PreCalculus KC 10 Review (Parametric & Polar Equations) Name: ____________________________

Calculator

1. An NFL place-kicker attempts a 40 yd field goal with an initial velocity of 70 ft/sec at a 45° angle with the horizontal.

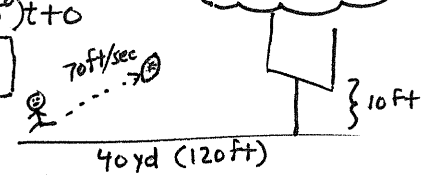
$g = -32 \text{ ft/sec}^2$, $v_0 = 70 \text{ ft/sec}$, $\theta = 45°$, $y_0 = x_0 = 0$

a) Model the motion of the football using parametric equations.

$x(t) = v_0 \cos\theta \, t + x_0$ $\qquad y(t) = \frac{1}{2} g t^2 + v_0 \sin\theta \, t + y_0$

$\boxed{x(t) = (70\cos 45°)t}$ $\qquad y(t) = \frac{1}{2}(-32)t^2 + (70\sin 45°)t + 0$

$\boxed{y(t) = -16t^2 + (70 \cdot \sin 45°)t}$

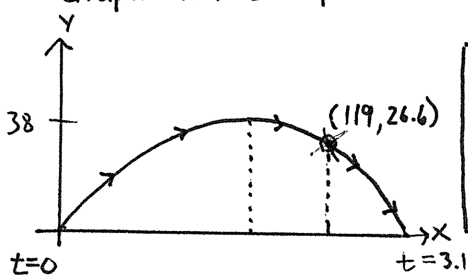


b) To make the field goal, the football will have to clear a 10 ft high crossbar. Will the kicker make the field goal if the kick sails "true"? By how much does the ball clear or fall short (vertically) of the crossbar?

Graph on the Nspire:

$0 \le t \le 5$, tstep = 0.01
Window: [0, 200] by [0, 50]

Graph trace:



Yes, the kicker will make the fieldgoal. After the ball has traveled 119 ft (almost 120ft) its height is at 26.6 ft, clearing the crossbar by 26.6 − 10 = 16.6 ft

- To be exact, set $x(t) = 120$.

$120 = 70\cos 45° t$

$\frac{120}{70\cos 45°} = t$ "store" ctrl var $t = 2.424$

$\therefore y(2.424) = -16(2.424)^2 + (70 \cdot \sin 45°)(2.424)$
$= 25.959$

Yes, clears by $25.959 - 10 = 15.959$ ft

c) What is the ball's maximum height above the field?

- Graph trace: The maximum height is $\boxed{38.2 \text{ ft}}$ (Approximation using trace feature)
- To be exact, recall the following equation: $t = \frac{-b}{2a}$ (vertical line of symmetry for parabola)

$y(t) = -16t^2 + (70 \cdot \sin 45°)t$; $a = -16$, $b = 70 \cdot \sin 45°$

$t = \frac{-70 \cdot \sin 45°}{2(-16)} = 1.547$ "store" ctrl var

$y(1.547) = -16(1.547)^2 + 70 \cdot \sin 45°(1.547) = \boxed{38.281 \text{ ft}}$

d) What is the ball's "hang time" (the total time the football is in the air)?

- Graph trace: "hang time" is $\boxed{3.1 \text{ seconds}}$
- To be exact, set the height (vertical) equal to 0.

$0 = -16t^2 + 70 \cdot \sin 45° t$ Factor out t

$0 = t(-16t + 70 \cdot \sin 45°)$

$\cancel{0 = t}$ or $0 = -16t + 70 \cdot \sin 45° \Rightarrow 16t = 70 \cdot \sin 45° \Rightarrow t = \frac{70 \cdot \sin 45°}{16} = \boxed{3.094 \text{ sec}}$

2. Find the rectangular coordinates of the polar coordinate point $\left(\underset{r}{-2}, \underset{\theta}{-\frac{14\pi}{5}}\right)$. Approximate the coordinates to the nearest thousandth (three decimal places).

$(x, y) = (r\cos\theta, r\sin\theta)$
$= \left(-2 \cdot \cos\left(-\frac{14\pi}{5}\right), -2 \cdot \sin\left(-\frac{14\pi}{5}\right)\right)$
$= (1.618, 1.176)$

$\therefore$ Rectangular coordinates: $(1.618, 1.176)$

3. Find the intersection(s) of the following polar equations: $r = 1 + 2\cos\theta$ and $r = 1 + \sqrt{3}$

$1+\sqrt{3} = 1+2\cos\theta$

$\sqrt{3} = 2\cos\theta$

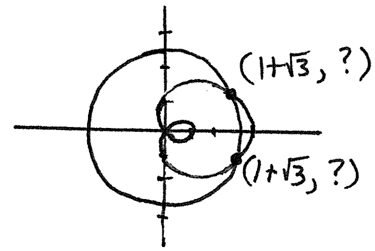
$\frac{\sqrt{3}}{2} = \cos\theta \xRightarrow{\text{inverse}} \theta = \frac{\pi}{6}, \frac{11\pi}{6}$

S	A
T	C

Recall cosine is positive in the 1st and 4th Quadrant

$\boxed{\therefore, \left(1+\sqrt{3}, \frac{\pi}{6}\right) \quad \left(1+\sqrt{3}, \frac{11\pi}{6}\right)}$

Graph polar equations to get visual:



4. Complete the table for the polar equation $r = 1 + 2\sin\theta$ and sketch the graph at the points in the table (only on the interval $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$).

θ	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
r	3	2.414	1	0	−0.732	−1

$r = 1+2\sin\left(\frac{\pi}{2}\right) = 1+2(1) = 3$

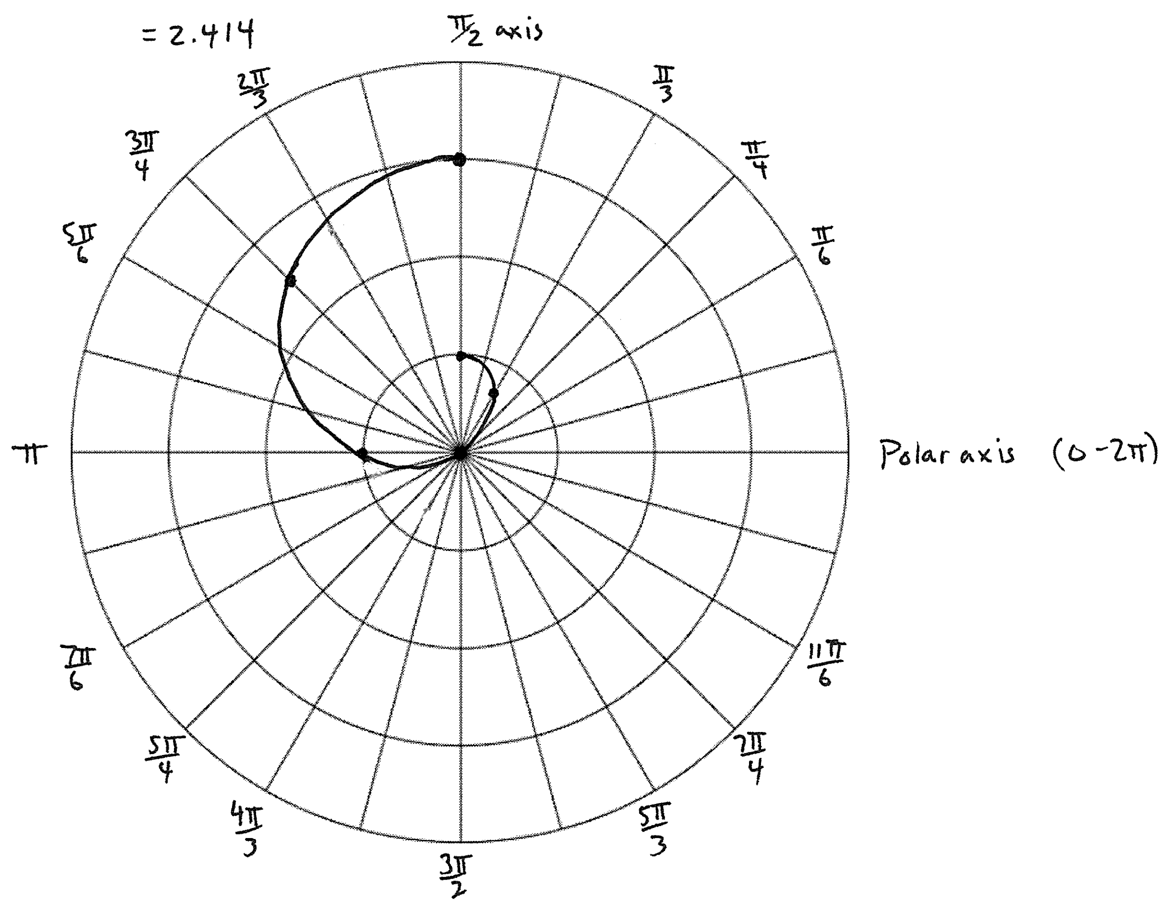
$r = 1+2\sin\left(\frac{3\pi}{4}\right) = 1+\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}} = 1+\sqrt{2} = 2.414$

$r = 1+2\sin(\pi) = 1+2(0) = 1$

$r = 1+2\sin\left(\frac{7\pi}{6}\right) = 1+2\cdot\left(-\frac{1}{2}\right) = 1-1 = 0$

$r = 1+2\sin\left(\frac{4\pi}{3}\right) = 1+2\left(-\frac{\sqrt{3}}{2}\right) = -0.732$

$r = 1+2\sin\left(\frac{3\pi}{2}\right) = 1+2(-1) = 1-2 = -1$



Non-calculator

5. Use an algebraic method to eliminate the parameter t and then identify the graph of the parametric curve.

$$x = 2\sin t \quad y = 2\cos t, \quad \underbrace{\frac{\pi}{2} \le t \le \pi}_{\text{restriction}}$$

* When $t = \frac{\pi}{2}$,
$x = 2\sin(\frac{\pi}{2}) = 2(1) = 2$
$y = 2\cos(\frac{\pi}{2}) = 2(0) = 0$ } $(2,0)$

* When $t = \pi$,
$x = 2\sin(\pi) = 2(0) = 0$
$y = 2\cos(\pi) = 2(-1) = -2$ } $(0,-2)$

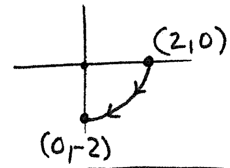
$\frac{x}{2} = \sin t \quad \frac{y}{2} = \cos t$

$(\sin t)^2 + (\cos t)^2 = 1$ Pyth. identity

$(\frac{x}{2})^2 + (\frac{y}{2})^2 = 1$

$\frac{x^2}{4} + \frac{y^2}{4} = 1$ Multiply each term in equation by 4.

$\boxed{x^2 + y^2 = 4}$



$\therefore$, Graph is a Quarter-circle, radius 2, center (0,0), 4th Quadrant

6. The graph of the parametric equations $x = t - 2$ and $y = -3t - 2$ where $3 \le t \le 7$ is a line segment. Find the endpoints of this line segment.

$x = 3 - 2 = 1$
$y = -3(3) - 2 = -9 - 2 = -11$ } $(1,-11)$

$x = 7 - 2 = 5$
$y = -3(7) - 2 = -21 - 2 = -23$ } $(5,-23)$

$\boxed{\therefore, \text{ End pts: } (1,-11) \text{ and } (5,-23)}$

7. Use an algebraic method to eliminate the parameter t and then identify the graph of the parametric curve.

$$x = 2 - 3t \quad y = -2t - 1$$ (No restriction(s) on t)

$x = 2 - 3t$

$x + 3t = 2$

$\frac{3t}{3} = \frac{2}{3} - \frac{x}{3}$

$t = \frac{2}{3} - \frac{1}{3}x$

$y = -2t - 1$

$y = -2(\frac{2}{3} - \frac{1}{3}x) - 1$

$y = -\frac{4}{3} + \frac{2}{3}x - 1$

$y = \frac{2}{3}x - \frac{7}{3}$

$-\frac{4}{3} - 1 = -\frac{4}{3} - \frac{3}{3} = \frac{-4-3}{3} = -\frac{7}{3}$

$\boxed{\therefore, \text{ Line: } y = \frac{2}{3}x - \frac{7}{3}}$

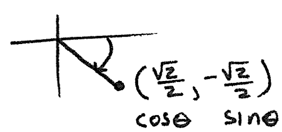
8. Convert $\left(10, -\frac{\pi}{4}\right)$ from polar to rectangular coordinates.

$(x, y) = (r\cos\theta, r\sin\theta)$

$= (10 \cdot \cos(-\frac{\pi}{4}), 10 \cdot \sin(-\frac{\pi}{4}))$

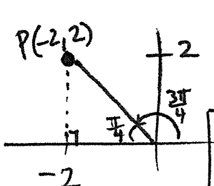
$= (10 \cdot \frac{\sqrt{2}}{2}, 10(-\frac{\sqrt{2}}{2}))$

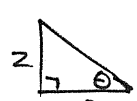
$= (5\sqrt{2}, -5\sqrt{2})$



$\boxed{\therefore, \text{ Rect. coord: } (5\sqrt{2}, -5\sqrt{2})}$

9. Find two polar coordinate pairs for the point with rectangular coordinates $P(\overset{x}{-2}, \overset{y}{2})$.





$\therefore$ Two possible polar coord pairs: $\boxed{(-2\sqrt{2}, -\frac{\pi}{4}) \text{ and } (2\sqrt{2}, \frac{3\pi}{4})}$

① $x^2 + y^2 = r^2$

$(-2)^2 + (2)^2 = r^2$

$4 + 4 = r^2$

$8 = r^2$

$\pm\sqrt{8} = r$

$\pm 2\sqrt{2} = r$

$\sqrt{8}$
$\sqrt{4}\sqrt{2}$
$2\sqrt{2}$

② $\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{2}{-2} = -1$

$\theta = \tan^{-1}(-1) = -\frac{\pi}{4}$

10. Convert each polar equation to rectangular form and identify the graph.

Conversion equations Remember!

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2=r^2$

a) $r = \frac{1}{2}\sec\theta$

$r = \frac{1}{2}\cdot\frac{1}{\cos\theta}$ Multiply by $\cos\theta$ to both sides.

$r\cos\theta = \frac{1}{2}\cdot\frac{1}{\cos\theta}\cdot\cancel{\cos\theta}$

$\underbrace{r\cos\theta} = \frac{1}{2}$

$x = \frac{1}{2}$

$\boxed{\therefore \text{ Vertical line: } x = \frac{1}{2}}$

b) $r = 3\sin\theta - 4\cos\theta$

$r\cdot r = 3r\sin\theta - 4r\cos\theta$ multiply by r to every term in equation.

$r^2 = 3r\sin\theta - 4r\cos\theta$

$x^2+y^2 = 3y - 4x$

$x^2+4x+y^2-3y=0$

Complete square
$\left(\frac{4}{2}\right)^2 = (2)^2 = 4$
$\left(\frac{-3}{2}\right)^2 = \frac{(-3)^2}{2^2} = \frac{9}{4}$

$\underbrace{x^2+4x+4}_{\text{Factor}} + \underbrace{y^2-3y+\frac{9}{4}}_{\text{Factor}} = 0+4+\frac{9}{4}$

$\frac{4}{1}\cdot\frac{4}{4}+\frac{9}{4} = \frac{16}{4}+\frac{9}{4} = \frac{25}{4}$

$(x+2)(x+2) + \left(y-\frac{3}{2}\right)\left(y-\frac{3}{2}\right) = \frac{25}{4}$

$(x+2)^2 + \left(y-\frac{3}{2}\right)^2 = \frac{25}{4}$

Circle
Center: $\left(-2, \frac{3}{2}\right)$
radius: $\sqrt{\frac{25}{4}} = \frac{5}{2}$

11. Transform each rectangular equation to a polar equation and identify the graph.

a) $x^2+y^2+6y=0$

$r^2+6r\sin\theta=0$ Factor out r.

$r(r+6\sin\theta)=0$

$\cancel{r=0}$ or $r+6\sin\theta=0$
point

$\boxed{r=-6\sin\theta}$

Circle
Center: $(0,-3)$
$r=3$

Check center: ☺
$x^2+\underline{y^2+6y+9} = 0+9$
$x^2+(y+3)(y+3)=9$
$x^2+(y+3)^2=9$
Center: $(0,-3)$ ✓

b) $(x-4)^2+y^2=16$

$(x-4)(x-4)+y^2=16$

$x^2-4x-4x+16+y^2=16$

$x^2-8x+\cancel{16}+y^2=\cancel{16}$
$\quad -16 \qquad\qquad -16$

$x^2+y^2-8x=0$

$r^2-8r\cos\theta=0$ Factor out r.

$r(r-8\cos\theta)=0$

$\cancel{r=0}$ or $r-8\cos\theta=0$
point

$\boxed{r=8\cos\theta}$

Circle
Center: $(4,0)$
$r=4$