original equation.	$\frac{1}{1-3} + \frac{1}{1+3} \stackrel{?}{=} \frac{2}{(1)^2-9}$ $\quad$ $\frac{-1}{-1-3} + \frac{-1}{-1+3} \stackrel{?}{=} \frac{2}{(-1)^2-9}$ $-\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$ ✔ $\quad$ $\frac{1}{4} + -\frac{1}{2} = -\frac{1}{4}$ ✔
Write the solutions.	The solutions are $x = 1$ and $x = -1$.

Problem 3 Using a Graphing Calculator to Solve a Rational Equation

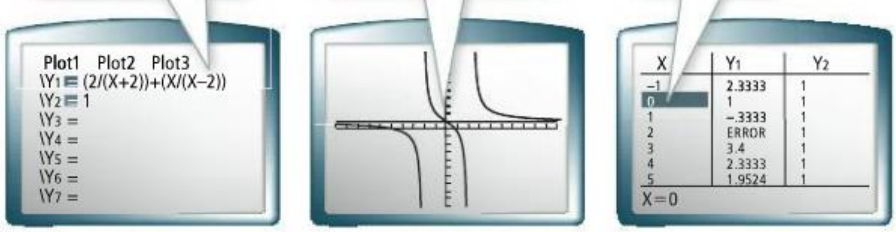
What are the solutions of the rational equation? Use a graphing calculator to solve.

$\frac{2}{x+2} + \frac{x}{x-2} = 1$

Enter one side of the equation as Y_1. Enter the other side as Y_2.

There appears to be only one intersection point, at $x = 0$.

$Y_1 = Y_2$ when $x = 0$.



The solution is $x = 0$.

Check $\frac{2}{x+2} + \frac{x}{x-2} = 1$

$\frac{2}{0+2} + \frac{0}{0-2} \stackrel{?}{=} 1$

$1 + 0 = 1$ ✔

Target 3D: Solve systems involving rational equations and identify, if any, extraneous solutions. [A.CED.3, A.REI.11]

You can solve systems with rational equations using some of same methods you used with linear systems.

Follow each direction to solve the system $\begin{cases} y = \dfrac{x}{3x-1} \\ y = \dfrac{1}{x+1} \end{cases}$.

1. Set the expressions for y equal to each other.
2. Solve for x.
3. Check your answer by substituting in the original system.

A) $y = \dfrac{\quad}{3x-1}$ $\qquad$ $y = \dfrac{\quad}{x+1}$

$(x+1)\cancel{(3x-1)}\dfrac{x}{\cancel{3x-1}} = \dfrac{1}{\cancel{x+1}}(3x-1)\cancel{(x+1)}$

$x^2 + x = 3x - 1$

$-3x+1 \quad -3x+1$

$x^2 - 2x + 1 = 0$

$(x-1)(x-1) = 0$

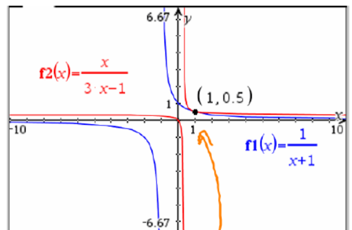
$\boxed{x = 1}$

$y = \dfrac{1}{x+1}$

$y = \dfrac{1}{(1)+1}$

$y = \dfrac{1}{2}$

$\boxed{\left(1, \frac{1}{2}\right)}$



Follow each direction to solve the system $\begin{cases} x - 2 = \dfrac{6}{y} \\ y + 1 = x \end{cases}$.

4. Solve each equation for y.
5. Set the resulting expressions equal to each other.
6. Solve for x.
7. Check your answer by substituting in the original system.

B) $x -$

substitu

$(y+1) - 2 = \dfrac{6}{y}$

$y(y-1) = \dfrac{6}{y}(y)$

$y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3 \quad y = -2$

$y + 1 = x$ $\qquad$ $y + 1 = x$

$(3) + 1 = x$ $\qquad$ $(-2) + 1 = x$

$4 = x$ $\qquad$ $-1 = x$

$\boxed{(4,3)}$ $\qquad$ $\boxed{(-1,-2)}$