Name ______________________________ Class ____________ Date ____________

Chapter 8 Performance Tasks

Task 1

a. Suppose you are at the center of a circle with a radius of 1 mi. How far left or right and how far up or down must you move in order to end up on the boundary of the circle, with a final angle of 22° in standard position with respect to the horizontal axis? **≈ 0.93 mi to the right, and ≈ 0.37 mi up**

b. Convert 22° to radians and redo part (a). How do your answers change? **≈ 0.38 radians; The answers stay the same.**

c. If you started 1 mi to the right of your original position, how far would you have to travel along the boundary of the circle to reach the same final point as in part (a)? Did you choose to work in radians or in degrees? Why? Explain your answer. **≈ 0.38 mi; in radians, the arc length of a circle with a radius of 1 is given by the numerical value of the angle that created the arc.**

d. Give an example of a real-world situation that could be modeled by part (a) or part (c). **Check students' work.**

[4] Student correctly finds the directions to move in part (a), correctly converts degrees to radians and finds the results are the same as in part (a), correctly determines arc length, and provides a reasonable example that could be modeled by this situation. The only errors are computational.

[3] Student's solution strategies are appropriate, but work contains errors.

[2] Some of student's solution strategies are appropriate, but work contains errors and some responses are incorrect or incomplete OR responses are correct, but without work shown.

[1] Much of response is incorrect or incomplete.

[0] no response OR incorrect response without work shown

Task 2

a. In general, does Earth have a constant period in its orbit around the sun? If so, what is the period? If not, explain why not. **Yes; in general, one 365-day year**

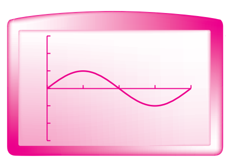
b. Explain whether Earth's orbit around the sun is best modeled by a quadratic, a periodic, or a rational function. **Periodic, since every 365 days Earth is at the same basic position with respect to the sun.**

c. In general terms, what is the amplitude of Earth's orbit around the sun? **The amplitude is one-half the greatest distance across Earth's orbit.**

d. Let $f(t)$ be a sine or cosine function representing the distance of Earth from the sun in its orbit around the sun, where t represents time in 365-day years. How does the amplitude, phase shift, and vertical shift of $f(t)$ affect the amount of time that elapses between the maximum and minimum distances the Earth is from the sun? **The amplitude, phase shift, and vertical shift of $f(t)$ does not affect the amount of time that elapses between the maximum and minimum distances the Earth is from the sun.**

e. Use the information and result from part (d) to write a sine function $f(t)$ that can be used to determine the amount of time that elapses between the maximum and minimum distances the Earth is from the sun. **Answers may vary. Sample: $f(t) = \sin 2\pi t$.**

f. Use your graphing calculator to graph the sine function in part (e). Use the graph to determine the amount of time that elapses between the maximum and minimum distances the Earth is from the sun.



; 0.5 yr

Xmin=0 Ymin=–3
Xmax=1 Ymax=3
Xscl=.25 Yscl=1

[4] Student uses appropriate methods. The only errors are computational.

[3] Student's solution strategies are appropriate, but work contains minor errors.

[2] Some solution strategies are appropriate, but work contains major errors and some responses are incorrect or incomplete OR responses are correct, but without work shown

[1] Much of response is incorrect or incomplete.

[0] Entire response is missing or inappropriate.

Chapter 8 Performance Tasks (continued)

Task 3

a. Write a sine function with amplitude 2 and period 4.

b. Write a cosine function with amplitude 3 and period $\frac{\pi}{2}$, which has been shifted up 1 unit.

c. Write a tangent function with period 3π.

d. Can the graph of a tangent function be shifted 3 units to the left? Explain your answer.

e. Can the graph of a secant function be shifted 2 units right and 5 units up? Explain your answer.

a. Answers may vary. Sample: $y = 2 \sin \frac{\pi}{2}x$
b. Answers may vary. Sample: $y = 3 \cos 4x + 1$
c. Answers may vary. Sample: $y = \tan \frac{1}{3}x$
d. yes; Check students' work.
e. yes; Check students' work.

[4] Student correctly writes three functions, and gives an accurate explanation of shifting tangent and secant functions. The only errors are computational.

[3] Student correctly writes three functions, and gives a reasonable explanation of shifting tangent and secant functions, with minor errors.

[2] Student writes two of three functions correctly, and gives a reasonable explanation of shifting tangent and secant functions, with minor errors.

[1] Student writes one of three functions correctly, or has minor errors in all three functions. Student gives an unclear or inaccurate explanation of shifting tangent and secant functions.

[0] Entire response is missing or inappropriate.

Task 4

a. Explain how you can determine the locations of the asymptotes for the graph of a tangent function. Give an example.

b. Explain how the graphs of $y = \sin 2\theta$ and $y = \csc 2\theta$ are related.

c. Explain how the graphs of $y = -3 \cos \theta$ and $y = -3 \sec \theta$ are related.

Check students' work.

[4] Student describes a correct process for locating asymptotes of a tangent function, gives an appropriate example, and gives accurate descriptions of the relationships between the reciprocal functions.

[3] Student describes a correct process for locating asymptotes of a tangent function, gives an inappropriate example, and gives accurate descriptions of the relationships between the reciprocal functions.

[2] Student has errors in the description of locating the asymptotes of the tangent function, gives an appropriate example, and has errors in the descriptions of the relationships between the reciprocal functions.

[1] Student has errors in the description of the locating the asymptotes of the tangent function, has an incorrect example or no example, and has errors in the descriptions of the relationships between the reciprocal functions.

[0] Entire response is missing or inappropriate.

Name ______________________ Class __________ Date __________

Chapter 8 Project: The Wave of the Future

Beginning the Chapter Project

The ebb and flow of the ocean tides contain tremendous amounts of energy. For centuries this energy has been used to run tidal mills. In the last few decades, utility companies have explored ways to use this energy to generate electricity.

The tides vary greatly, but in predictable, repetitive ways that can be utilized. Tides are periodic, as are the functions in Chapter 13. To determine where a tidal power plant might be feasible, accurate predictions of the tides are essential.

In this project, you will consider how tides are modeled and how the periodic nature of tides creates special problems in the design of tidal power plants. Then you will summarize what you have learned and discuss whether you think tidal power plants will be a practical source of electricity in the future.

List of Materials

- Calculator
- Graph paper

Activities

Activity 1: Estimating

You can use a periodic function to approximate the cycle of tides. Every day at many locations around the world, people record the height of the tide above a level called *mean low water*. The following table shows possible data at two such locations. Estimate the period and amplitude of the function that models the tidal cycle at each location.

Location 1		Location 2	
Time	Tide Height (ft)	Time	Tide Height (ft)
11:30 A.M.	0.6	4:46 P.M.	−2.4
5:42 P.M.	4.8	10:59 P.M.	3.3
11:55 P.M.	0.6	5:11 A.M.	−2.4
6:07 A.M.	4.8	11:24 A.M.	3.3

Location 1: 12 h 25 min, 2.1 ft; Location 2: 12 h 25 min, 2.85 ft

Chapter 8 Project: The Wave of the Future (continued)

Activity 2: Modeling

The range of the tides is affected by the relative positions of the sun and the moon. During the new moon and full moon, the highest high tides and the lowest low tides occur. During the first and third quarters of the lunar month, the lowest high tides and highest low tides occur. Throughout the month, the tidal range gradually increases and decreases between the minimum and maximum values of the range. **Check students' work.**

- Sketch a graph of a tidal range as a function of time, showing what you think the shape of a tidal cycle might look like for one month. On your graph, indicate where you think each phase of the moon occurs.
- Research how the phases of the moon and the forces that control the tides are related. Include an illustration with your explanation.

Activity 3: Researching

To harness tidal power, a dam is built across the narrow neck of a bay where there is a large difference between high and low tide. Power is generated by both the incoming and outgoing tides as water flows through the dam. However, near the times of high and low tide, other sources of energy must be used to supplement tidal power. **Check students' work.**

- How does the periodic nature of tides explain why tidal power is not a steady source of energy?
- Determine in what parts of the world it is practical to harness tidal power.
- How do utility companies that use tidal power provide energy when their customers need it?

Finishing the Project

The answers to the activities should help you complete your project. Write a brief essay describing how the tides can be harnessed to create electrical power. Discuss the advantages and disadvantages of tidal power. Based on your research and analysis, do you think tidal power plants are practical sources of electricity for the future? Support your conclusions.

Reflect and Revise

Find someone in your class who reached a different conclusion about tidal power than you did. Read each other's essays. Discuss your differences of opinion. Have you changed your mind about your conclusions? If necessary, do further research and revise your paper.

Extending the Project

Research other factors that affect the cycle of tides such as the changing distances of the moon and sun from Earth. Describe these factors in terms of periodic functions, if possible.

Name ______________________ Class ____________ Date ____________

Additional Vocabulary Support

Exploring Periodic Data

Choose the word from the list that best completes each sentence.

amplitude	**cycle**	**period**	**periodic function**

1. A ____cycle____ is one complete pattern in a periodic function.

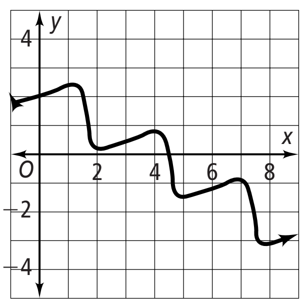
2. A ____periodic function____ repeats a pattern of y-values at regular intervals.

3. The ____amplitude____ of a periodic function is found by calculating half the difference between the maximum and minimum values of the function.

4. The horizontal length of one cycle is called the ____period____ of a function.

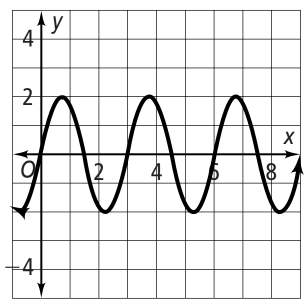
Determine whether the following functions are periodic. If the function is periodic, name the period.

5.



____not periodic____

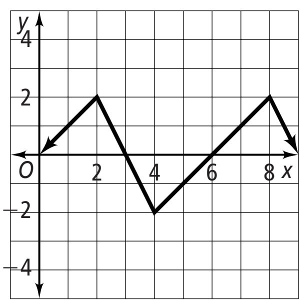
6.



____periodic; 3____

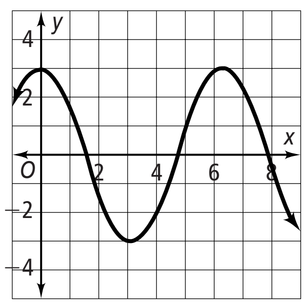
Find the amplitude of the following periodic functions.

7.



2

8.



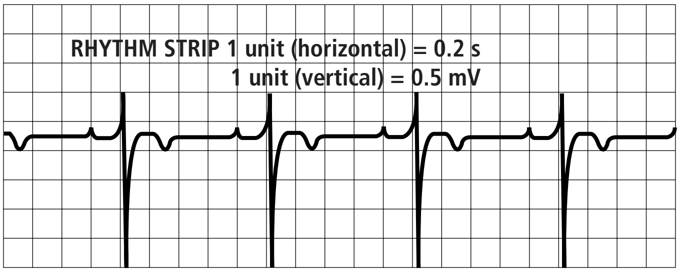
3

Name ______________________ Class ____________ Date ____________

Think About a Plan

Exploring Periodic Data

Health An electrocardiogram (EKG or ECG) measures the electrical activity of a person's heart in millivolts over time.



a. What is the period of the EKG shown above?
b. What is the amplitude of the EKG?

Know

1. One horizontal unit on the graph represents $\boxed{\text{0.2 s}}$.

2. One vertical unit on the graph represents $\boxed{\text{0.5 mV}}$.

3. The EKG represents a periodic function.

Need

4. To solve the problem, I need to find the period and amplitude of the periodic function represented by the graph.

Plan

5. One cycle of the function has a length of $\boxed{\text{5 units}}$.

6. What is the period of the function? 1 s

7. What is the definition of amplitude? amplitude $= \frac{1}{2}$(maximum value − minimum value)

8. The amplitude of the function is $\boxed{\text{1.5 mV}}$.

Name ______________________________ Class ____________ Date ____________

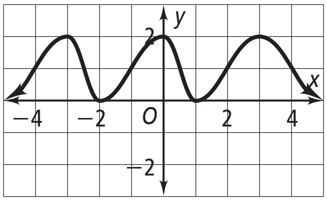
Practice

Form G

Exploring Periodic Data

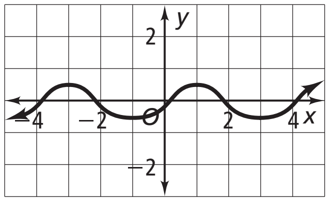
Name one cycle in two different ways. Then determine the period of the function.

1.



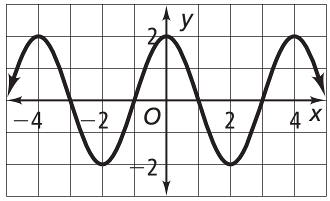
Answers may vary.
Sample: from 0 to 3; 3

2.



Answers may vary.
Sample: from 0 to 4; 4

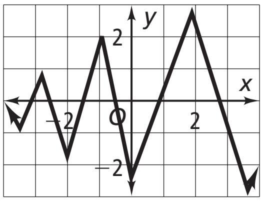
3.



Answers may vary.
Sample: from 0 to 4; 4

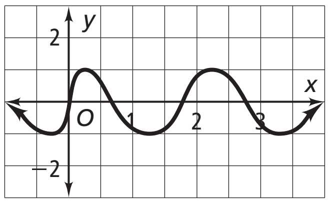
Determine whether each function *is* or *is not* periodic. If it is, find the period.

4.



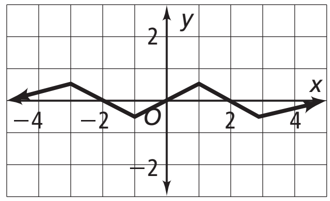
not periodic

5.



periodic; 2

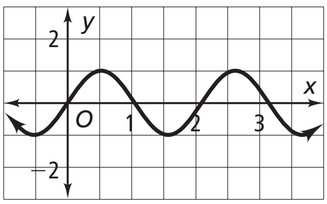
6.



periodic; 4

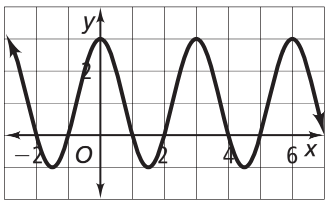
Find the amplitude of each periodic function.

7.



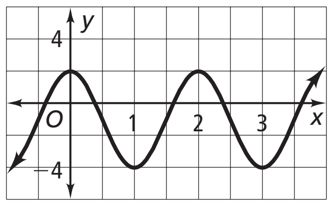
1

8.



2

9.

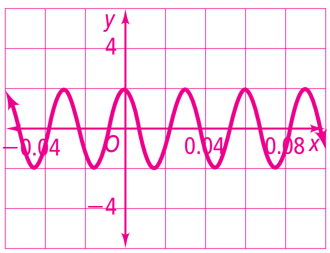


3

Sketch the graph of a sound wave with the given period and amplitude.

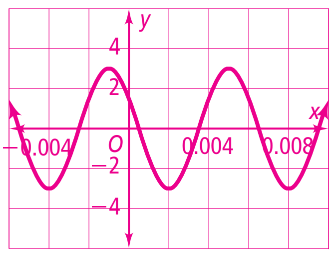
10. period 0.03, amplitude 2

Answers may vary. Sample:



11. period 0.006, amplitude 3

Answers may vary. Sample:



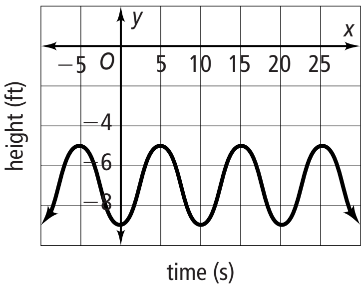
Name ______________________ Class ____________ Date ____________

Practice (continued) *Form G*

Exploring Periodic Data

12. Open-Ended Describe a situation that you could represent with a periodic function. **Answers may vary. Sample: the height above the ground of the last car of an amusement park roller coaster, recorded during each ride for one day**

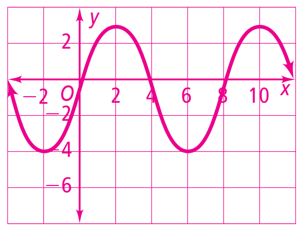
13. The graph below shows the height of ocean waves below the deck of a platform.



a. What is the period of the graph? **10 s**

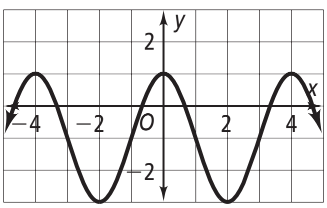
b. What is the amplitude of the graph? **2 ft**

14. Open-Ended Sketch a graph of a periodic function that has a period of 8 and an amplitude of $3\frac{1}{2}$. **Answers may vary. Sample:**

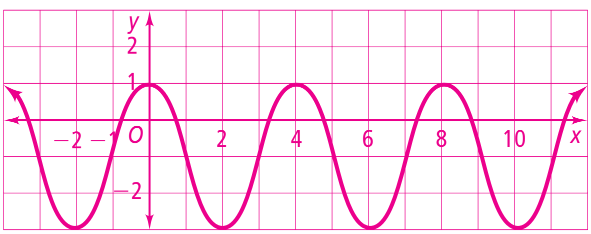


Find the maximum, minimum, and period of each periodic function. Then copy the graph and sketch two more cycles.

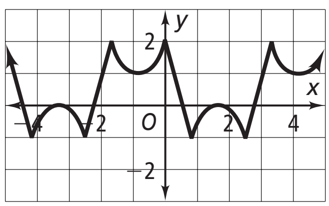
15.



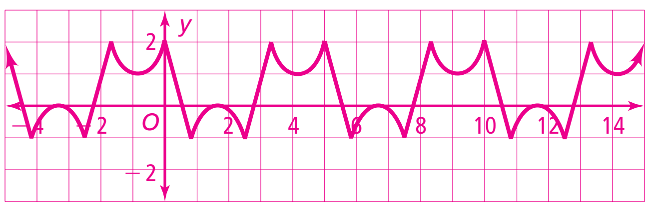
1; −3; 4



16.



2; −1; 5



Name ______________________________ Class ____________ Date ____________

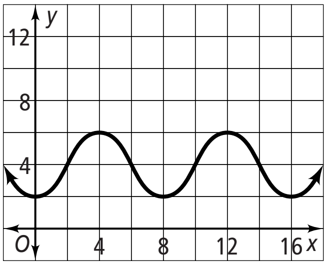
Practice

Form K

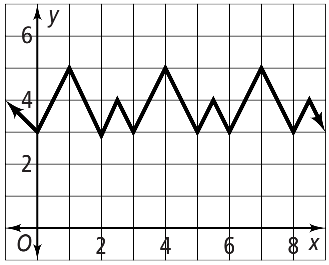
Exploring Periodic Data

Trace two complete cycles of the functions below. Answers may vary. Check that students have traced two cycles.

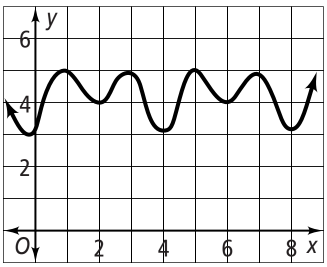
1.



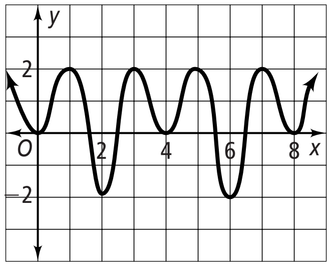
2.



3.

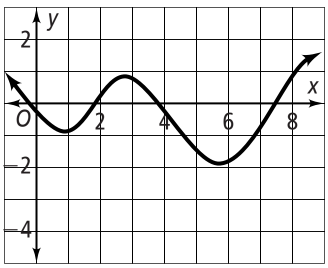


4.



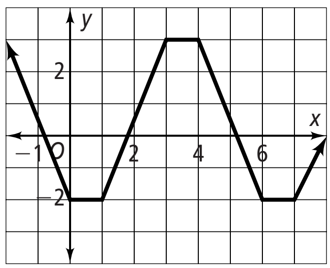
Tell whether the function is periodic. Find the period if the function is periodic.

5.



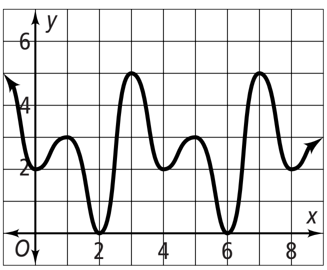
not periodic

6.



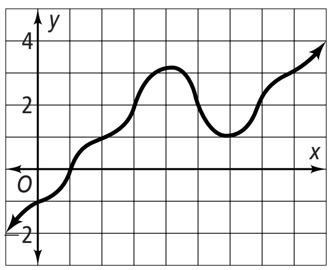
periodic; period: 6

7.



periodic; period: 4

8.



not periodic

Name ______________________ Class ____________ Date ____________

Practice (continued) *Form K*

Exploring Periodic Data

9. The amplitude of a function is $\frac{1}{2}$ (maximum value – minimum value).

The maximum value of a periodic function is 12.

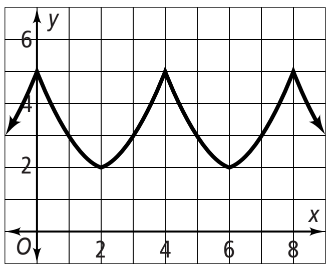
The minimum value of a periodic function is 4.

What is the amplitude? **4**

10. Reasoning Do you use x-values or y-values to calculate the amplitude of a periodic function? Explain.
***y*-values; the amplitude is defined by the values or outputs of the function, not the inputs.**

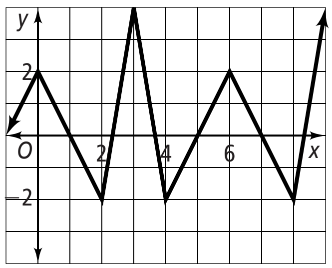
Find the maximum, minimum, and amplitude of each periodic function.

11.



max: 5; min: 2; amp: 1.5

12.



max: 4; min: –2; amp: 3

13. Suppose f is a function with a period of 7. $f(1)$ is 10. What is $f(8)$? How do you know?
10; *f* is a periodic function, so each corresponding point in any given period has a matching output value.

14. Open-Ended Draw a periodic function on the coordinate axes with an amplitude of 3. Label your graph. What are the period, maximum, and minimum values of the function? **Check students' work.**

Name ______________________________ Class ____________ Date ____________

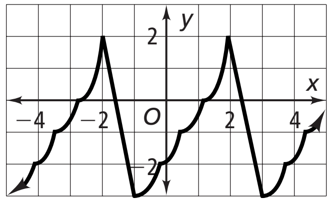
Standardized Test Prep

Exploring Periodic Data

Multiple Choice

For Exercises 1–3, choose the correct letter.

1. Which pair of coordinates names one complete cycle of the periodic function? D



Ⓐ $(-5, -3)$ to $(-2, 2)$

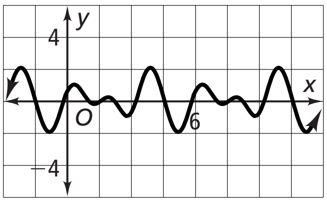
Ⓑ $(-5, -3)$ to $(5, 0)$

Ⓒ $(-2, 2)$ to $(-1, -3)$

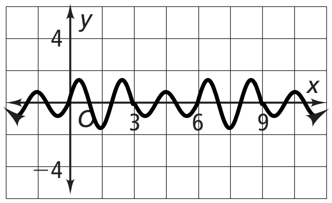
Ⓓ $(-1, -3)$ to $(3, -3)$

2. Which graph is NOT the graph of a periodic function? G

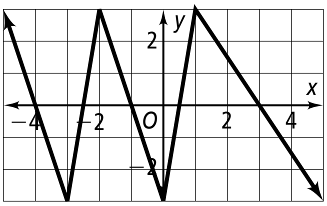
Ⓕ



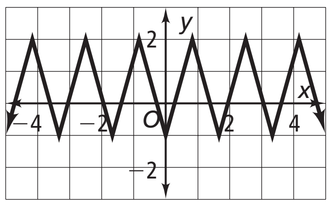
Ⓗ



Ⓖ



Ⓘ

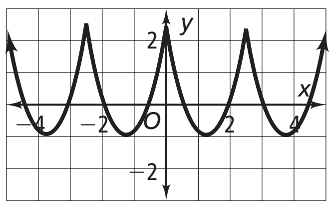


3. A periodic function has a period of 12 s. How many cycles does it go through in 40 s? A

Ⓐ $3\frac{1}{3}$ cycles   Ⓑ $\frac{3}{10}$ cycle   Ⓒ 28 cycles   Ⓓ 480 cycles

Short Response

4. The graph at the right represents a periodic function.



a. What is the period of the function?

b. What is the amplitude of the function?

[2] a. 2.5 b. 1.75

[1] incorrect period OR incorrect amplitude

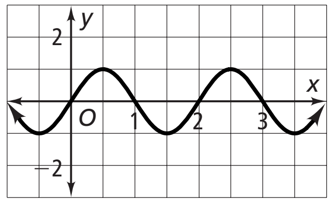
[0] no answers given

Name ______________________________ Class ____________ Date ____________

Enrichment

Exploring Periodic Data

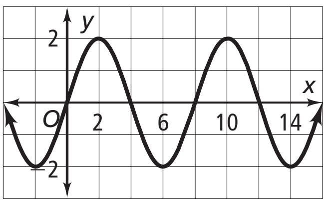
A periodic function repeats a pattern of outputs at regular intervals. There are several ways to describe this pattern. You have already learned that a cycle is one complete pattern. The period is the horizontal length of one cycle and the amplitude measures the amount of variation in the function values. One additional way to describe a periodic function is by its *frequency*.



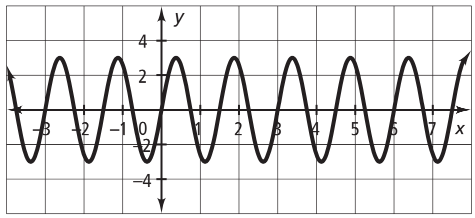
1. Determine the period and the amplitude for the graph at the right. **2; 1**

2. The frequency is defined as the number of cycles completed in one unit. For this graph, what part of a cycle is completed in 1 unit? $\frac{1}{2}$

Determine the period and the frequency of each graph.

3. 

8; $\frac{1}{8}$

4. 

$\frac{3}{2}$, $\frac{2}{3}$

5. Describe the relationship between the period and the frequency.
Answers may vary. Sample: They are reciprocals of each other.

6. If a graph has a period of 12 units, what is its frequency? $\frac{1}{12}$

7. If a graph has a frequency of $\frac{3}{4}$, what is the period? $\frac{4}{3}$

8. The period of visible light ranges from 1.27×10^{-15} s to 2.54×10^{-15} s.

a. What is the range of frequencies for visible light? **approximately 3.94×10^{14} to 7.87×10^{14} cycles per s**

b. What is the physical meaning of this range?
Answers may vary. Sample: Visible light is periodic and oscillates approximately 3.94×10^{14} to 7.87×10^{14} times while traveling 1 s.

Name ____________________ Class __________ Date __________

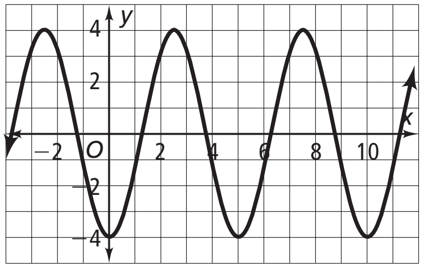
Reteaching

Exploring Periodic Data

The graph of a periodic function shows a repeating pattern of y-values. One complete pattern is a *cycle*. The horizontal distance from one point on the graph to the point where the pattern begins repeating is called the *period* of the function.

Problem

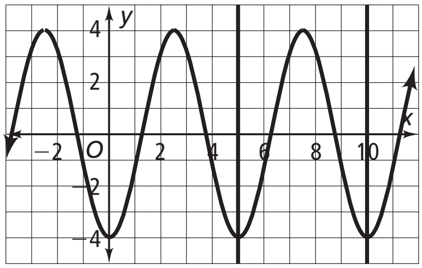
Is the function periodic? If it is, what is the period?



The repeating pattern of y-values shows that this function is periodic.

Name one cycle:

- Draw a vertical line through a point where the graph reaches its minimum y-value.
- Trace the graph with your finger until you feel the pattern repeat.
- Draw a second vertical line through the point where the pattern starts to repeat. The vertical lines mark the beginning and end of one cycle.



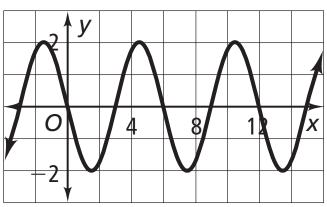
Find the period of the function:

- Find the points where the vertical lines intersect the graph: $(5, -4)$ and $(10, -4)$.
- Subtract the x-values to find the horizontal length of one cycle: $10 - 5 = 5$.

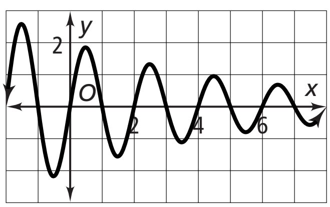
The period of the function is 5.

Exercises

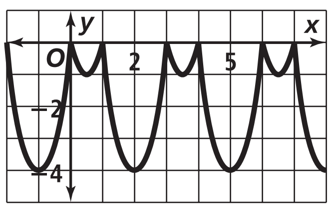
Determine whether each function *is* or *is not* periodic. If it is, find the period.

1. 

periodic; 6

2. 

not periodic

3. 

periodic; 3

Name ______________________ Class ____________ Date ____________

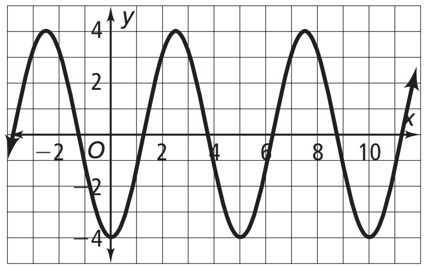
Reteaching (continued)

Exploring Periodic Data

You can measure the amount of variation in the y-values of a periodic function. The *amplitude* of a periodic function is the difference between the maximum and minimum values, divided by 2: $A = \frac{1}{2}$(maximum value − minimum value).

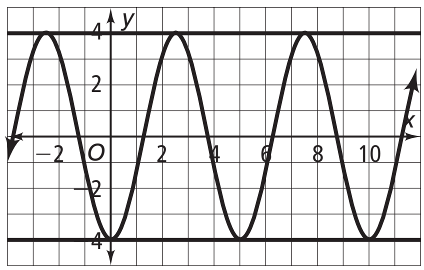
Problem

What is the amplitude of the periodic function?



Name the maximum and minimum y-values:

- Draw one horizontal line across the highest points on the graph.
- Draw a second horizontal line across the lowest points on the graph.



Find the amplitude of the function:

- Find a point where the first horizontal line intersects the graph: (7.5, 4). The y-value, 4, is the maximum value.
- Find a point where the second horizontal line intersects the graph: (10, −4). The y-value, −4, is the minimum value.
- Use the amplitude formula: $A = \frac{1}{2}$(maximum value − minimum value)

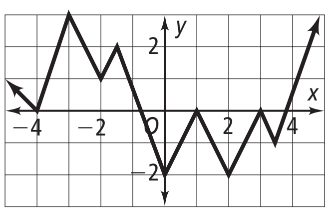
$$A = \frac{1}{2}(4 - (-4)) = \frac{1}{2}(8) = 4$$

The amplitude of the function is 4.

Exercises

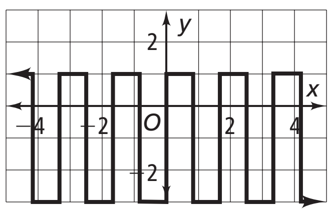
Determine whether each function *is* or *is not* periodic. If it is, find the amplitude.

4.



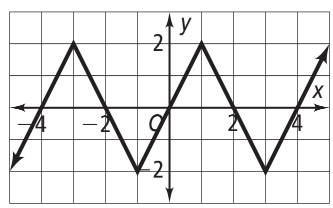
not periodic

5.



periodic; 2

6.



periodic; 2

Name ______________________ Class ____________ Date ____________

Additional Vocabulary Support

Angles and the Unit Circle

For Exercises 1–7, draw a line from each word in Column A to the matching item in Column B.

Column A	Column B
1. standard position	A. two angles in standard position with the same terminal side
2. initial side	B. has a radius of 1 unit and center at the origin
3. terminal side	C. the ray on the x-axis in an angle in standard position
4. coterminal angles	D. describes an angle with a vertex at the origin and one ray on the positive x-axis
5. unit circle	E. for 90°, it is 1
6. cosine of θ	F. the ray not on the x-axis in an angle in standard position
7. sine of θ	G. for 90°, it is 0

Circle the angle measure that is *not* coterminal with the other angles.

8. a. 90° b. −270° c. 270° d. 450°

9. a. 150° b. −210° c. 510° d. 210°

10. a. 25° b. 390° c. −335° d. 385°

Use a unit circle to solve the following problems.

11. sin 270° = $\boxed{-1}$

12. cos 180° = $\boxed{-1}$

Name ______________________ Class __________ Date __________

Think About a Plan

Angles and the Unit Circle

Time The time is 2:46 P.M. What is the measure of the angle that the minute hand swept through since 2:00 P.M.?

Understanding the Problem

1. How many minutes have passed since 2:00 P.M.? **46 min**

2. How many minutes does a full-circle sweep of the minute hand represent? **60 min**

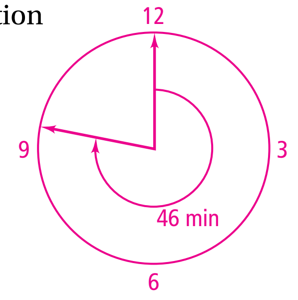
3. How many degrees are in a circle? **360°**

4. What is the problem asking you to determine?
The measure of the angle formed by a minute hand pointing at the 12 and a minute hand pointing at 46 min past the hour.

Planning the Solution

5. How can a drawing help you understand the problem?
Answers may vary. Sample: A drawing can help me understand the angles involved.

6. Make a drawing that represents the clock and the starting and ending position of the minute hand.



7. Write a proportion that you can use to determine the measure of the angle that the minute hand swept through since 2:00 P.M.

$$\boxed{\frac{46}{60} = \frac{x}{360}}$$

8. Is the angle positive or negative? Explain.
Negative; the minute hand moves clockwise, so the angle it sweeps through is negative.

Getting an Answer

9. Solve your proportion to find the measure of the angle. **–276°**

Name ______________________ Class ____________ Date ____________

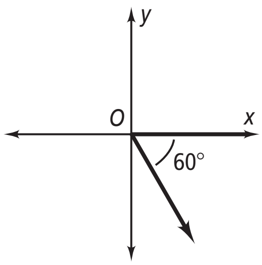
Practice

Form G

Angles and the Unit Circle

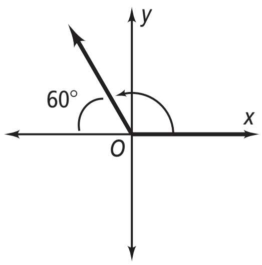
Find the measure of each angle in standard position.

1.



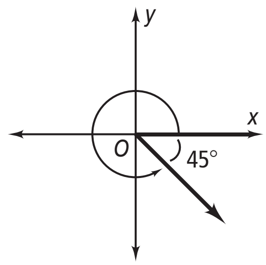
–60°

2.



120°

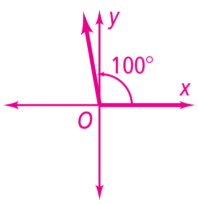
3.



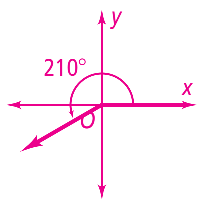
315°

Sketch each angle in standard position.

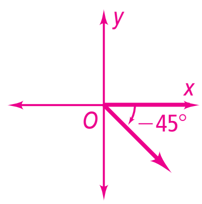
4. 100°



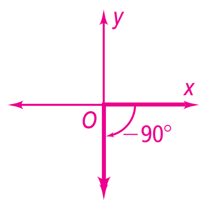
5. 210°



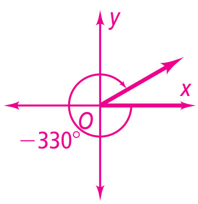
6. -45°



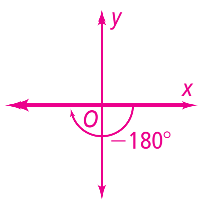
7. -90°



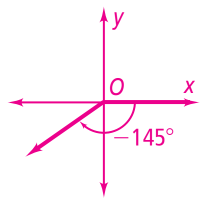
8. -330°



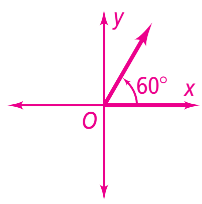
9. -180°



10. -145°



11. 60°



Find the measure of an angle between 0° and 360° coterminal with each given angle.

12. -100° **260°**
13. -60° **300°**
14. -225° **135°**
15. -145° **215°**
16. 372° **12°**

17. -15° **345°**
18. 482° **122°**
19. 484° **124°**
20. -20° **340°**
21. 421° **61°**

22. 409° **49°**
23. -38° **322°**
24. 376° **16°**
25. -210° **150°**
26. 387° **27°**

27. 390° **30°**
28. 660° **300°**
29. 440° **80°**
30. -170° **190°**
31. 370° **10°**

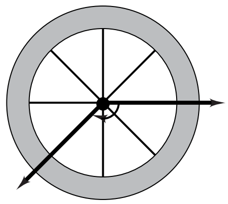
32. -700° **20°**
33. 458° **98°**
34. 480° **120°**
35. 406° **46°**
36. -120° **240°**

37. 460° **100°**
38. -222° **138°**
39. -330° **30°**
40. -127° **233°**
41. 377° **17°**

Practice (continued) *Form G*

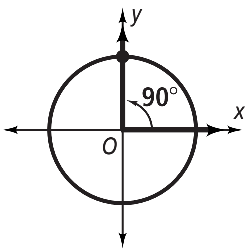
Angles and the Unit Circle

42. The spokes shown on the bicycle wheel at the right form an angle. Estimate the measures of two coterminal angles that coincide with the angle at the right. **Answers may vary. Sample: 225°, −135°**



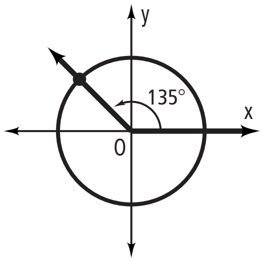
Find the exact values of the cosine and sine of each angle. Then find the decimal values. Round your answers to the nearest hundredth.

43.



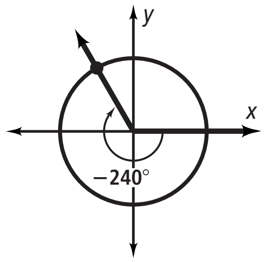
0, 1

44.



$-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$; **−0.71, 0.71**

45.



$-\frac{1}{2}, \frac{\sqrt{3}}{2}$; **−0.5, 0.87**

46. 45° $\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$; **0.71, 0.71**

47. −150° $-\frac{\sqrt{3}}{2}, -\frac{1}{2}$; **−0.87, −0.5**

48. 720° **1, 0**

Graphing Calculator For each angle θ, find the values of cos θ and sin θ. Round your answers to the nearest hundredth.

49. 225° **(−0.71, −0.71)**

50. −225° **(−0.71, 0.71)**

51. −45° **(0.71, −0.71)**

52. 330° **(0.87, −0.5)**

53. −330° **(0.87, 0.5)**

54. 150° **(−0.87, 0.5)**

Open-Ended Find a positive and a negative coterminal angle for the given angle.

55. 50° **410°, −310°**

56. −130° **230°, −490°**

57. −680° **40°, −320°**

58. 395° **35°, −325°**

59. −38° **322°, −398°**

60. −434° **286°, −74°**

61. a. Suppose you know the terminal side of angle θ lies in Quadrant II. What is the sign of cos θ? sin θ? **negative, positive**

b. Writing Describe the reasoning you followed to answer part (a). **Answers may vary. Sample: Cos θ is the *x*-coordinate of the point where the terminal side of θ intersects the unit circle, and sin θ is the *y*-coordinate. In Quadrant II, *x*-values are negative and *y*-values are positive.**

Name ______________________ Class ____________ Date ____________

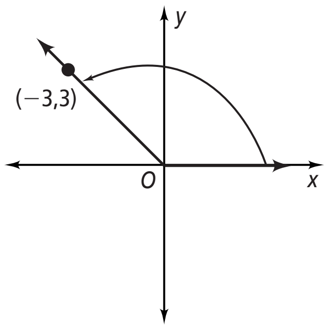
Practice

Form K

Angles and the Unit Circle

Find the measure of each angle in standard position.

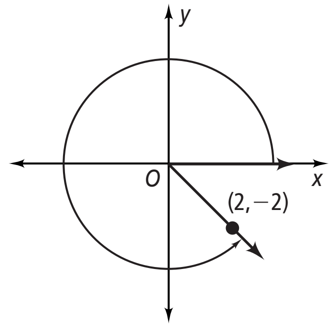
1.



The terminal side of this angle passes through point $(-3, 3)$, so it is 45° beyond a right angle.

$90° + 45° = \boxed{135°}$

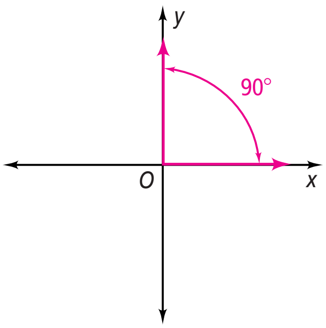
2.



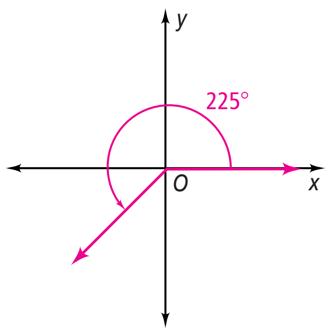
315°

Draw a sketch of each angle in standard position. Remember, the measure of an angle is positive when the angle opens in a counterclockwise direction. The measure is negative when the angle opens in a clockwise direction.

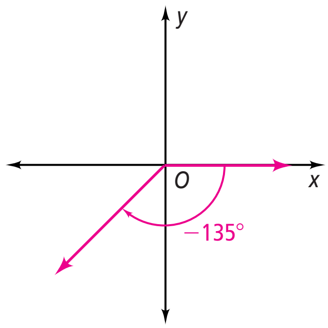
3. 90°



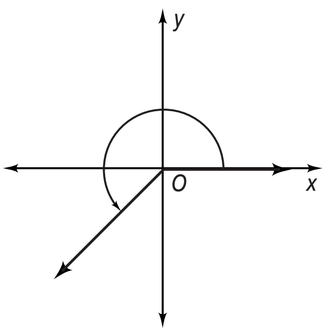
4. 225°



5. −135°



6. Error Analysis Your classmate believes that the angle shown below measures −225°. What error did your classmate make? What is the correct measure of the angle?



Your classmate classified this as a negative angle, when it is actually a positive angle. The measure is 225°.

Name ______________________ Class ____________ Date ____________

Practice (continued) *Form K*

Angles and the Unit Circle

Use your knowledge of coterminal angles to answer the following questions. Remember, coterminal angles share a terminal side.

7. Which of the following angles is not coterminal with the other three angles? **C**

(A) 210° (B) −150° (C) 150° (D) 570°

8. Which of the following angles is not coterminal with the other three angles? **F**

(F) 165° (G) 555° (H) 195° (I) −525°

Use a unit circle to find the sine and the cosine of the following angles.

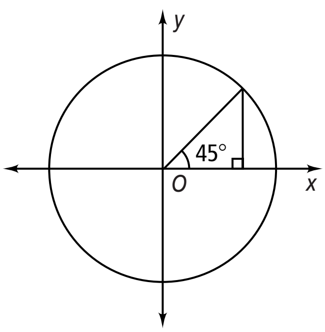
9. −270°
sin (−270°) = 1;
cos (−270°) = 0

10. 180°
sin 180° = 0;
cos 180° = −1

11. 360°
sin 360° = 0;
cos 360° = 1

Find the exact sine and cosine of the following angles.

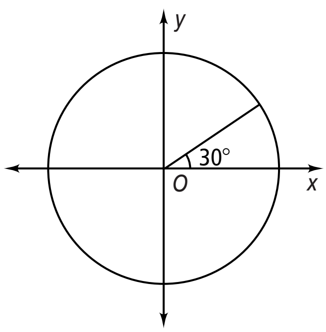
12.



Remember, the lengths of the legs of a 45°-45°-90° triangle are $\frac{\sqrt{2}}{2}$ times the length of the hypotenuse.

$\sin 45° = \frac{\sqrt{2}}{2}$; $\cos 45° = \frac{\sqrt{2}}{2}$

13.



$\cos 30° = \frac{\sqrt{3}}{2}$; $\sin 30° = \frac{1}{2}$

Name ______________________________ Class ____________ Date ____________

Standardized Test Prep

Angles and the Unit Circle

Multiple Choice

For Exercises 1–4, choose the correct letter.

1. Which angle, in standard position, is coterminal with an angle in standard position measuring 152°? **D**

(A) 28° (B) 62° (C) −152° (D) −208°

2. Which could be the measure of an angle θ where $\sin \theta$ is $-\frac{\sqrt{3}}{2}$? **G**

(F) −330° (G) 240° (H) 60° (I) 150°

3. An angle in standard position intersects the unit circle at $(0, -1)$. Which could be the measure of the angle? **C**

(A) 90° (B) −270° (C) −450° (D) 540°

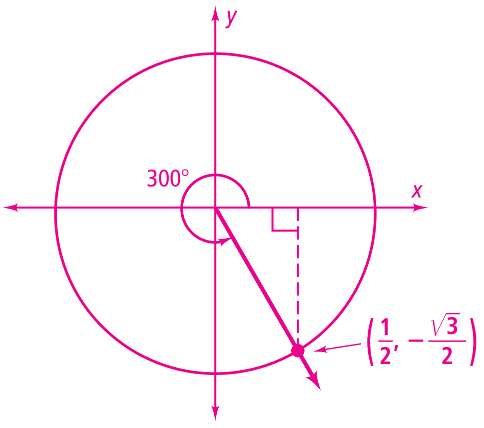
4. What are the coordinates of the point where the terminal side of a 135° angle intersects the unit circle? **F**

(F) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ (G) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ (H) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ (I) $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

Short Response

5. What is the exact value of $\sin(300°)$? Show your work.

[2] $\sin(300°) = -\frac{\sqrt{3}}{2}$



[1] incorrect answer OR incorrect work OR work not shown
[0] incorrect answers and no work shown OR answers not given

Name ______________________________ Class ____________ Date ____________

Enrichment

Angles and the Unit Circle

Nautical Miles

Recall that 1° is $\frac{1}{360}$ of a full 360° rotation. You can break down a degree even further. If you divide 1° into 60 equal parts, each one of the parts is called 1 minute, denoted $1'$. One minute is $\frac{1}{60}$ of a degree; there are 60 minutes in every degree.

If a central angle with its vertex at the center of the earth has a measure of $1'$, then the arc on the surface of the earth that is cut off by this angle has a measure of 1 nautical mile.

For the following problems, assume that the radius of the earth is 4000 miles.

1. Find the number of regular (statute) miles in 1 nautical mile to the nearest hundredth of a mile. **about 1.16 miles**

2. If two ships are 20 nautical miles apart on the ocean, how many statute miles apart are they? Use the result from Exercise 1 in your calculation. **23.2 miles**

3. Two islands are in the ocean. If the central angle with vertex at the center of the earth and rays that pass through these two islands measures $12'$, how many statute miles apart are they? Use the result from Exercise 1 in your calculation. **13.92 miles**

4. Los Angeles and San Francisco are approximately 450 miles apart on the surface of the earth. Find the measure of the central angle with its vertex at the center of the earth, one ray that passes through Los Angeles, and another ray that passes through San Francisco. **about 0.113 radians or about 6.5°**

5. Los Angeles and New York City are approximately 2500 miles apart on the surface of the earth. Find the measure of the central angle with its vertex at the center of the earth, one ray that passes through Los Angeles, and another ray that passes through New York City. **0.625 radian or about 35.8°**

Name ______________________ Class ____________ Date ____________

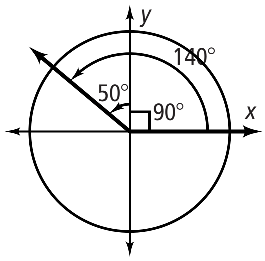
Reteaching

Angles and the Unit Circle

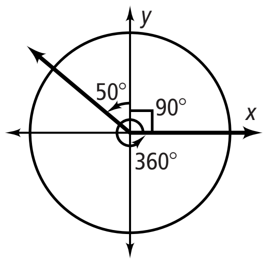
A measurement of an angle in standard position is the measurement of the *rotation* from the initial side of the angle to the terminal side of the angle. Coterminal angles have the same terminal side.

Problem

What are two angles that are coterminal with a 140° angle?

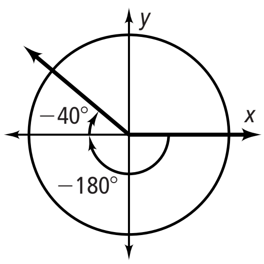


Step 1 Sketch a 140° angle in standard position.
The rotation from the initial side of the angle to the positive y-axis is 90°. So, the rotation from the positive y-axis to the terminal side of the angle is 50°. $(140 - 90 = 50)$



Step 2 Put your finger on the point where the initial side intersects the unit circle. Trace one rotation *counterclockwise* around the circle. Count the degrees of rotation (90°, 180°, 270°, 360°) as you pass each axis. Keep tracing to the positive y-axis again. The degree of rotation is now 450°. $(360 + 90 = 450)$

Continue tracing to the terminal side of the angle. Now the degree of rotation is 500°. $(450 + 50 = 500)$
A 500° angle is coterminal with a 140° angle.



Step 3 Put your finger on the point where the initial side intersects the unit circle. Trace the circle *clockwise*, counting the *negative* degrees of rotation as you pass each axis. $(-90^\circ, -180^\circ)$

Keep tracing until you reach the terminal side of the angle. The rotation from the negative x-axis to the terminal side of the angle is -40°. $(140 - 180 = -40)$
So, the total rotation is -220°. $(-180 + (-40) = -220)$
A -220° angle is coterminal with a 140° angle.

Exercises

Give one positive angle and one negative angle coterminal with the given angle. Answers may vary. Samples are given.

1. 20° **380°, -340°** **2.** 265° **625°, -95°** **3.** 305° **665°, -55°**

Name ______________________ Class ____________ Date ____________

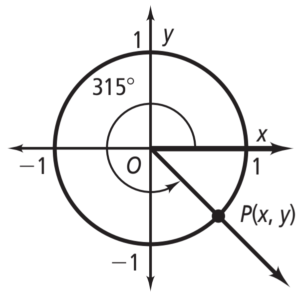
Reteaching (continued)

Angles and the Unit Circle

The coordinates of the point where the terminal side of the angle intersects the unit circle are the cosine and sine of the angle.

Problem

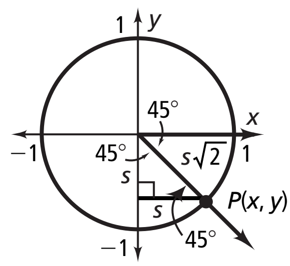
What are the coordinates of the point where the terminal side of a 315° angle intersects the unit circle?



Step 1 Use a compass to draw a unit circle. Use a protractor to sketch the angle. Have the terminal side of the angle intersect the circle.

Step 2 Because the terminal side is in the fourth quadrant, x is positive and y is negative.

Step 3 Use a ruler to draw the horizontal leg of the right triangle. The hypotenuse lies on the terminal side of the angle. The other leg lies on the negative y-axis.



Step 4 Because $360 - 315 = 45$, you can label the acute angles of the triangle as 45°. Use properties of special right triangles. The length of the hypotenuse is $\sqrt{2}$ times the length of a leg. Label each leg s.

Step 5 The unit circle has a radius of 1 unit.

hypotenuse $= 1$

$s\sqrt{2} = 1$ Substitute $s\sqrt{2}$ for the length of the hypotenuse.

$s = \frac{1}{\sqrt{2}}$ Divide both sides by $\sqrt{2}$.

$s = \frac{\sqrt{2}}{2}$ Rationalize the denominator by multiplying the fraction by $\frac{\sqrt{2}}{\sqrt{2}}$.

each leg $= \frac{\sqrt{2}}{2}$

The coordinates of the point of intersection are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Exercises

Find the coordinates of the point where the terminal side of each angle intersects the unit circle. These are the cosine and sine of the angle.

4. $-150°$ $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

5. $30°$ $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

6. $-330°$ $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

7. $-45°$ $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

8. $120°$ $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

9. $225°$ $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

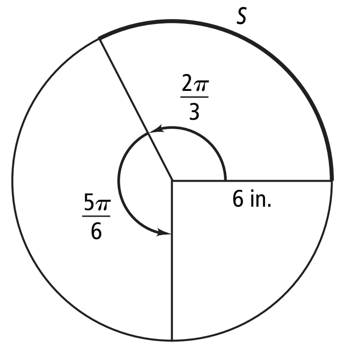
Name ______________________ Class ____________ Date ____________

Additional Vocabulary Support

Radian Measure

Problem

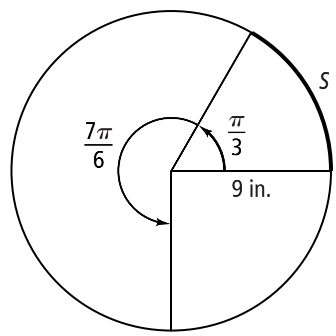
Use the circle at the right. What is length s to the nearest tenth?



	$s = r\theta$	Use the formula for the length of an intercepted arc.
	$s = 6 \cdot \frac{2\pi}{3}$	Substitute 6 for r and $\frac{2\pi}{3}$ for θ.
	$s = \frac{12\pi}{3} = 4\pi$	Simplify.
	$s \approx 12.6$ in.	Use a calculator.
Check	$12.6 = r\frac{2\pi}{3}$	Substitute 12.6 and $\frac{2\pi}{3}$ into the formula.
	$12.6 = 2.1r$	Simplify $\frac{2\pi}{3}$ and round to the nearest tenth.
	$r = \frac{12.6}{2.1} = 6$	Solve for r.

Exercise

Use the circle at the right. What is length s to the nearest tenth?



	$s = r\theta$	**Use the formula for the length of an intercepted arc.**
	$s = 9 \cdot \frac{\pi}{3}$	**Substitute 9 for r and $\frac{\pi}{3}$ for θ.**
	$s = \frac{9\pi}{3} = 3\pi$	**Simplify.**
	$s \approx 9.4$ in.	**Use a calculator.**
Check	$9.4 = r\frac{\pi}{3}$	**Substitute 9.4 and $\frac{\pi}{3}$ into the formula.**
	$9.4 = 1.05r$	**Simplify $\frac{\pi}{3}$ and round to the nearest hundredth.**
	$r = \frac{9.4}{1.05} \approx 9$	**Solve for r.**

Name ______________________ Class ____________ Date ____________

Think About a Plan

Radian Measure

Transportation Suppose the radius of a bicycle wheel is 13 in. (measured to the outside of the tire). Find the number of radians through which a point on the tire turns when the bicycle has moved forward a distance of 12 ft.

Know

1. The radius of the tire is $\boxed{\text{13 in.}}$.

2. The bicycle moves forward a distance of $\boxed{\text{12 ft}}$.

3. The formula for the circumference of a circle is $\boxed{C = 2\pi r}$.

Need

4. To solve the problem I need to find the number of radians a point on the tire turns when the bicycle travels forward 12 ft.

Plan

5. The circumference of the tire is $\boxed{26\pi \text{ in.}}$.

6. The distance the bicycle travels forward is $\boxed{144}$ in.

7. The number of radians a point on the tire turns in one complete rotation is $\boxed{2\pi}$.

8. What proportion can you use to find the radians through which a point on the tire turns when the bicycle has moved forward a distance of 12 ft?

$$\frac{26\pi \text{ in.}}{2\pi \text{ radians}} = \frac{144 \text{ in.}}{x \text{ radians}}$$

9. Solve your proportion to find the radians through which a point on the tire turns when the bicycle has moved forward a distance of 12 ft. about 11 radians

Name ______________________ Class ____________ Date ____________

Practice

Form G

Radian Measure

Write each measure in radians. Express your answer in terms of π and as a decimal rounded to the nearest hundredth.

1. $45°$ $\frac{\pi}{4}$; 0.79

2. $90°$ $\frac{\pi}{2}$; 1.57

3. $30°$ $\frac{\pi}{6}$; 0.52

4. $-150°$ $-\frac{5\pi}{6}$; -2.62

5. $180°$ π; 3.14

6. $-240°$ $-\frac{4\pi}{3}$; -4.19

7. $270°$ $\frac{3\pi}{2}$; 4.71

8. $300°$ $\frac{5\pi}{3}$; 5.24

Write each measure in degrees. Round your answer to the nearest degree, if necessary.

9. $\frac{\pi}{6}$ radians $30°$

10. $-\frac{7\pi}{6}$ radians $-210°$

11. $\frac{7\pi}{4}$ radians $315°$

12. -4 radians $-229°$

13. 1.8 radians $103°$

14. 0.45 radians $26°$

The measure θ of an angle in standard position is given. Find the exact values of $\cos\theta$ and $\sin\theta$ for each angle measure.

15. $\frac{\pi}{6}$ $\frac{\sqrt{3}}{2}, \frac{1}{2}$

16. $\frac{\pi}{3}$ $\frac{1}{2}; \frac{\sqrt{3}}{2}$

17. $-\frac{3\pi}{4}$ $-\frac{\sqrt{2}}{2}; -\frac{\sqrt{2}}{2}$

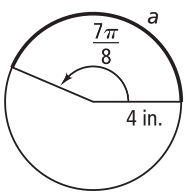
18. $\frac{7\pi}{4}$ $\frac{\sqrt{2}}{2}; -\frac{\sqrt{2}}{2}$

19. $\frac{11\pi}{6}$ $\frac{\sqrt{3}}{2}; -\frac{1}{2}$

20. $-\frac{2\pi}{3}$ $-\frac{1}{2}; -\frac{\sqrt{3}}{2}$

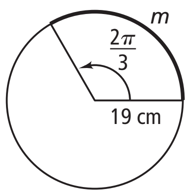
Use each circle to find the length of the indicated arc. Round your answer to the nearest tenth.

21.



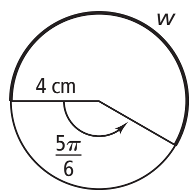
11.0 in.

22.



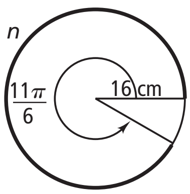
39.8 cm

23.



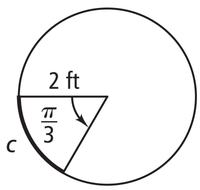
10.5 cm

24.



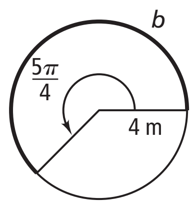
92.2 cm

25.



2.1 ft

26.



15.7 m

Name ______________________ Class __________ Date __________

Practice (continued)

Form G

Radian Measure

27. The minute hand of a clock is 8 in. long.

a. What distance does the tip of the minute hand travel in 10 min? **about 8.38 in.**

b. What distance does the tip of the minute hand travel in 40.5 min? **about 33.93 in.**

c. What distance does the tip of the minute hand travel in 3.25 h? **about 163.36 in.**

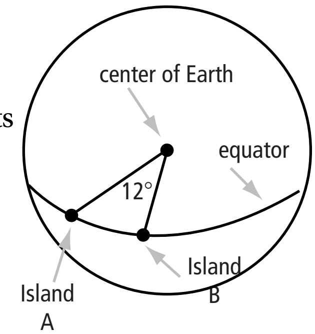
d. Reasoning After approximately how many hours has the tip of the minute hand traveled 100 ft? **about 24 h**

28. A 0.8 m pendulum swings through an angle of 86°. What distance does the tip of the pendulum travel? **about 1.2 m**

29. A scientist studies two islands shown at the right. The distance from the center of the Earth to the equator is about 3960 mi.

a. What is the measure in radians of the central angle that intercepts the arc along the equator between the islands? **$\frac{\pi}{15}$ radians**

b. About how far apart are the two islands? **about 829.38 mi**



Determine the quadrant or axis where the terminal side of each angle lies.

30. $\frac{\pi}{5}$ **I**

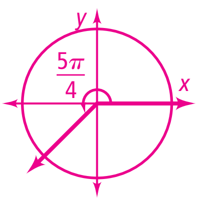
31. $-\frac{5\pi}{2}$ **negative *y*-axis**

32. $\frac{5\pi}{3}$ **IV**

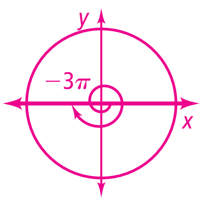
33. $\frac{8\pi}{7}$ **III**

Draw an angle in standard position with each given measure. Then find the values of the cosine and sine of the angle to the nearest hundredth.

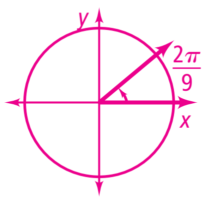
34. $\frac{5\pi}{4}$ **−0.71; −0.71**



35. -3π **−1; 0**



36. $\frac{2\pi}{9}$ **0.77; 0.64**



37. Error Analysis A student wanted to convert 75° to radians. His calculation is shown at the right. What error did he make? What is the correct conversion?

$$\frac{(75 \times 180)}{\pi} \approx 4297.18 \text{ radians}$$

He inverted the conversion factor; $\frac{5\pi}{12}$ radians

Name ______________________ Class ____________ Date ____________

Practice

Form K

Radian Measure

Find the measure of each angle in radians.

1. 30°

$30^\circ \cdot \frac{\pi \text{ radians}}{180^\circ}$

$\frac{\pi}{6}$ radians

2. 140°

$\frac{7\pi}{9}$ radians

3. 300°

$\frac{5\pi}{3}$ radians

4. 15°

$\frac{\pi}{12}$ radians

5. 60°

$\frac{\pi}{3}$ radians

6. 260°

$\frac{13\pi}{9}$ radians

Find the measure of each angle in degrees.

7. 2π radians

$2\pi \cdot \frac{180^\circ}{\pi \text{ radians}}$

360°

8. $\frac{2\pi}{3}$ radians

120°

9. $\frac{\pi}{4}$ radians

45°

Find the exact values of sin θ and cos θ for the following angles.

10. π

sine: 0, cosine: –1

11. $\frac{3\pi}{4}$

sine: $\frac{\sqrt{2}}{2}$, cosine: $-\frac{\sqrt{2}}{2}$

12. $\frac{4\pi}{3}$

sine: $-\frac{\sqrt{3}}{2}$, cosine: $-\frac{1}{2}$

13. $\frac{\pi}{3}$

sine: $\frac{\sqrt{3}}{2}$, cosine: $\frac{1}{2}$

14. $\frac{11\pi}{6}$

sine: $-\frac{1}{2}$, cosine: $\frac{\sqrt{3}}{2}$

15. $\frac{7\pi}{6}$

sine: $-\frac{1}{2}$, cosine: $-\frac{\sqrt{3}}{2}$

Practice (continued)

Form K

Radian Measure

16. Reasoning Why are radian angle measures sometimes more useful than degree measures? **Answers may vary. Sample: Radian measures allow you to find the length of an arc of a circle using a simple equation.**

Find the length of an arc of a circle, given the radius and angle measure.

17. radius $= 4$

$\theta = \frac{\pi}{2}$

$s = r\theta$

$s = 4\left(\frac{\pi}{2}\right)$

$s = 2\pi$

$s \approx \boxed{6.28}$

18. radius $= 7$

$\theta = \frac{3\pi}{20}$

$s \approx 3.3$

19. A large pizza with diameter of 18 in. is cut into 8 equal slices. How long is the crust of one slice of pizza? **about 7.1 in.**

20. Writing When does it make sense to keep your answers in terms of π? When do you need to simplify?
Answers may vary. Sample: Leaving answers in terms of π is more exact than estimating for π, so it is better to leave your answers in terms of π unless you are asked for a specific measurement that must be expressed in decimal form.

Name ______________________________ Class ____________ Date ____________

Standardized Test Prep

Radian Measure

Multiple Choice

For Exercises 1–4, choose the correct letter.

1. Which angle measure is equivalent to $\frac{4\pi}{3}$ radians? **D**

(A) 60° (B) 120° (C) 135° (D) 240°

2. If $\sin\theta = \frac{\sqrt{3}}{2}$, which could be the value of θ? **F**

(F) $\frac{2\pi}{3}$ radians (G) $\frac{3\pi}{4}$ radians (H) $\frac{4\pi}{3}$ radians (I) $\frac{3\pi}{2}$ radians

3. In a circle with a 12 mm radius, a central angle measuring $\frac{7\pi}{6}$ radians intercepts an arc. What is the length of the arc? **D**

(A) $\frac{2\pi}{7}$ mm (B) $\frac{72\pi}{7}$ mm (C) 12π mm (D) 14π mm

4. Circle X has a central angle of $\frac{3\pi}{8}$ radians intercepting an arc 3π ft long. Circle Y has a central angle of $\frac{3\pi}{4}$ radians intercepting an arc 3π ft long. Which best describes the radii of circle X and circle Y? **G**

(F) The radius of circle X is half as long as the radius of circle Y.

(G) The radius of circle X is twice as long as the radius of circle Y.

(H) The radius of circle X is the same length as the radius of circle Y.

(I) The radius of circle X is more than twice as long as the radius of circle Y.

Short Response

5. Describe the relationship between the total number of radians in a circle and the circumference of the circle.

[2] A central angle measuring 1 radian intercepts an arc the same length as the radius of the circle. Because the circumference of a circle is $2\pi r$, there are 2π radians in a circle.

[1] incomplete explanation

[0] no answer given

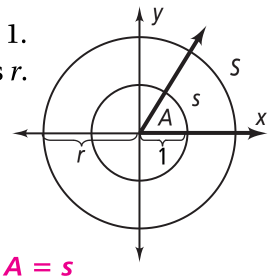
Name ______________________ Class ____________ Date ____________

Enrichment

Radian Measure

Conversion Formulas

Radian measure results in an easy-to-remember formula for computing the length of an arc of a circle intercepted by a given angle. Suppose you are given two concentric circles, one of radius r and one of radius 1, and a central angle A.



Let s denote the length of the arc intercepted by $\angle A$ in the circle of radius 1. Let S denote the length of the arc intercepted by $\angle A$ in the circle of radius r.

1. Write an equation involving the ratios of the arc lengths to the radii of the circles. $\frac{s}{1} = \frac{S}{r}$

2. If A represents the measure of $\angle A$ in radians, express A in terms of s. $A = s$

3. Express A in terms of S and r. $A = \frac{S}{r}$

4. Use your results to find S in terms of r and A. $S = rA$

5. In a circle of radius 3, find the length of the arc intercepted by a central angle of 60°. π

6. Recall that there are 360° or 2π radians in a circle. If A represents the number of radians and D represents the number of degrees in $\angle A$, write a proportion that can be used to convert between degrees and radians. $\frac{D}{360} = \frac{A}{2\pi}$

7. Derive a formula for the length of the arc S intercepted by $\angle A$ in terms of D. $s = \frac{\pi}{180}rD$

8. What is the length of the arc S intercepted by an angle of V revolutions along a circle of radius r? $S = 2\pi Vr$

9. Revolutions are often used to express rates of angular rotation. For example, the rate of angular rotation of a long-playing record is $33\frac{1}{3}$ rpm. Express this rate in radians per second. $\frac{10\pi}{9}$ radians/s

Name ______________________ Class ____________ Date ____________

Reteaching

Radian Measure

- A central angle that measures π radians intercepts an arc that forms a semicircle. It is a 180° rotation from the initial side to the terminal side of the angle.
- When converting radians to degrees or degrees to radians, use the proportion $\frac{\text{degree measure}}{360} = \frac{\text{radian measure}}{2\pi}$.

Problem

What is the radian measure of an angle of 225°?

$\frac{225}{360} = \frac{x}{2\pi}$ Substitute 225 for degree measure and a variable for radian measure.

$360x = 450\pi$ Cross multiply.

$x = \frac{450\pi}{360}$ Divide both sides by 360.

$x = \frac{5\pi}{4}$ Simplify.

$x \approx 3.93$ Use a calculator.

Check $\frac{\theta}{360} = \frac{\frac{5}{4}\pi}{2\pi}$ Check by substituting the radians into the proportion and solving for degrees.

$\frac{\theta}{360} = \frac{\frac{5}{4}\cancel{\pi}}{2\cancel{\pi}}$ Cancel π since it is in the numerator and denominator.

$2\theta = 450$ Cross multiply.

$\theta = 225$ Divide both sides by 2. This gives the degree measure.

An angle of 225° measures about 3.93 radians.

Exercises

Write each measure in radians and check.

1. 20° $\frac{\pi}{9} \approx 0.35$

2. 150° $\frac{5\pi}{6} \approx 2.62$

3. 45° $\frac{\pi}{4} \approx 0.79$

4. -110° $-\frac{11\pi}{18} \approx -1.92$

5. 315° $\frac{7\pi}{4} \approx 5.50$

6. 320° $\frac{16\pi}{9} \approx 5.59$

Write each measure in degrees and check.

7. $-\frac{3\pi}{2}$ -270°

8. $\frac{5\pi}{3}$ 300°

9. $\frac{\pi}{12}$ 15°

10. $\frac{8\pi}{5}$ 288°

11. $-\frac{7\pi}{6}$ -210°

12. $\frac{9\pi}{2}$ 810°

Reteaching (continued)

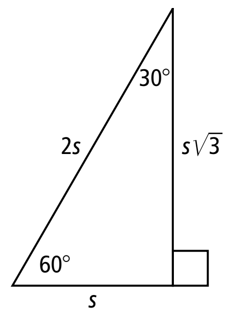
Radian Measure

You can use Special Right Triangles to find the exact values for the cosine and sine of radian measures.

Problem

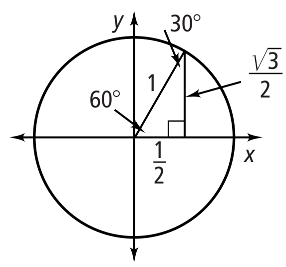
What are the exact values of $\cos\left(\frac{\pi}{3}\text{ radians}\right)$ and $\sin\left(\frac{\pi}{3}\text{ radians}\right)$?

Step 1 Find the angle measure in degrees. $\frac{\pi}{3} = \frac{180^\circ}{3} = 60^\circ$



Step 2 Recall the 30°-60°-90° triangle from Geometry.
In the unit circle, the hypotenuse is 1. So, $2s = 1$, or $s = \frac{1}{2}$.
Therefore the side opposite the 30° angle is $\frac{1}{2}$ and the side opposite the 60° angle is $\frac{\sqrt{3}}{2}$.

Step 3 Draw the angle on the unit circle. Complete a 30°-60°-90° triangle. Label the sides of the triangle.



Step 4 The cosine is the x-coordinate of the point at which the terminal side of the angle intersects the unit circle.
The sine is the y-coordinate.
$\cos\left(\frac{\pi}{3}\text{ radians}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\text{ radians}\right) = \frac{\sqrt{3}}{2}$.

Exercises

The measure θ of an angle in standard position is given. Find the exact values of $\cos\theta$ and $\sin\theta$ for each angle measure.

13. $-\frac{\pi}{6}$ radians

$\frac{\sqrt{3}}{2}, -\frac{1}{2}$

14. $\frac{3\pi}{4}$ radians

$-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$

15. $-\frac{\pi}{3}$ radians

$\frac{1}{2}, -\frac{\sqrt{3}}{2}$

16. $-\frac{2\pi}{3}$ radians

$-\frac{1}{2}, -\frac{\sqrt{3}}{2}$

17. $\frac{5\pi}{6}$ radians

$-\frac{\sqrt{3}}{2}, \frac{1}{2}$

18. $\frac{7\pi}{4}$ radians

$\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$

Name ______________________ Class ____________ Date ____________

Additional Vocabulary Support

The Sine Function

Properties of Sine Functions
Suppose $y = a \sin b\theta$, with $a \neq 0$, $b > 0$, and θ in radians. • $\lvert a \rvert$ is the amplitude of the function. • b is the number of cycles in the interval from 0 to 2π. • $\frac{2\pi}{b}$ is the period of the function.

Find the amplitude, number of cycles in the interval from 0 to 2π, and period of each function.

1. $y = 4 \sin \frac{1}{2}\theta$ amplitude: $\boxed{4}$ cycles: $\boxed{\frac{1}{2}}$ period: $\boxed{4\pi}$

2. $y = -3 \sin 2\pi\theta$ amplitude: $\boxed{3}$ cycles: $\boxed{2\pi}$ period: $\boxed{1}$

3. $y = 2 \sin \left(\frac{\pi}{2} \cdot \theta\right)$ amplitude: $\boxed{2}$ cycles: $\boxed{\frac{\pi}{2}}$ period: $\boxed{4}$

Write sine functions with the following properties.

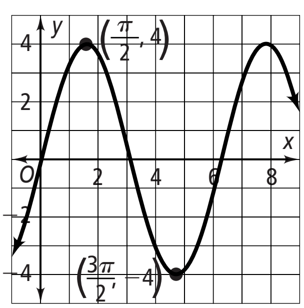
4. an amplitude of 6 and 4π cycles $y = 6 \sin 4\pi\theta$ or $y = -6 \sin 4\pi\theta$

5. an amplitude of 1 and $\frac{\pi}{2}$ cycles $y = \sin \left(\frac{\pi}{2} \cdot \theta\right)$ or $y = -\sin \left(\frac{\pi}{2} \cdot \theta\right)$

6. an amplitude of 2 and 6 cycles $y = 2 \sin 6\theta$ or $y = -2 \sin 6\theta$

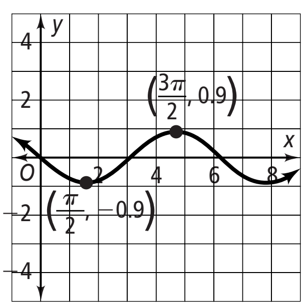
Find the amplitude of each sine curve. Amplitude equals half the difference of the maximum and minimum values of the function.

7.



4

8.



0.9

Name ______________________ Class ____________ Date ____________

Think About a Plan

The Sine Function

Music The sound wave for a certain pitch fork can be modeled by the function $y = 0.001 \sin 1320\pi\theta$. Sketch a graph of the sine curve.

Understanding the Problem

1. What is the function that models the sound wave? $y = 0.001 \sin 1320\pi\theta$

2. What is the standard form of a sine function? $y = a \sin b\theta$

3. What is the problem asking you to determine?

   the graph of the sine curve that models the sound wave of a tuning fork

Planning the Solution

4. How can you find the period and amplitude from the function rule?

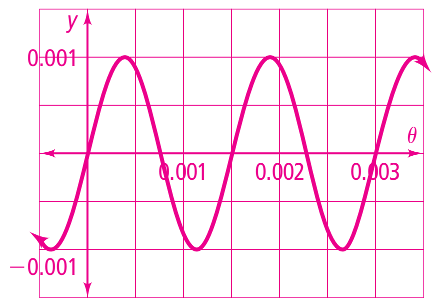
   Compare the function rule to the standard form of a sine function; $\frac{2\pi}{b}$ = the period and $|a|$ = the amplitude of the function

5. What are the period and amplitude of the function? $\frac{1}{660}$; 0.001

6. How many cycles of the graph are between 0 and 2π? 1320π

Getting an Answer

7. Sketch a graph of the function $y = 0.001 \sin 1320\pi\theta$. Adjust the scale of the axes to make the function easier to draw.



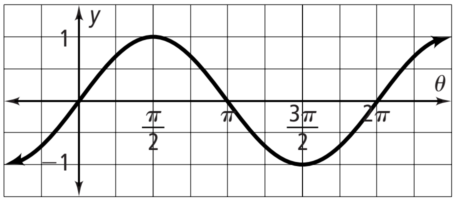
Name ______________________ Class ____________ Date ____________

Practice

Form G

The Sine Function

Use the graph at the right to find the value of $y = \sin\theta$ for each value of θ.



1. 5.5 radians **about −0.7**

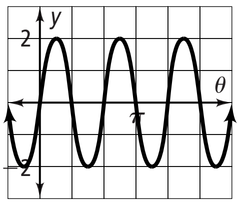
2. 1 radian **about 0.8**

3. $\frac{3\pi}{4}$ radians **about 0.7**

4. $\frac{\pi}{2}$ radians **1**

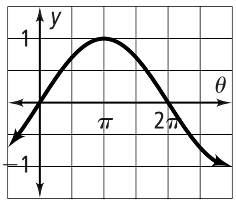
Determine the number of cycles each sine function has in the interval from 0 to 2π. Find the amplitude and period of each function.

5.



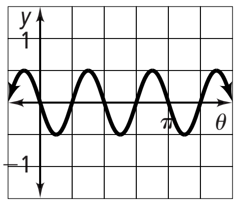
3; 2; $\frac{2\pi}{3}$

6.



$\frac{1}{2}$; 1; 4π

7.



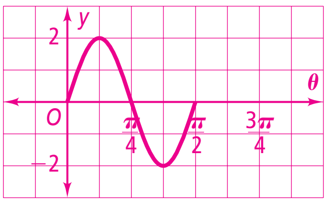
4; $\frac{1}{2}$; $\frac{\pi}{2}$

Sketch one cycle of each sine curve. Assume $a > 0$. Write an equation for each graph.

8. amplitude $= 2$

period $= \frac{\pi}{2}$

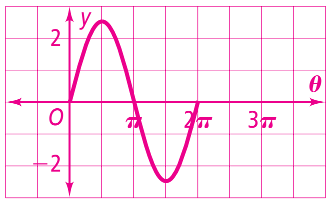
$y = 2\sin 4\theta$



9. amplitude $= 2.5$

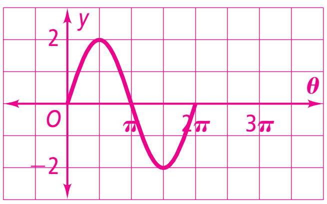
period $= 2\pi$

$y = 2.5\sin\theta$

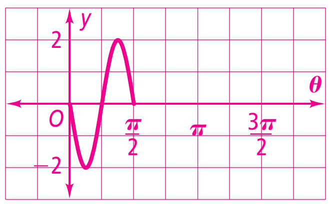


Sketch one cycle of the graph of each sine function.

10. $y = 2\sin\theta$

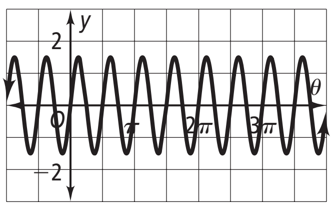


11. $y = -2\sin 4\theta$



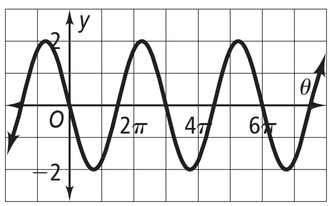
Find the period of each sine curve. Then write an equation for each sine function.

12.



$\frac{\pi}{2}$; $y = 1.5\sin 4\theta$

13.



3π; $y = -2\sin\frac{2}{3}\theta$

Practice (continued) *Form G*

The Sine Function

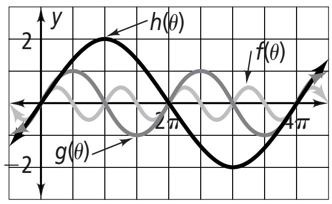
Determine the number of cycles each sine function has in the interval from 0 to 2π. Find the amplitude and period of each function.

14. $y = \sin 2\theta$ **2; 1; π**

15. $y = -3 \sin 2\theta$ **2; 3; π**

16. $y = 4 \sin 5\theta$ **5; 4; $\frac{2\pi}{5}$**

17. Reasoning The graph at the right shows the sine functions $f(\theta)$, $g(\theta)$, and $h(\theta)$. For each function, $a > 0$.



a. Order the functions by increasing value of a. **$f(\theta)$, $g(\theta)$, $h(\theta)$**

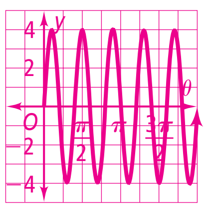
b. Order the functions by increasing value of b. **$h(\theta)$, $g(\theta)$, $f(\theta)$**

18. Writing Compare and contrast the graphs of $p(\theta) = a \sin b\theta$, $q(\theta) = -a \sin b\theta$, and $r(\theta) = -a \sin(-b\theta)$. Assume $a > 0$ and $b > 0$.

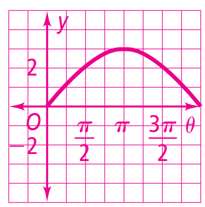
The graphs all have the same amplitude and period. Function $r(\theta)$ is identical to $p(\theta)$. Function $q(\theta)$ is a reflection of $p(\theta)$ in the x-axis.

Find the period and amplitude of each sine function. Then sketch each function from 0 to 2π.

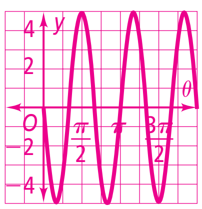
19. $y = 4 \sin 5\theta$ **$\frac{2\pi}{5}$; 4**



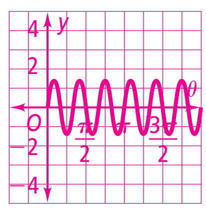
20. $y = 3 \sin \frac{\theta}{2}$ **4π; 3**



21. $y = -5 \sin 3\theta$ **$\frac{2\pi}{3}$; 5**



22. $y = 1.5 \sin 6\theta$ **$\frac{\pi}{3}$; 1.5**



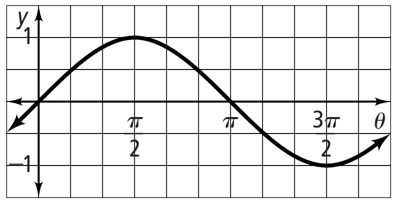
Name ______________________ Class ____________ Date ____________

Practice

Form K

The Sine Function

Use the graph at the right to find or estimate the value of $y = \sin\theta$ for each value of θ.



1. π radians
0

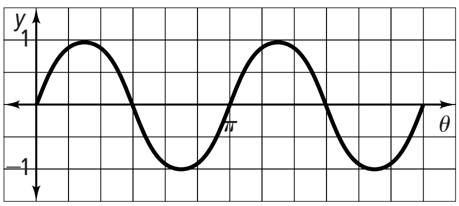
2. 2 radians
≈ 0.9

3. $\frac{3\pi}{2}$ radians
-1

4. 4 radians
≈ -0.8

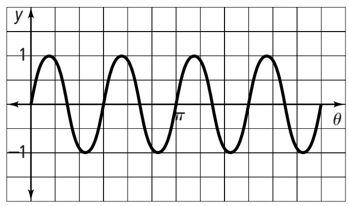
Determine the number of cycles each sine function has in the interval from 0 to 2π. Then find the period of each function.

5.



two cycles; period: π

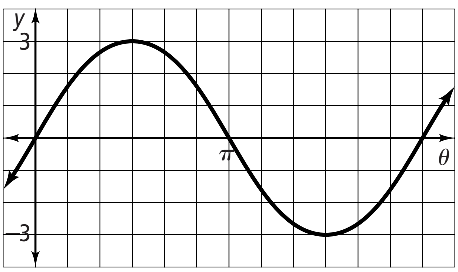
6.



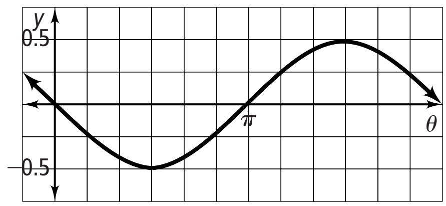
four cycles; period: $\frac{\pi}{2}$

Find the amplitude of the following sine curves.

7. $y = 3\sin x$ 3



8. $y = -0.5\sin x$ 0.5



9. Error Analysis Your friend said that the amplitude of $y = -4\sin x$ is -4. What error did she make? What is the correct amplitude?
She did not use the absolute value of *a*. The amplitude is 4.

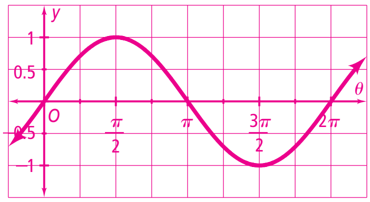
Name ______________________ Class ____________ Date ____________

Practice (continued) *Form K*

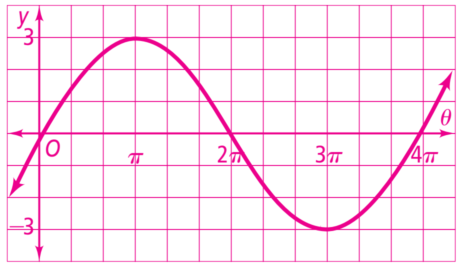
The Sine Function

Sketch one cycle of each of the following sine curves. Assume $a > 0$.

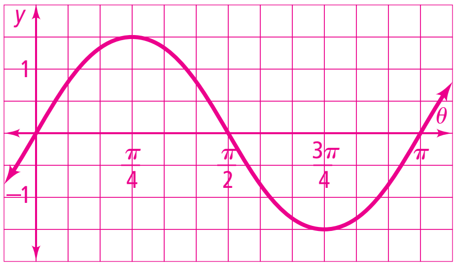
10. amplitude 1, period 2π



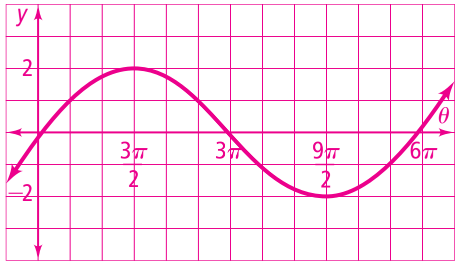
11. amplitude 3, period 4π



12. amplitude 1.5, period π



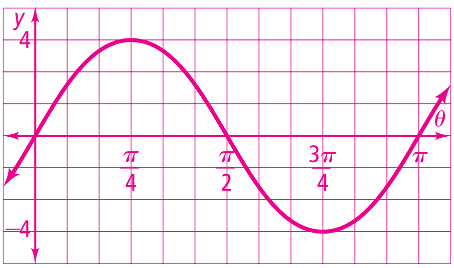
13. amplitude 2, period 6π



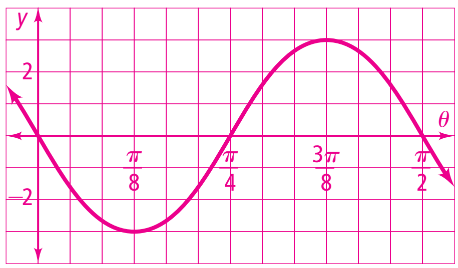
Sketch one cycle of the graph of each sine function.

14. $y = 4 \sin 2\theta$

amplitude: 4
cycles from 0 to 2π: 2
period: π



15. $y = -3 \sin 4\theta$



Name ______________________ Class ____________ Date ____________

Standardized Test Prep

The Sine Function

Multiple Choice

For Exercises 1–5, choose the correct letter.

1. Which expressions have the same value? C

I. $\sin(-30^\circ)$ II. $\sin 390^\circ$ III. $\sin 30^\circ$

A I and II B I and III C II and III D I, II, and III

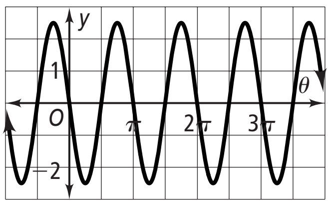
2. What is the period of the function $y = -\frac{2}{5}\sin 6\pi\theta$? F

F $\frac{1}{3}$ G $\frac{2}{5}$ H $\frac{1}{3}\pi$ I 6π

3. Which function has an amplitude of 3 and a period of 3π? D

A $y = \frac{2}{3}\sin 3\theta$ B $y = \frac{3}{2}\sin\frac{2}{3}\theta$ C $y = 3\sin 3\pi\theta$ D $y = 3\sin\frac{2}{3}\theta$

4. What is the amplitude and period of the sine curve shown at the right? I



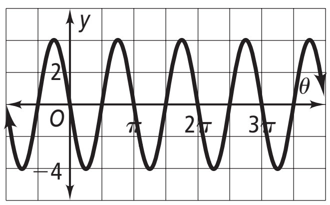
F amplitude −2.5, period 4π

H amplitude −2.5, period π

G amplitude 2.5, period $\frac{3}{2}\pi$

I amplitude 2.5, period π

5. Which function represents the sine curve shown at the right? A



A $y = -4\sin 2\theta$

C $y = -4\sin\theta$

B $y = 4\sin\pi\theta$

D $y = 4\sin 2\pi\theta$

Extended Response

6. The function $y = \frac{2}{3}\sin\frac{7\pi}{9}\theta$ represents a sine curve. Find the amplitude of the sine curve and its period in radians. Show your work.

[4] amplitude: $\frac{2}{3}$; period: $\frac{2\pi}{b} = \frac{2\pi}{\frac{7\pi}{9}} = 2\pi \cdot \frac{9}{7\pi} = 2\frac{4}{7}$

[3] appropriate methods, but with one computational error

[2] incorrect amplitude with correct period OR correct amplitude with period calculated incorrectly

[1] correct amplitude and period, without work shown

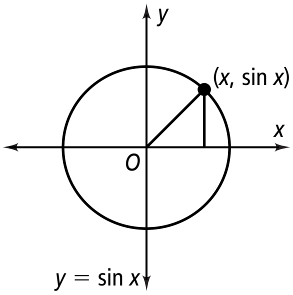
[0] incorrect answers and no work shown OR no answers given

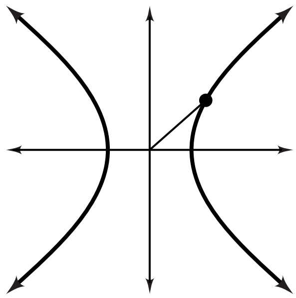
Name ______________________ Class __________ Date __________

Enrichment

The Sine Function

Recall that the sine function describes the y-coordinate of a point on a unit circle for any given angle. Another function called *sinh* or *hyperbolic sine* relates to the unit rectangular hyperbola $x^2 - y^2 = 1$.



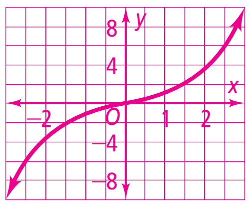


You can evaluate the sinh function algebraically with the equation $\sinh x = \frac{e^x - e^{-x}}{2}$.

1. Use this equation to complete the table of values for $y = \sinh x$. Round to the nearest tenth.

x	−3	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2	3
y	−10	−3.6	−2.1	−1.2	−0.5	0	0.5	1.2	2.1	3.6	10

2. Use the values from your table to sketch a graph of $y = \sinh x$.



3. What are the domain and range of $y = \sinh x$?
The domain and range are both all real numbers.

4. Use your graph to make a reasonable estimate of sinh 2.5. **about 6**

5. Use the equation to verify your estimate. **about 6.05**

6. Use your graph to make a reasonable estimate of a solution to the equation $0.75 = \sinh x$. **about 0.7**

7. Explain how you would use the equation to verify your estimate to the equation $0.75 = \sinh x$. **Answers may vary. Sample: Solve the equation $0.75 = \frac{e^x - e^{-x}}{2}$ for x.**

Name ______________________ Class ____________ Date ____________

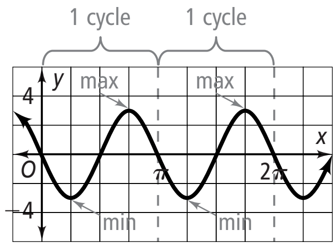
Reteaching

The Sine Function

A sine curve is the graph of a sine function. You can identify a sine curve by its *amplitude* and *period*. Amplitude is one-half the vertical distance between the maximum and minimum values. The period is the horizontal length of one cycle.

Problem

Use the graph of $y = -3\sin 2x$, where x is measured in radians, at the right. What are the amplitude and period of the sine curve?



Amplitude

The maximum value of the sine curve is 3.

The minimum value of the sine curve is −3.

One-half the difference of these values is $\frac{(3-(-3))}{2} = \frac{6}{2} = 3$.

The amplitude of the curve is 3.

> The **amplitude** equals the absolute value of the coefficient of the function.

Period

Between 0 and 2π, the graph cycles 2 times.

To get the length of one cycle, divide 2π by the number of cycles between 0 and 2π.

The period of the curve is $\frac{2\pi}{2} = \pi$.

> The **number of cycles** between 0 and 2π equals the coefficient of x in the function.

Summary

For all sine functions written in the form $y = a \sin b\theta$, where $a \neq 0$, $b > 0$, and θ is measured in radians:

$$\text{amplitude} = |a| \qquad \text{period} = \frac{2\pi}{b}$$

Exercises

Find the amplitude and period of each sine function.

1. $y = \frac{1}{2}\sin 3\theta$ $\frac{1}{2}; \frac{2\pi}{3}$

2. $y = \sin 5\theta$ $1; \frac{2\pi}{5}$

3. $y = 4\sin \frac{4}{3}\pi\theta$ $4; \frac{3}{2}$

4. $y = \frac{3}{2}\sin \theta$ $\frac{3}{2}; 2\pi$

5. $y = -2\sin \frac{3}{4}\theta$ $2; \frac{8\pi}{3}$

6. $y = \pi \sin 2\theta$ π, π

Name ______________________ Class ____________ Date ____________

Reteaching (continued)

The Sine Function

Problem

What is the graph of two cycles of $y = 2 \sin \frac{1}{2}\theta$?

Step 1 Compare the function to $y = a \sin b\theta$. $a = 2$ and $b = \frac{1}{2}$

Find the amplitude. $|a| = |2| = 2$

Find the period of the curve. $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

Step 2 Find the minimum and maximum of the curve.
Because the amplitude is 2, the maximum is 2 and the minimum is -2.

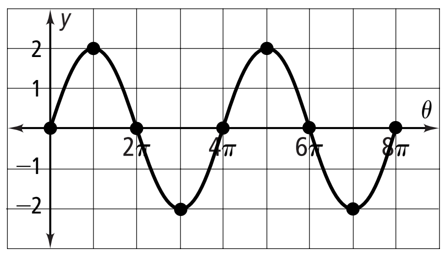
Step 3 Make a table of values. Choose θ-values at intervals of one-fourth the period: $\frac{4\pi}{4} = \pi$.

The y-values cycle through the pattern *zero-max-zero-min-zero*.

θ	0	π	2π	3π	4π	5π	6π	7π	8π
y	0	2	0	-2	0	2	0	-2	0

Step 4 Plot the points from the table.

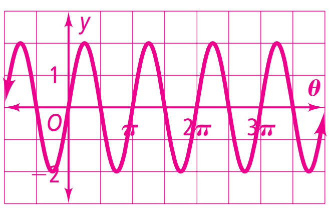
Step 5 Draw a smooth curve through the points.



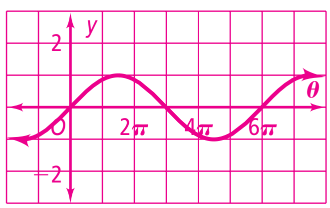
Exercises

Graph each function.

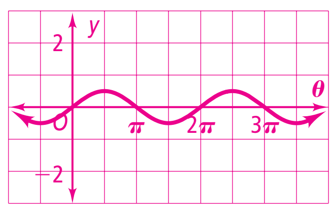
7. $y = 2 \sin 2\theta$



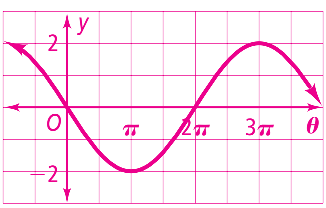
8. $y = \sin \frac{1}{3}\theta$



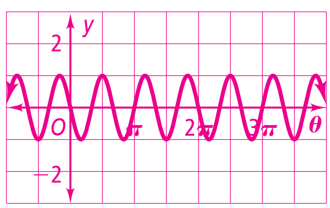
9. $y = \frac{1}{2} \sin \theta$



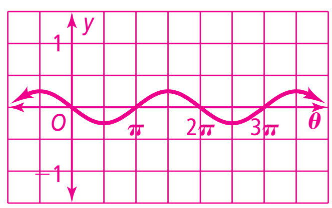
10. $y = -2 \sin \frac{1}{2}\theta$



11. $y = -\sin 3\theta$



12. $y = -\frac{1}{4} \sin \theta$



Name ______________________ Class ____________ Date ____________

Additional Vocabulary Support

The Cosine Function

Properties of Cosine Functions
Suppose $y = a \cos b\theta$, with $a \neq 0$, $b > 0$, and θ in radians.
• $\lvert a \rvert$ is the amplitude of the function.
• b is the number of cycles in the interval from 0 to 2π.
• $\frac{2\pi}{b}$ is the period of the function.

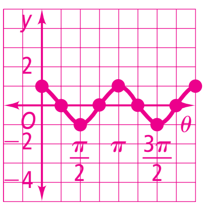
Name the amplitude, number of cycles in the interval from 0 to 2π, and period for the following functions.

1. $y = -1.4 \cos \frac{1}{4}\theta$ amplitude: $\boxed{1.4}$ cycles: $\boxed{\frac{1}{4}}$ period: $\boxed{8\pi}$

2. $y = 3 \cos \frac{\pi}{2}\theta$ amplitude: $\boxed{3}$ cycles: $\boxed{\frac{\pi}{2}}$ period: $\boxed{4}$

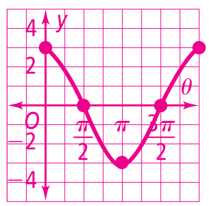
Sketch the graph of each function in the interval from 0 to 2π. Name the amplitude, number of cycles, and period of each function.

3. $y = \cos 2\theta$



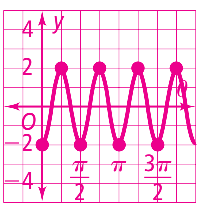
amplitude: 1; cycles: 2; period: π

4. $y = 3 \cos\theta$



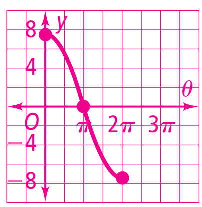
amplitude: 3; cycles: 1; period: 2π

5. $y = -2 \cos 4\theta$



amplitude: 2; cycles: 4; period: $\frac{\pi}{2}$

6. $y = 7.5 \cos \frac{1}{2}\theta$



amplitude: 7.5; cycles: $\frac{1}{2}$; period: 4π

Name ______________________ Class ____________ Date ____________

Think About a Plan

The Cosine Function

Tides The table at the right shows the times for high tide and low tide of one day. The markings on the side of a local pier showed a high tide of 7 ft and a low tide of 4 ft on the previous day.

Tide Table	
High Tide	4:03 A.M.
Low Tide	10:14 A.M.
High Tide	4:25 P.M.
Low Tide	10:36 P.M.

a. What is the average depth of water at the pier? What is the amplitude of the variation from the average depth?

b. How long is one cycle of the tide?

c. Write a cosine function that models the relationship between the depth of water and the time of day. Use $y = 0$ to represent the average depth of water. Use $t = 0$ to represent the time 4:03 A.M.

d. Reasoning Suppose your boat needs at least 5 ft of water to approach or leave the pier. Between what times could you come and go?

1. What is the average depth of water at the pier? **5.5 ft**

2. How can you find the amplitude of the variation from the average depth? What is the amplitude?

   (0.5)(maximum − minimum) = 1.5 ft

3. How can you find the length of one cycle of the tide? What is the cycle length in minutes?

   One cycle is one complete pattern, such as between 2 consecutive maximums, or high tides. One cycle is 742 min long.

4. How can you find a cosine function that models the relationship between the depth of water and the time of day? Write the cosine function.

   Start with the standard form $y = a \cos b\theta$. Substitute 1.5 for a and $\frac{2\pi}{742}$ for b;

   $y = 1.5 \cos \frac{2\pi}{742} t$

5. How can you use a graph to find the times of day when the water depth is at least 5 ft?

   Answers may vary. Sample: Graph the cosine function and the line $y = -0.5$ on a graphing calculator. The parts of the cosine function that are above the line represent a water depth greater than 5 ft. Find the points of intersection.

6. Over what domain should you graph the cosine function to represent the entire day? **−243 min to 1197 min**

7. Between what times could you come and go? **12:17 A.M. to 7:49 A.M. and 12:39 P.M. to 8:11 P.M.**

Name ______________________ Class ____________ Date ____________

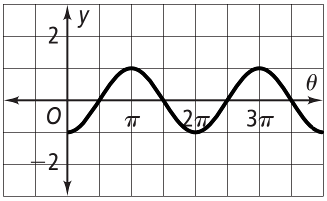
Practice

Form G

The Cosine Function

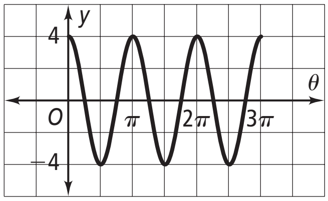
Find the period and amplitude of each cosine function. Determine the values of θ for $0 \le \theta < 2\pi$ that the maximum value(s), minimum value(s), and zeros occur.

1.



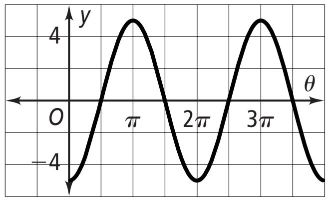
Period: 2π; amp.: 1; max.: π, min: 0; zeros: $\frac{\pi}{2}$; $\frac{3\pi}{2}$

2.



Period: π; amp.: 4; max.: 0, π; min.: $\frac{\pi}{2}$, $\frac{3\pi}{2}$; zeros: $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$

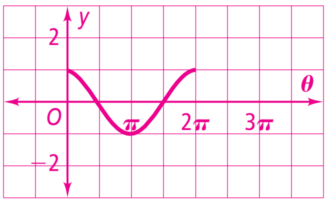
3.



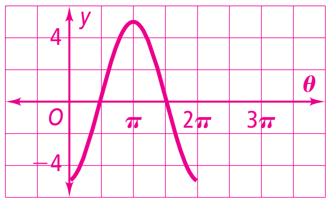
Period: 2π; amp.: 5; max.: π; min.: 0; zeros: $\frac{\pi}{2}$; $\frac{3\pi}{2}$

Sketch the graph of each function in the interval from 0 to 2π.

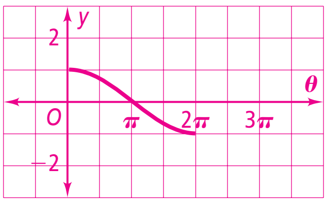
4. $y = \cos\theta$



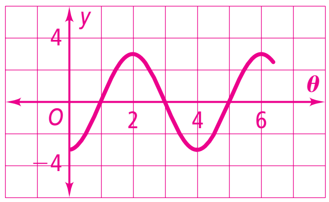
5. $y = -5\cos\theta$



6. $y = \cos\frac{\theta}{2}$



7. $y = -3\cos\frac{\pi}{2}\theta$



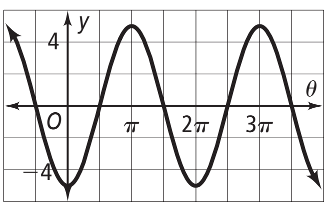
Write a cosine function for each description. Assume that $a > 0$.

8. amplitude 2π, period 1 **$y = 2\pi\cos 2\pi\theta$**

9. amplitude $\frac{1}{2}$, period π **$y = \frac{1}{2}\cos 2\theta$**

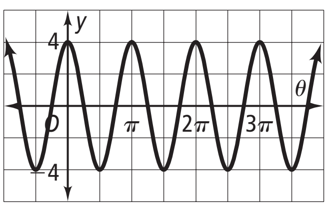
Write an equation of a cosine function for each graph.

10.



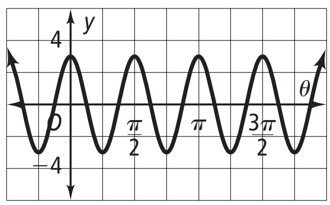
$y = -5\cos\theta$

11.



$y = 4\cos 2\theta$

12.



$y = 3\cos 4\theta$

Name ______________________ Class ____________ Date ____________

Practice (continued) Form G

The Cosine Function

Solve each equation in the interval from 0 to 2π. Round your answer to the nearest hundredth.

13. $2 \cos 3\theta = 1.5$ **0.24, 1.85, 2.34, 3.95, 4.43, 6.04**

14. $\cos \frac{\theta}{3} = 1$ **0**

15. $1.5 \cos \pi\theta = -1.5$ **1.00, 3.00, 5.00**

16. $3 \cos \frac{\pi}{5}\theta = 2$ **1.34**

17. $3 \cos\theta = 2$ **0.84, 5.44**

18. $0.5 \cos\frac{\theta}{2} = 0.5$ **0**

19. $4 \cos \frac{\pi}{4}\theta = -2$ **2.67, 5.33**

20. $3 \cos\frac{\theta}{4} = 1.5$ **4.19**

21. $3 \cos\theta = -3$ **3.14**

22. $\sin\theta = -0.4$ **3.55, 5.87**

23. $5 \sin \frac{1}{2}\theta = 2$ **0.82, 5.46**

24. $\sin \frac{\pi}{3}\theta = \frac{2}{3}$ **0.70, 2.30**

Identify the period, range, and amplitude of each function.

25. $y = -\cos \theta$
2π; $-1 \le y \le 1$; 1

26. $y = \cos 2\pi\, \theta$
1; $-1 \le y \le 1$; 1

27. $y = -2 \cos 2\theta$
π; $-2 \le y \le 2$; 2

28. $y = 3 \cos 4\theta$
$\frac{\pi}{2}$; $-3 \le y \le 3$; 3

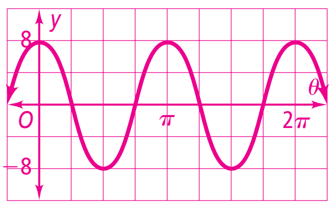
29. $y = 3 \cos 8\theta$
$\frac{\pi}{4}$; $-3 \le y \le 3$; 3

30. $y = -4 \cos \pi\theta$
2; $-4 \le y \le 4$; 4

31. Reasoning Let the variable n represent a solution of $y = a \sin b\theta$ in the interval from 0 to 2π. Write a solution in terms of n for the equation in the interval from 2π to 4π. ***$n + 2\pi$***

32. A carousel horse can move up to 8 in. above or below its starting position. The equation $y = 8 \cos 2\theta$ describes the horse's vertical movement as the carousel revolves.

a. Graph the equation.



b. If the horse starts at its maximum height, how many times does it reach its minimum height in one full revolution of the carousel?
2 times

33. A helicopter lowers a rope ladder to a scuba diver floating on the ocean surface. The waves crest at 4 ft above the lowest level of the water every 8 s.

a. Write a cosine equation to describe the height of the diver as a function of time t. **$y = 2 \cos \frac{\pi}{4}t$**

b. Writing The diver can reach 2 ft above her. The lowest rung of the ladder is 3 ft above the average level of the water. For about how many consecutive seconds will the ladder be within the diver's reach? Explain.

Answers may vary. Sample: The height of the diver is $y = 2 \cos \frac{\pi}{4}t + 2$. The height of the ladder is $y = 3$. I graphed $y = 3$ and $y = 2 \cos \frac{\pi}{4}t + 2$. They intersect at $t \approx 1.33$ and $t \approx 6.67$. The ladder is within the diver's reach for about 1.33 s before and after she reaches maximum height. This gives the diver a total of 2.66 s to grab the ladder.

Name ______________________________ Class ____________ Date ____________

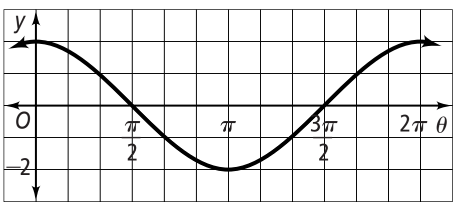
Practice

Form K

The Cosine Function

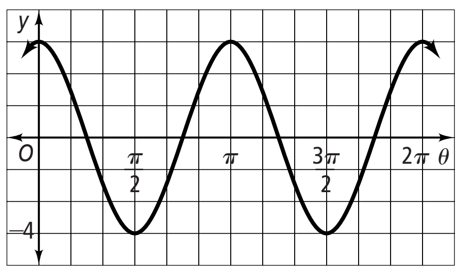
Find the period and the amplitude of each cosine function. Then determine where the maximum values, minimum values, and zeros occur in the interval from 0 to 2π.

1.



period: 2π, amplitude: 2, maximums: 0 and 2π, minimum: π, zeros: $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

2.

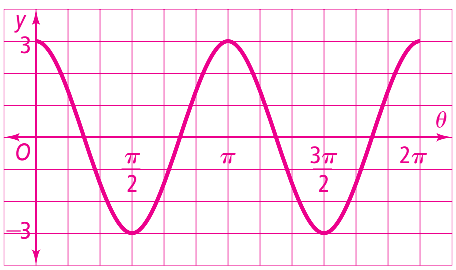


period: π; amplitude: 4; maximums: 0, π, and 2π; minimums: $\frac{\pi}{2}$ and $\frac{3\pi}{2}$; zeros: $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$

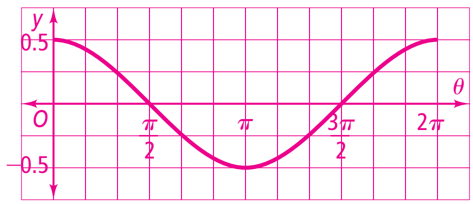
Sketch the graph of each function in the interval from 0 to 2π.

3. $y = 3\cos 2\theta$

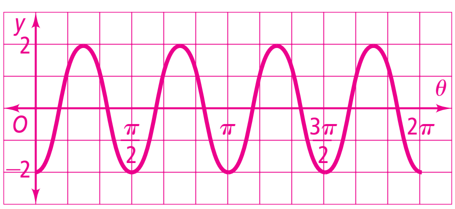
amplitude: 3
cycles: 2
period: π



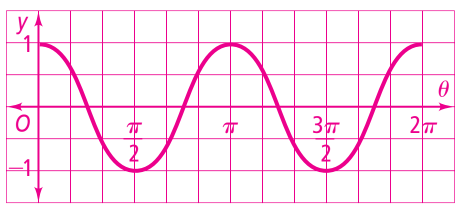
4. $y = 0.5\cos\theta$



5. $y = -2\cos 4\theta$



6. $y = \cos 2\theta$



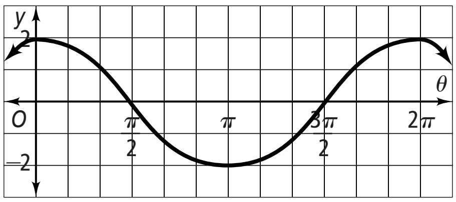
Name ______________________________ Class ____________ Date ____________

Practice (continued) *Form K*

The Cosine Function

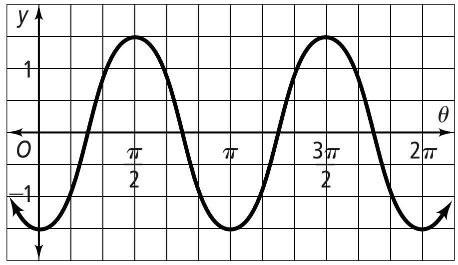
Write an equation of a cosine function for each graph.

7.



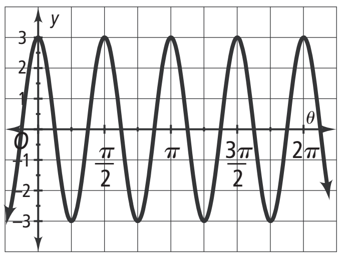
$y = 2 \cos \theta$

8.



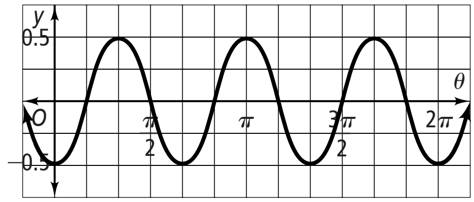
$y = -1.5 \cos 2\theta$

9.



$y = 3 \cos 4\theta$

10.



$y = -0.5 \cos 3\theta$

Solve each equation in the interval from 0 to 2π. Round your answers to the nearest hundredth.

11. $\cos 2t = 0.25$
0.66, 2.48, 3.8, and 5.62

12. $5 \cos t = 1$
1.37 and 4.91

13. $4 \cos \frac{t}{4} = 2$
4.19

14. $-3 \cos t = 2.5$
2.56 and 3.73

15. $2 \cos 0.5t = -0.5$
3.65

16. $-0.5 \cos 2t = 0.3$
1.11, 2.03, 4.25, and 5.18

Name ______________________ Class ____________ Date ____________

Standardized Test Prep

The Cosine Function

Multiple Choice

For Exercises 1–5, choose the correct letter.

1. Which is equivalent to $-\cos(\theta + 2\pi)$? C

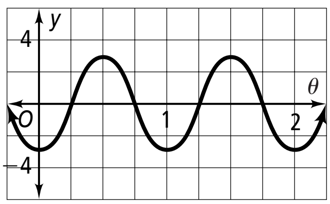
(A) $\cos\theta$ (B) $-\cos\pi\theta$ (C) $\cos(\theta + \pi)$ (D) $\cos(\theta + 2\pi)$

2. Which function has the same period as $f(x) = 2\cos 3\pi\theta$? I

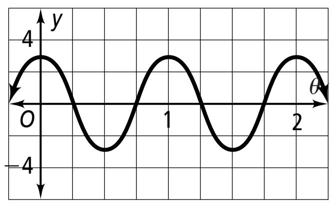
(F) $f(x) = 2\cos\theta$ (G) $f(x) = \cos 3\theta$ (H) $f(x) = 2\cos\pi\theta$ (I) $f(x) = \cos 3\pi\theta$

3. Which graph represents $y = -3\cos 2\pi\theta$? A

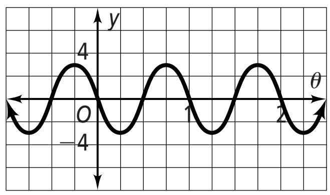
(A)



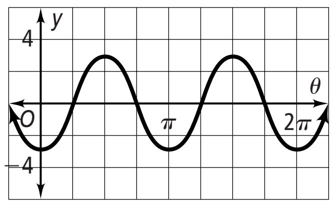
(C)



(B)



(D)



4. Which equation has the greatest number of solutions for $0 \leq \theta \leq 2\pi$? H

(F) $\cos\theta = 1$ (G) $2\cos\theta = 1$ (H) $\cos 2\theta = 1$ (I) $-\cos 2\theta = 1$

5. Which approximate value of θ is a solution of $-4\cos 2\theta = 3$ for $0 \leq \theta \leq 2\pi$? A

(A) 1.2 (B) 1.6 (C) 2.4 (D) 3.1

Short Response

6. Solve $-\cos 2\theta = 0$ for $0 \leq \theta \leq 2\pi$.

[2] $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

[1] incomplete answer

[0] no answer given

Name ____________________ Class __________ Date __________

Enrichment

The Cosine Function

Calculating Trigonometric Functions

How are values of trigonometric functions of other angles, such as 26°, calculated? One way is to construct a right triangle in which one of the angles is 26°. Then measure the sides and perform the necessary arithmetic. This approach has its drawbacks. First, accuracy is limited to the accuracy with which the angles and sides can be measured. Second, it is time-consuming.

Another way, which can be carried out to any degree of accuracy, involves the use of certain polynomials. For example, the following three polynomials can be used to approximate the sine of an angle measured in radians.

$S_3(x) = x - \frac{x^3}{6}$ $\quad S_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$ $\quad S_7(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$

By using a calculator to complete the following table, you can see how these polynomials approximate the sine function to various degrees of accuracy.

Let D be the number of degrees in the angle. Let x be the *radian* measure of $\angle D$. Find the value of each polynomial to six decimal places using x. Then find $\sin x$ to six decimal places.

	D	x	$S_3(x)$	$S_5(x)$	$S_7(x)$	$\sin x$
1.	5°	$\frac{\pi}{36}$	0.087156	**0.087156**	**0.087156**	**0.087156**
2.	11°	$\frac{11\pi}{180}$	**0.190807**	**0.190809**	**0.190809**	**0.190809**
3.	26°	$\frac{13\pi}{90}$	**0.438212**	**0.438372**	**0.438371**	**0.438371**
4.	37°	$\frac{37\pi}{180}$	**0.600888**	**0.601824**	**0.601815**	**0.601815**

5. Discuss the results of the table. **Answers may vary. Sample: The results indicate that $S_n(x)$ approximates sin x to a greater degree of accuracy as n increases. For x small, $S_3(x)$ is a good approximation to sin x; indicating that $\sin x \approx x$ for x small.**

These three polynomials can be used to approximate the cosine of any angle:

$C_2(x) = 1 - \frac{x^2}{2}$ $\quad C_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ $\quad C_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$

Fill in the following table as you did for Exercises 1–4.

	D	x	$C_2(x)$	$C_4(x)$	$C_6(x)$	$\cos x$
6.	5°	$\frac{\pi}{36}$	0.996192	**0.996192**	**0.996195**	**0.996195**
7.	11°	$\frac{11\pi}{180}$	**0.981571**	**0.981627**	**0.981627**	**0.981627**
8.	26°	$\frac{13\pi}{90}$	**0.897039**	**0.898806**	**0.898794**	**0.898794**
9.	37°	$\frac{37\pi}{180}$	**0.791489**	**0.798735**	**0.798635**	**0.798636**

10. Discuss the results of the table. **Answers may vary. Sample: The results indicate that $C_n(x)$ approximates cos x to a greater degree of accuracy as n increases. For x small, $C_2(x)$ is a good approximation to cos x; indicating that $\cos x \approx 1$ for x small.**

Name ______________________ Class __________ Date __________

Reteaching

The Cosine Function

Problem

What is the graph of $y = 3\cos\frac{\pi}{2}\theta$ in the interval from 0 to 2π?

Step 1 Compare the function to $y = a\cos b\theta$. $a = 3$ and $b = \frac{\pi}{2}$

Find the amplitude. $|a| = |3| = 3$

Find the period of the curve. $\frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{2}} = 4$

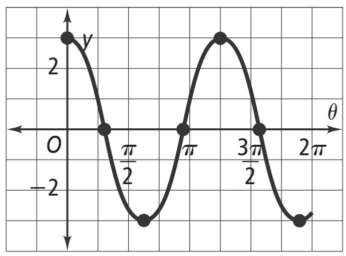
Step 2 Find the minimum and maximum of the curve.
Because the amplitude is 3, the maximum is 3 and the minimum is -3.

Step 3 Make a table of values. Choose θ-values at intervals of one-fourth the period: $\frac{4}{4} = 1$.
The y-values cycle through the pattern *max-zero-min-zero-max*.

θ	0	1	2	3	4	5	6
y	3	0	-3	0	3	0	-3

Step 4 Plot the points from the table.

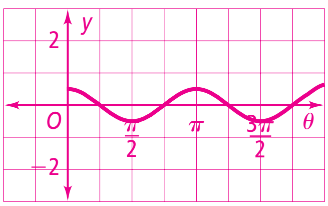
Step 5 Draw a smooth curve through the points.



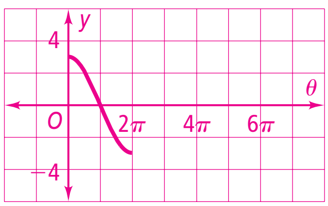
Exercises

Sketch the graph of each function in the interval from 0 to2π.

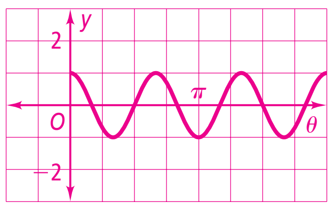
1. $y = \frac{1}{2}\cos 2\theta$



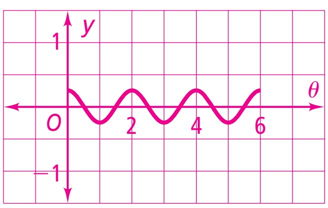
2. $y = 3\cos\frac{1}{2}\theta$



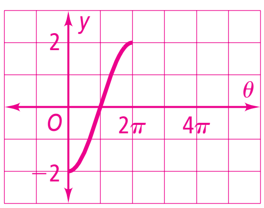
3. $y = \cos 3\theta$



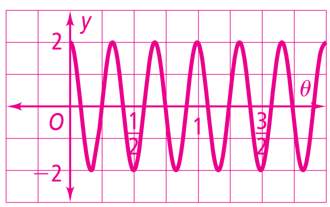
4. $y = \frac{1}{4}\cos\pi\theta$



5. $y = -2\cos\frac{1}{2}\theta$



6. $y = 2\cos 6\pi\theta$



Reteaching (continued)

The Cosine Function

Solving a sine or cosine equation is similar to solving a system of two linear equations. You can graph each side of the equation. The solutions will be the points where the graphs intersect.

Problem

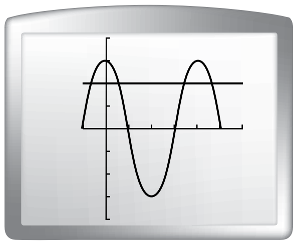
What are the solutions of $3\cos\frac{1}{2}\theta = 2$ in the interval 0 to 4π?

Step 1 Set each side of the equation equal to y.

$y = 3\cos\frac{1}{2}\theta$

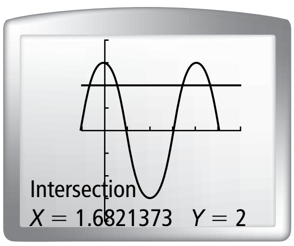
$y = 2$

Step 2 Graph each equation on the same grid.

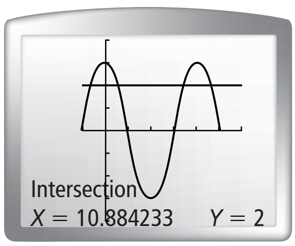


x Scale: π y Scale: 1

Step 3 Between $\theta = 0$ and $\theta = 4\pi$, the graphs intersect 2 times. Use the **Intersect** feature to find the coordinates of these points.



X Scale: π Y Scale: 1



X Scale: π Y Scale: 1

The solutions of $3\cos\frac{1}{2}\theta = 2$ in the interval 0 to 4π are $\theta \approx 1.68$ and 10.88.

Exercises

Find all solutions in the interval from 0 to 2π. Round to the nearest hundredth.

7. $-\cos\theta = \frac{3}{4}$

2.42, 3.86

8. $2\cos\theta = 1$

1.05, 5.24

9. $3\cos\pi\theta = 2$

0.27, 1.73, 2.27, 3.73, 4.27, 5.73, 6.27

10. $\cos\frac{1}{2}\pi\theta = -0.5$

1.33, 2.67, 5.33

11. $\frac{1}{2}\cos 4\theta = 0$

0.39, 1.18, 1.96, 2.75, 3.53, 4.32, 5.11, 5.89

12. $-3\cos 2\pi\theta = 2.5$

0.41, 0.59, 1.41, 1.59, 2.41, 2.59, 3.41, 3.59, 4.41, 4.59, 5.41, 5.59

13. $5\cos 4\theta = 3$

0.23, 1.34, 1.80, 2.91, 3.37, 4.48, 4.94, 6.05

14. $\frac{3}{4}\cos\frac{1}{2}\pi\theta = \frac{1}{2}$

0.54, 3.46, 4.54

15. $-4\cos 2\theta = 2$

1.05, 2.09, 4.19, 5.24

Name ______________________ Class ____________ Date ____________

Additional Vocabulary Support

The Tangent Function

Your teacher made a set of note cards describing how to graph the function $y = \tan \frac{1}{2}\pi\,\theta$. The cards got mixed up as he was about to show them to your class.

Divide the period into fourths and locate 3 points between the asymptotes. Plot the points and sketch the curve.

Find the asymptotes of the function. The asymptotes occur at each end of the cycle.

Determine the cycle of the function by using the formula $-\frac{\pi}{2b}$ to $\frac{\pi}{2b}$.

Use the formula for the period $p = \frac{\pi}{b}$. Substitute $\frac{1}{2}\pi$ for b and simplify.

Use the note cards to write the steps in order.

1. First, use the formula for the period $p = \frac{\pi}{b}$. Substitute $\frac{1}{2}\pi$ for b and simplify.
2. Second, determine the cycle of the function by using the formula $-\frac{\pi}{2b}$ to $\frac{\pi}{2b}$.
3. Then, find the asymptotes of the function. The asymptotes occur at each end of the cycle.
4. Finally, divide the period into fourths and locate 3 points between the asymptotes. Plot the points and sketch the curve.

Name ______________________ Class ______________ Date ______________

Think About a Plan

The Tangent Function

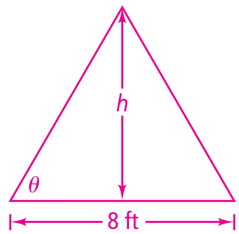
Construction An architect is designing a hexagonal gazebo. The floor is a hexagon made up of six isosceles triangles. The function $y = 4 \tan \theta$ models the height of one triangle, where θ is the measure of one of the base angles and the base of the triangle is 8 ft long.

a. Graph the function. Find the height of one triangle when $\theta = 60°$.

b. Find the area of one triangle in square feet when $\theta = 60°$.

c. Find the area of the gazebo floor in square feet when the triangles forming the hexagon are equilateral.

1. Make a sketch of one triangle in the hexagonal floor.



2. Graph the function on your calculator. **Check student's graph.**

3. How can the graph of the function help you find the height of each triangle?

Answers may vary. Sample: You can use the TABLE feature to find the height of the triangle for different values of θ.

4. What is the height of one triangle when $\theta = 60°$? **about 6.9282 ft**

5. What is the formula for the area of a triangle? $A = \frac{1}{2}bh$

6. What is the area of one triangle in square feet when $\theta = 60°$? **about 27.7128 ft^2**

7. What is θ when the triangle is equilateral? Explain.

60°; all of the angles in an equilateral triangle are 60°.

8. How many triangles make up the hexagonal floor? **6 triangles**

9. What is the area of the gazebo floor in square feet when the triangles forming the hexagon are equilateral? **about 166.3 ft^2**

Name ______________________ Class ____________ Date ____________

Practice

Form G

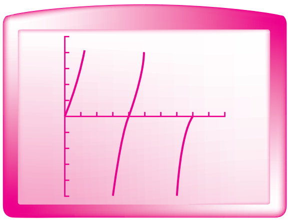
The Tangent Function

Find each value without using a calculator.

1. $\tan \frac{\pi}{4}$ **1**   **2.** $\tan 3\pi$ **0**   **3.** $\tan\left(-\frac{\pi}{4}\right)$ **−1**   **4.** $\tan\left(-\frac{3\pi}{2}\right)$ **undefined**

Graphing Calculator **Graph each function on the interval $0 \le x \le 2\pi$ and $-200 \le y \le 200$. Evaluate each function at $x = \frac{\pi}{4}, \frac{\pi}{2}$, and $\frac{3\pi}{4}$. Round to the nearest tenth, if necessary.**

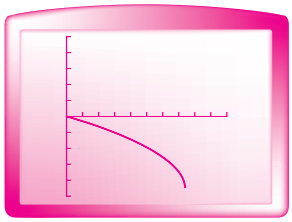
5. $y = 200 \tan x$



Xmin = 0 Ymin = −200
Xmax = 6.28 Ymax = 200
Xscl = 0.785 Yscl = 50

200, undefined, −200

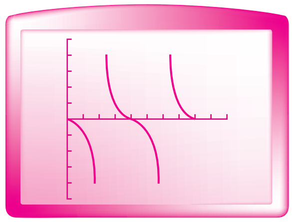
6. $y = -75 \tan\left(\frac{1}{4}x\right)$



Xmin = 0 Ymin = −200
Xmax = 6.28 Ymax = 200
Xscl = 0.785 Yscl = 50

−14.9, −31.1, −50.1

7. $y = -50 \tan x$

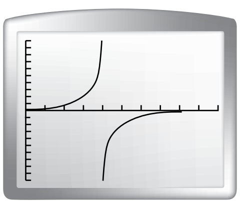


Xmin = 0 Ymin = −200
Xmax = 6.28 Ymax = 200
Xscl = 0.785 Yscl = 50

−50, undefined, 50

Each graphing calculator screen shows the interval 0 to 2π. What is the period of each graph?

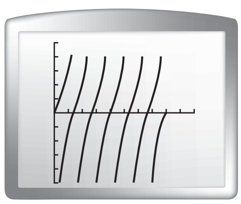
8.



x Scale: $\frac{\pi}{4}$
y Scale: 1

2π

9.



x Scale: $\frac{\pi}{3}$
y Scale: 1

$\frac{\pi}{3}$

Identify the period and determine where two asymptotes occur for each function. **Answers may vary. Samples given.**

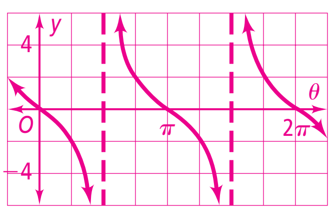
10. $y = 2 \tan \frac{\theta}{2}$ **2π; $\theta = \pi$, $\theta = 3\pi$**

11. $y = -\tan \frac{\pi}{2}\theta$ **2; $\theta = 1$, $\theta = 3$**

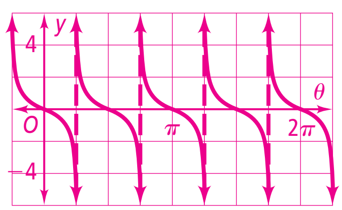
12. $y = 4 \tan 2\theta$ **$\frac{\pi}{2}$; $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$**

Sketch the graph of each tangent curve in the interval from 0 to 2π.

13. $y = -2 \tan \theta$



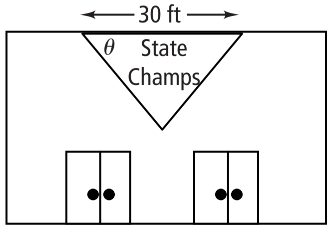
14. $y = -0.5 \tan 2\theta$



Practice (continued) *Form G*

The Tangent Function

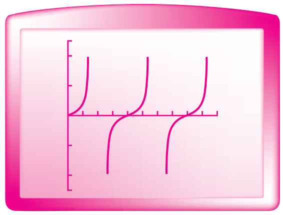
15. Graphing Calculator A banner hangs from the ceiling of a school gym as shown at the right. The function $y = 15 \tan \theta$ models the perpendicular distance from the ceiling to the tip of the banner. The base of the banner is 30 ft wide.



a. Graph the function on a graphing calculator.

b. How far down from the ceiling does the banner hang when $\theta = 30°$? **≈8.7 ft**

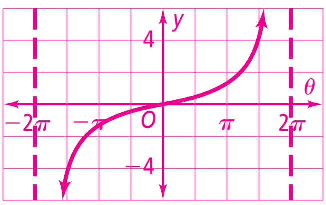
c. How far down from the ceiling does the banner hang when $\theta = 35°$? **≈10.5 ft**



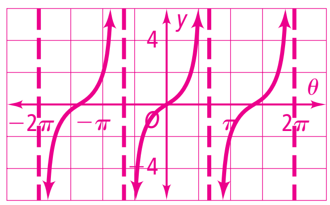
Xmin = 0
Xmax = 450
Xscl = 50
Ymin = −200
Ymax = 200
Yscl = 90

Identify the period for each tangent function. Then graph each function in the interval from -2π to 2π.

16. $y = \tan \frac{1}{4}\theta$ **4π**



17. $y = \tan(0.75\theta)$ **$\frac{4\pi}{3}$**



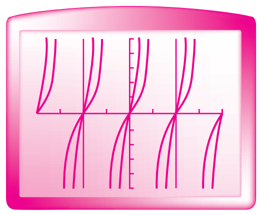
Graphing Calculator Solve each equation in the interval from 0 to 2π. Round your answers to the nearest hundredth.

18. $\tan \theta = \frac{1}{2}$ **0.46, 3.61**

19. $\tan \theta = -1$ **2.36, 5.50**

20. $3 \tan \theta = 1$ **0.32, 3.46**

21. a. Graphing Calculator Graph the functions $y = \tan x$, $y = 5 \tan x$, and $y = 25 \tan x$ on the same set of axes on the interval $-2\pi \le x \le 2\pi$ and $-4 \le y \le 4$.



X Scale: $\frac{\pi}{2}$ *Y Scale:* 1

b. Writing Describe the relationship between the values of y for each function for a given x-value.

c. Reasoning Without using a calculator, predict the value of $y = 125 \tan x$ for $x = 4$. **about 144.73**

b. For a given x-value, when the coefficient of the tangent function is multiplied by n, the value of y is multiplied by n.

22. a. Open-Ended Write a tangent function that has an asymptote through $\theta = \pi$.

b. Graph the function on the interval -2π to 2π. **Check students' work.**

Use the function $y = 150 \tan x$ on the interval $0° \le x \le 141°$. Complete each ordered pair. Round your answers to the nearest whole number.

23. $(45°, \square)$ **150**

24. $(\square°, -150)$ **135**

25. $(141°, \square)$ **−121**

26. $(\square°, 8594)$ **89**

Name ______________________ Class ____________ Date ____________

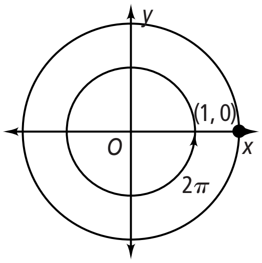
Practice

Form K

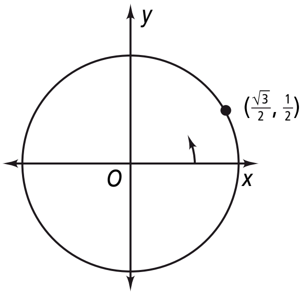
The Tangent Function

Use the unit circle to find the value of each expression.

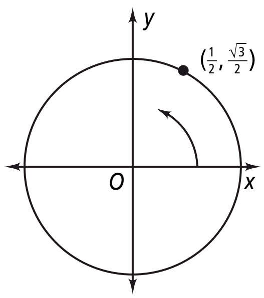
1. $\tan 2\pi$ **0**



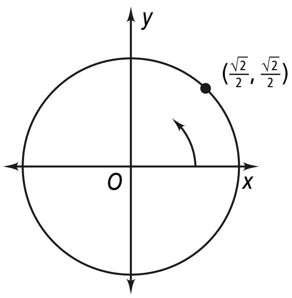
2. $\tan \frac{\pi}{6}$ $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$



3. $\tan \frac{\pi}{3}$ $\sqrt{3}$



4. $\tan \frac{\pi}{4}$ **1**



Identify the period and determine where two asymptotes occur for each function.

5. $y = \tan 3\theta$

period: $\frac{\pi}{3}$;
sample asymptotes:
$\theta = -\frac{\pi}{6}$ and $\theta = \frac{\pi}{6}$

6. $y = \tan \frac{2}{3}\theta$

period: $\frac{3\pi}{2}$;
sample asymptotes:
$\theta = -\frac{3\pi}{4}$ and $\theta = \frac{3\pi}{4}$

7. $y = \tan 3\pi\theta$

period: $\frac{1}{3}$;
sample asymptotes:
$\theta = -\frac{1}{6}$ and $\theta = \frac{1}{6}$

8. $y = \tan \frac{1}{2}\theta$

period: 2π;
sample asymptotes:
$\theta = -\pi$ and $\theta = \pi$

9. $y = \tan 6\theta$

period: $\frac{\pi}{6}$;
sample asymptotes:
$\theta = -\frac{\pi}{12}$ and $\theta = \frac{\pi}{12}$

10. $y = \tan \frac{\pi}{2}\theta$

period: 2;
sample asymptotes:
$\theta = -1$ and $\theta = 1$

Name ______________________________ Class ____________ Date ____________

Practice (continued) *Form K*

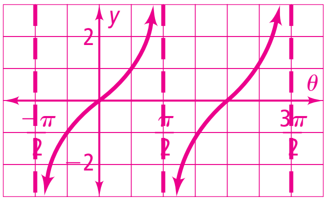
The Tangent Function

Graph two cycles of each of the following tangent functions.

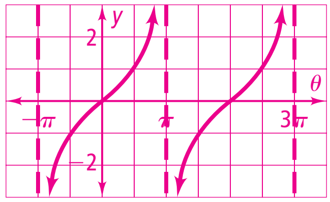
11. $y = \tan\theta$

period: π

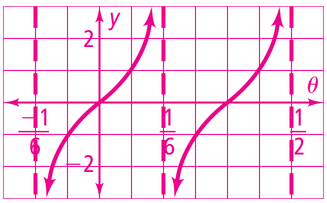
asymptotes: $\theta = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$, and $\theta = \frac{3\pi}{2}$



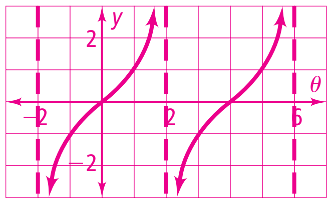
12. $y = \tan\frac{1}{2}\theta$



13. $y = \tan 3\pi\theta$



14. $y = \tan\frac{\pi}{4}\theta$



Use a graphing calculator to solve the following problems.

15. A carpenter is building a frame shaped like an isosceles triangle. The function $y = 6\tan\theta$ models the height of the frame, where θ is the measure of one of the base angles. What is the height of the frame when $\theta = 40°$? Express your answer in feet. **about 5.03 ft**

16. Jonah plans to build a pool shaped like an isosceles triangle. The base of the triangle will be 50 ft. The function $y = 25\tan\theta$ models the height of the triangle, where θ is the measure of one of the base angles. What is the height of the triangle when $\theta = 30°$? **about 14.43 ft**

Name ______________________ Class ____________ Date ____________

Standardized Test Prep

The Tangent Function

Multiple Choice

For Exercises 1–5, choose the correct letter.

1. Which value is *not* defined? B

(A) $\tan \pi$ (B) $\tan\left(-\frac{\pi}{2}\right)$ (C) $\tan\left(\frac{2\pi}{3}\right)$ (D) $\tan(-2\pi)$

2. What is the exact value of $\tan\left(\frac{3\pi}{4}\right)$? H

(F) 1 (G) $\frac{\sqrt{2}}{2}$ (H) -1 (I) $-\frac{\sqrt{2}}{2}$

3. Which pair of values are equal? D

(A) $\tan\frac{\pi}{4}, \tan\left(-\frac{\pi}{4}\right)$

(B) $\tan\frac{\pi}{4}, -\tan\frac{\pi}{4}$

(C) $\tan\frac{\pi}{4}, \tan\frac{3\pi}{4}$

(D) $\tan\frac{\pi}{4}, -\tan\left(-\frac{\pi}{4}\right)$

4. Which equation represents a vertical asymptote of the graph of $y = \tan 2\theta$? F

(F) $\theta = \frac{\pi}{4}$ (G) $\theta = \pi$ (H) $\theta = -\frac{\pi}{2}$ (I) $\theta = -2\pi$

5. Which function has a period of $\frac{2\pi}{3}$? B

(A) $y = \frac{3}{2}\tan\theta$ (B) $y = \tan\frac{3}{2}\theta$ (C) $y = \tan\frac{2\pi}{3}\theta$ (D) $y = \frac{2}{3}\tan\pi\theta$

Short Response

6. Error Analysis A student says that the graph of the function $y = 2\tan\theta$ has an amplitude of 2. Is the student correct? Explain.

[2] No; the graph of a tangent function does not have a maximum or minimum value.
[1] correct answer with incomplete or incorrect explanation
[0] incorrect answer and no explanation OR no answer given

Name ______________________ Class ____________ Date ____________

Enrichment

The Tangent Function

Even and Odd Functions

Functions that meet certain criteria are classified as *even* functions or *odd* functions.

An *even* function is a function for which $f(-x) = f(x)$ for all x in the domain of f.

1. If a function is even, then every time the point (a, b) is on the graph, so is the point $(-a, b)$.

An *odd* function is a function for which $f(-x) = -f(x)$ for all x in the domain of f.

2. If a function is odd, then every time the point (a, b) is on the graph, so is the point $(-a, -b)$.

You can explore even and odd functions in relation to the trigonometric functions using the unit circle.

3. $\sin 30° = \frac{1}{2}$ $\sin(-30°) = -\frac{1}{2}$

$\sin 45° = \frac{\sqrt{2}}{2}$ $\sin(-45°) = -\frac{\sqrt{2}}{2}$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\sin(-60°) = -\frac{\sqrt{3}}{2}$

Is the sine function even or odd? How do you know? **odd; $f(-x) = -f(x)$**

4. $\cos 30° = \frac{\sqrt{3}}{2}$ $\cos(-30°) = \frac{\sqrt{3}}{2}$

$\cos 45° = \frac{\sqrt{2}}{2}$ $\cos(-45°) = \frac{\sqrt{2}}{2}$

$\cos 60° = \frac{1}{2}$ $\cos(-60°) = \frac{1}{2}$

Is the cosine function even or odd? How do you know? **even; $f(-x) = f(x)$**

The results of Exercises 1 and 2 show that even and odd functions can be described in terms of symmetry. Even functions are symmetric with respect to the y-axis and odd functions are symmetric with respect to the origin.

5. Examine the graph of $y = \sin x$. What special kind of symmetry does the graph appear to have? From your answer, is the function even or odd?
symmetrical about the origin; odd

6. Examine the graph of $y = \cos x$. What special kind of symmetry does the graph appear to have? From your answer, is the function even or odd?
symmetrical about the y-axis; even

7. Examine the graph of $y = \tan x$. What special kind of symmetry does the graph appear to have? From your answer, is the function even or odd?
symmetrical about the origin; odd

Name ____________________ Class ____________ Date ____________

Reteaching

The Tangent Function

Like the sine and cosine functions, the standard form of a tangent function is $y = a\ \tan\ b\ \theta$, where $a \neq 0$, $b > 0$, and θ is measured in radians. However, the graph of a tangent function is different in several important ways.

- The amplitude is undefined.
- The period is $\frac{\pi}{b}$.
- One cycle occurs between vertical asymptotes at $0 = -\frac{\pi}{2b}$ and $0 = \frac{\pi}{2b}$.
- This cycle also passes through $\left(-\frac{\pi}{4b}, -a\right)$ and $\left(\frac{\pi}{4b}, a\right)$.
- Vertical asymptotes occur at the end of each cycle.

Problem

What are the period and asymptotes of the graph of $y = -2\tan(3\theta)$ in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ Find two points on the graph that are not on the x-axis.

$a = -2, b = 3$	Compare $y = -2\tan(3\theta)$ to $y = a\tan b\theta$.
$\frac{\pi}{b} = \frac{\pi}{3}$	Calculate the period.
$\theta = -\frac{\pi}{2b} = -\frac{\pi}{2(3)} = -\frac{\pi}{6}$	Find one asymptote.
$\theta = \frac{\pi}{2b} = \frac{\pi}{2(3)} = \frac{\pi}{6}$	Find another asymptote.
$\left(-\frac{\pi}{4b}, -a\right) = \left(-\frac{\pi}{4(3)}, -(-2)\right) = \left(-\frac{\pi}{12}, 2\right)$	Find one point on the graph.
$\left(\frac{\pi}{4b}, a\right) = \left(\frac{\pi}{4(3)}, -2\right) = \left(\frac{\pi}{12}, -2\right)$	Find another point on the graph.

Exercises

Find the period and two asymptotes of the graph of each tangent function. Then find two points on each graph that are not on the x-axis.

1. $y = 4\tan\theta$

   Sample: π; $\theta = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$; $\left(-\frac{\pi}{4}, -4\right)$, $\left(\frac{\pi}{4}, 4\right)$

2. $y = -\tan 2\theta$

   Sample: $\frac{\pi}{2}$, $\theta = -\frac{\pi}{2}$; $\theta = \frac{\pi}{4}$; $\left(-\frac{\pi}{8}, 1\right)$, $\left(\frac{\pi}{8}, -1\right)$

3. $y = \tan\frac{1}{2}\theta$

   Sample: 2π; $\theta = -\pi$, $\theta = \pi$; $\left(-\frac{\pi}{2}, -1\right)$, $\left(\frac{\pi}{2}, 1\right)$

Name ______________________________ Class ____________ Date ____________

Reteaching (continued)

The Tangent Function

The tangent function is a discontinuous periodic function. Each cycle of the graph occurs between vertical asymptotes. To graph a tangent function:

- Find the period, asymptotes, and two points on one cycle of the graph.
- Graph this cycle.
- Find additional asymptotes by adding positive and negative multiples of the period to the first two asymptotes.
- Find additional points on the graph by adding positive and negative multiples of the period to the *x*-coordinates of the first two points you found.

Problem

Sketch four cycles of the graph of $y = -2\tan(3\theta)$.

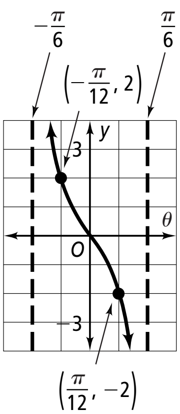
Step 1

Find the period, asymptotes, and two points on one cycle.

The period is $\frac{\pi}{3}$, asymptotes are at $\theta = -\frac{\pi}{6}$ and $\theta = \frac{\pi}{6}$, and two points on the graph are $\left(-\frac{\pi}{12}, 2\right)$ and $\left(\frac{\pi}{12}, -2\right)$.

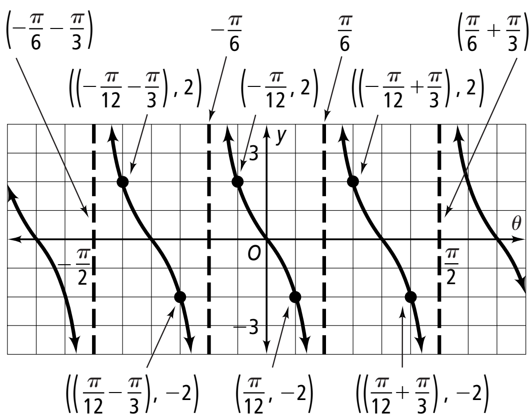
Step 2

Graph the first cycle.



Step 3

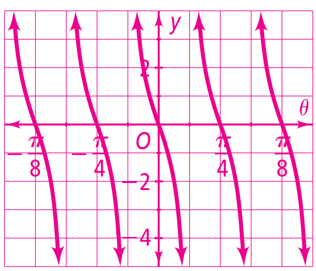
Graph additional cycles.



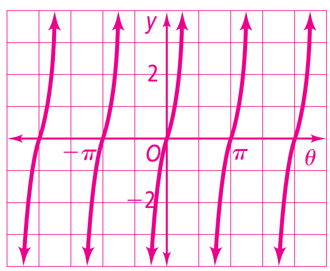
Exercises

Graph at least three cycles of each tangent function.

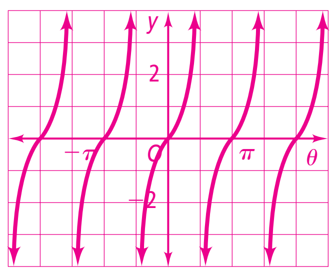
4. $y = -2\tan 4\theta$



5. $y = 3\tan\theta$



6. $y = \tan\theta$



Name ______________________ Class __________ Date __________

Additional Vocabulary Support

Translating Sine and Cosine Functions

For Exercises 1–5, draw a line from each item in Column A to the matching item in Column B.

Column A	Column B
1. h	**A.** vertical shift
2. $\lvert a \rvert$	**B.** period
3. $y = \sin x$	**C.** phase shift or horizontal shift
4. k	**D.** parent function
5. $\frac{2\pi}{b}$	**E.** amplitude

Name each of the following functions as a *phase shift* or a *vertical shift*.

6. $y = \sin x + 5$

vertical shift

7. $y = \cos(x - 3)$

phase shift

8. $y = 2\sin(x + 7)$

phase shift

9. $y = 0.5\cos x - 4$

vertical shift

10. $y = \sin\left(x + \frac{\pi}{2}\right)$

phase shift

11. $y = \cos x - \frac{2\pi}{3}$

vertical shift

Describe any phase shift or vertical shift in the graphs of the following functions.

12. $y = 3\sin x + 5$

Shift the graph of the parent function up 5 units.

13. $y = 0.5\cos\left(x - \frac{\pi}{4}\right)$

Shift the graph of the parent function $\frac{\pi}{4}$ units to the right.

14. $y = \sin(x - \pi) - 7$

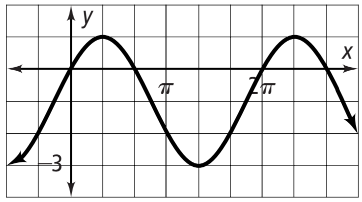
Shift the graph of the parent function π units to the right and 7 units down.

Name ______________________ Class ____________ Date ____________

Think About a Plan

Translating Sine and Cosine Functions

Write a cosine function for the graph. Then write a sine function for the graph.



Know

1. The equations for transformed cosine and sine functions are $\boxed{y = a \cos b(x - h) + k}$ and $\boxed{y = a \sin b(x - h) + k}$.

Need

2. To solve the problem I need to find

the same *a*, *b*, and *k* for both functions; a different *h* for the sine and cosine functions.

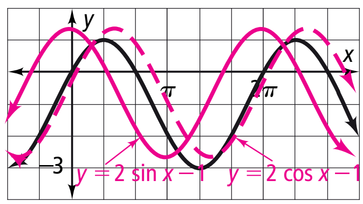
Plan

3. The amplitude of the graph is $\boxed{2 \text{ units}}$. a is $\boxed{2 \text{ units}}$.

4. The period of the graph is $\boxed{2\pi \text{ units}}$. b is $\boxed{1 \text{ unit}}$.

5. The vertical shift is $\boxed{1 \text{ unit down}}$. k is $\boxed{-1 \text{ unit}}$.

6. Sketch the cosine and sine functions on the grid, substituting the values for *a*, *b*, and *k*.



7. What horizontal shifts will transform the cosine function and the sine function into the function in the graph? $\frac{\pi}{3}$; $-\frac{\pi}{6}$

8. A cosine function for the graph is $\boxed{y = 2\cos\left(x - \frac{\pi}{3}\right) - 1}$.

9. A sine function for the graph is $\boxed{y = 2\sin\left(x + \frac{\pi}{6}\right) - 1}$.

Name ______________________ Class ____________ Date ____________

Practice

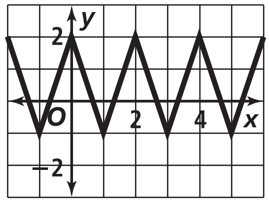
Form G

Translating Sine and Cosine Functions

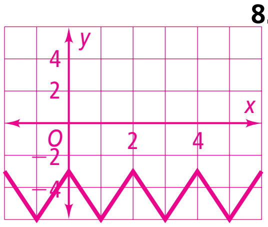
Determine the value of h in each translation. Describe each phase shift (use a phrase like *3 units to the left*).

1. $g(x) = f(x + 2)$
   −2; 2 units to the left
2. $g(x) = f(x - 1)$
   1; 1 unit to the right
3. $h(t) = f(t + 1.5)$
   −1.5; 1.5 units to the left
4. $f(x) = g(x - 1)$
   1; 1 unit to the right
5. $y = \cos\left(x - \frac{\pi}{2}\right)$
   $\frac{\pi}{2}$; $\frac{\pi}{2}$ units to the right
6. $y = \cos(x + \pi)$
   $-\pi$; π units to the left

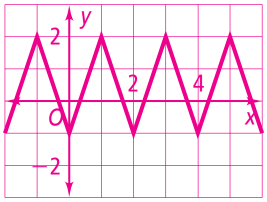
Use the function $f(x)$ at the right. Graph each translation.



7. $f(x) - 5$

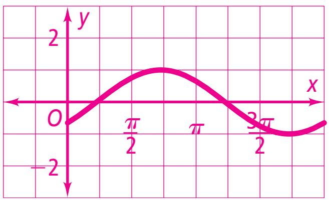


8. $f(x + 3)$

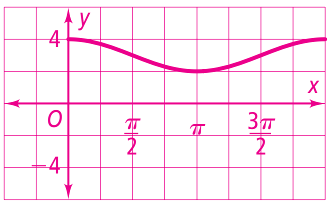


Graph each translation of $y = \cos x$ in the interval from 0 to 2π.

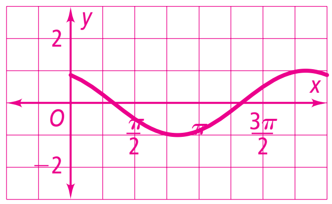
9. $y = \cos(x + 4)$



10. $y = \cos x + 3$



11. $y = \cos\left(x + \frac{\pi}{6}\right)$

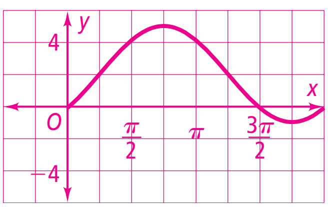


Describe any phase shift and vertical shift in the graph.

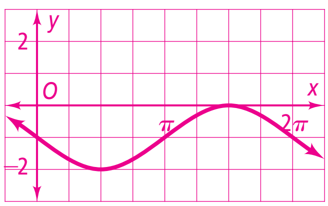
12. $y = 3\cos x + 2$
    none; 2 units up
13. $y = 2\cos(x - 1) + 3$
    1 unit right; 3 units up
14. $y = \sin\left(x + \frac{3\pi}{2}\right) - 1$
    $\frac{3\pi}{2}$ units left; 1 unit down

Graph each function in the interval from 0 to 2π.

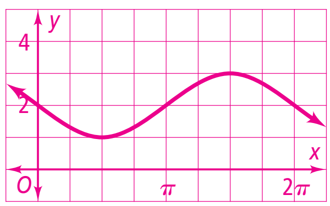
15. $y = 3\sin\left(x - \frac{\pi}{4}\right) + 2$



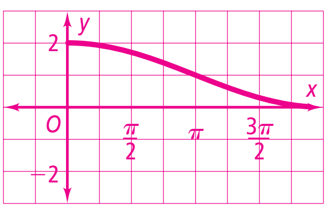
16. $y = \cos\left(x + \frac{\pi}{2}\right) - 1$



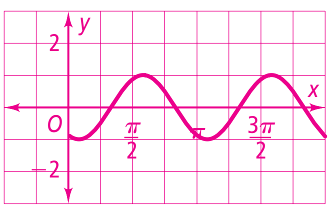
17. $y = \sin(x - \pi) + 2$



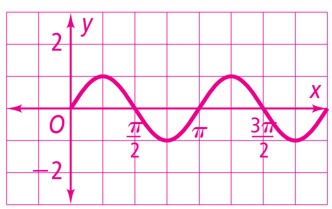
18. $y = \cos\frac{1}{2}x + 1$



19. $y = \sin 2\left(x - \frac{\pi}{3}\right)$



20. $y = -\cos 2\left(x + \frac{\pi}{4}\right)$



Name ______________________ Class ____________ Date ____________

Practice (continued) *Form G*

Translating Sine and Cosine Functions

Write an equation for each translation.

21. $y = \sin x$, 2 units down

$y = \sin x - 2$

22. $y = \cos x$, π units to the left

$y = \cos(x + \pi)$

23. $y = \cos x$, $\frac{\pi}{4}$ units up

$y = \cos x + \frac{\pi}{4}$

24. $y = \sin x$, 3.2 units to the right

$y = \sin(x - 3.2)$

25. $y = \sin x$; 3 units to the left, 1 unit down

$y = \sin(x + 3) - 1$

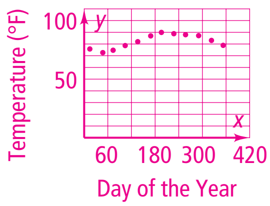
26. $y = \cos x$; $\frac{\pi}{2}$ units to the right, 2 units up

$y = \cos\left(x - \frac{\pi}{2}\right) + 2$

27. The table below shows the temperatures at a weather station on several days of the year.

Day of the Year	15	48	73	104	136	169	196	228	257	290	323	352
Temp. (°F)	76	73	75	79	82	87	90	89	88	87	83	79

a. Plot the data

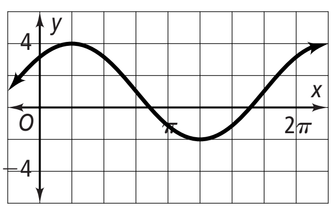


b. Write a cosine model for the data

$y \approx 8.5 \cos \frac{2\pi}{365}(x - 196) + 81.5$

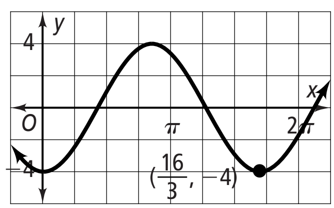
Write a cosine function for each graph. Then write a sine function for each graph.

28.



$y = 3\cos\left(x - \frac{\pi}{4}\right) + 1$; $y = 3\sin\left(x + \frac{\pi}{4}\right) + 1$

29.



$y = -4\cos\frac{3\pi}{8}x$; $y = -4\sin\left(\frac{3\pi}{8}x + \frac{\pi}{2}\right)$

30. a. Write a sine function to model the weather station data in Exercise 27.

b. Writing How do the cosine and sine models differ?

c. Estimation Use your sine model to estimate the temperature at the weather station on December 31 (day 365). about 73°F

a. $y \approx 8.5 \sin\left(\frac{2\pi}{365}(x - 196) + \frac{\pi}{2}\right) + 81.5$

b. The sine model has a different phase shift compared to its parent function than the cosine model does.

Name ______________________ Class ____________ Date ____________

Practice

Form K

Translating Sine and Cosine Functions

Determine the value of *h* in each translation. Then describe each phase shift.

1. $g(x) = f(x - 5)$

h = 5; the phase shift is 5 units to the right.

2. $y = \sin(x - 3)$

h = 3; the phase shift is 3 units to the right.

3. $g(x) = f(x + 7)$

h = −7; the phase shift is 7 units to the left.

4. $y = \cos(x + 0.5)$

h = −0.5; the phase shift is 0.5 units to the left.

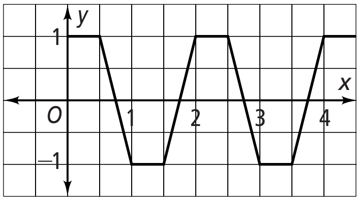
5. $g(x) = f(x - 2.3)$

h = 2.3; the phase shift is 2.3 units to the right.

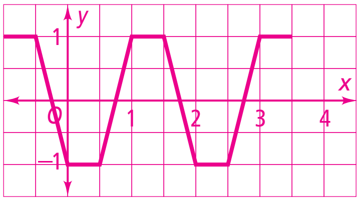
6. $y = \sin\left(x + \frac{3\pi}{2}\right)$

$h = -\frac{3\pi}{2}$; the phase shift is $\frac{3\pi}{2}$ units to the left.

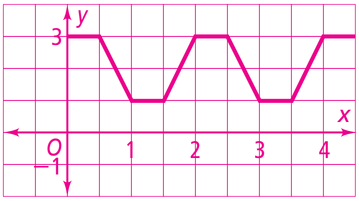
Use the function *f*(*x*) shown below. Graph each translation.



7. $g(x) = f(x + 2)$



8. $g(x) = f(x) + 2$



9. Writing Write a cosine function that has amplitude 2, period 2π, phase shift 3, and vertical shift −7. **$y = 2\cos(x - 3) - 7$**

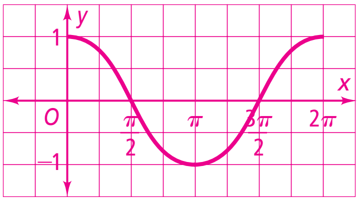
Name ________________________ Class ____________ Date ____________

Practice (continued) *Form K*

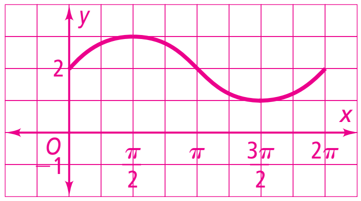
Translating Sine and Cosine Functions

Graph each translation of $y = \sin x$ in the interval from 0 to 2π.

10. $y = \sin\left(x + \frac{\pi}{2}\right)$

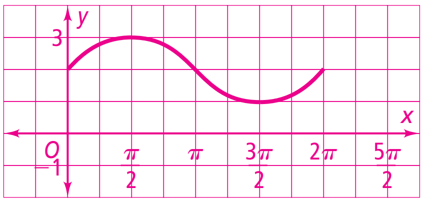


11. $y = \sin x + 2$

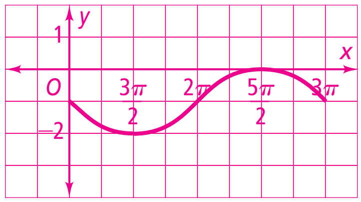


Graph each function in the interval from 0 to 2π.

12. $y = \cos\left(x - \frac{\pi}{2}\right) + 2$



13. $y = \sin(x - \pi) - 1$



Write an equation for each of the following translations.

14. $y = \cos x$, 6 units down
$y = \cos x - 6$

15. $y = \sin x$, 2π units right
$y = \sin(x - 2\pi)$

16. $y = \cos x$, $\frac{\pi}{2}$ units left
$y = \cos\left(x + \frac{\pi}{2}\right)$

17. $y = \sin x$, 3 units up and 5 units right
$y = \sin(x - 5) + 3$

18. $y = \cos x$, 2 units down and $\frac{\pi}{6}$ units left
$y = \cos\left(x + \frac{\pi}{6}\right) - 2$

19. Error Analysis Your classmate said that the function $y = \cos(x + 5)$ is a translation 5 units to the right. What error did he make? What translation does this function represent? **Answers may vary. Sample: He interpreted $(x + 5)$ as a phase shift to the right, when it is actually a phase shift to the left. This function represents a phase shift 5 units to the left.**

Name ________________________ Class ____________ Date ____________

Standardized Test Prep

Translating Sine and Cosine Functions

Multiple Choice

For Exercises 1–4, choose the correct letter.

1. Which function is a phase shift of $y = \cos\theta$ by 3 units to the right? C

(A) $y = 3\cos\theta$ (B) $y = \cos\theta - 3$ (C) $y = \cos(\theta - 3)$ (D) $y = \cos 3\theta$

2. Which function is a translation of $y = \sin\theta$ by 3 units up? H

(F) $y = 3\sin\theta$ (G) $y = \sin(\theta + 3)$ (H) $y = \sin\theta + 3$ (I) $y = \sin 3\theta$

3. Which function is a translation of $y = \cos\theta$ by $\frac{\pi}{4}$ units down and π units to the left? C

(A) $y = -\frac{\pi}{4}\cos\pi\theta$

(B) $y = \cos\left(\theta - \frac{\pi}{4}\right) + \pi$

(C) $y = \cos(\theta + \pi) - \frac{\pi}{4}$

(D) $y = \pi\cos\left(-\frac{\pi}{4}\theta\right)$

4. Which best describes the function $y = \sin 2\left(x - \frac{2\pi}{3}\right)$? G

(F) a translation of $y = \sin 2x$ $\frac{2\pi}{3}$ units to the left

(G) a translation of $y = \sin 2x$ $\frac{2\pi}{3}$ units to the right

(H) a translation of $y = \sin x$ $\frac{2\pi}{3}$ units to the left and 2 units up

(I) a translation of $y = \sin x$ $\frac{2\pi}{3}$ units to the right and 2 units up

Short Response

5. Write a function that is a transformation of $y = \cos\theta$ so that its amplitude is 9 and its minimum value is 3. Show your work.

[2] Sample: $y = 9\cos\theta + 12$; For $y = 9\cos\theta$, amplitude is 9 and min value is -9. For min value to be 3, the graph must be translated 12 units up.

[1] incorrect or incomplete work shown

[0] incorrect answer and no work shown OR no answer given

Name ______________________ Class __________ Date __________

Enrichment

Translating Sine and Cosine Functions

Ferris Wheel

The first Ferris wheel was designed and built by an American engineer named George W. G. Ferris in 1893. It had 36 cars that each held 40 passengers. The diameter of the wheel was 250 feet. The top of the wheel was 264 feet above the ground and it took 20 minutes to complete one revolution.

You can use trigonometric functions to model the position of a rider on a Ferris wheel.

Begin by making a table of values. Let y represent the height of the rider above the ground x minutes into the ride.

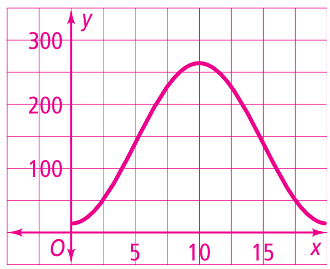
x	y
0	
5	
10	
15	
20	

1. When $x = 0$ or 20, the rider is at the start of the ride. Find the appropriate value of y and place it in the table. **14**

2. When $x = 10$, the rider is at the top of the wheel. Find the appropriate value of y and place it in the table. **264**

3. When $x = 5$ or 15, the rider is even with the center of the wheel. Find the appropriate value of y and place it in the table. **139**

Plot the ordered pairs and connect them with a smooth curve. The shape of the curve matches that of a cosine curve. Therefore, the equation of the curve is of the form $y = a \cos b(x - h) + k$, where $|a| =$ amplitide, $\frac{2\pi}{b} =$ period (when x is in radians, and $b > 0$), $h =$ phase shift, and $k =$ vertical shift.



4. Find the amplitude. **125**

5. Find the period. What value for b produces this period? **20; $\frac{\pi}{10}$**

6. Using the results of Exercises 4 and 5 and choosing $a > 0$, write an equation of the form $y = a \cos bx$. **$y = 125 \cos \frac{\pi}{10}x$**

7. Find the values of h and k by comparing the graph of the equation found in Exercise 6 with the plot of the data. **10; 139**

8. Write an equation that models the height of the rider at any time x during the ride. **$y = 125 \cos \frac{\pi}{10}(x - 10) + 139$**

Name ______________________ Class ____________ Date ____________

Reteaching

Translating Sine and Cosine Functions

You can translate the graphs of sine and cosine functions both horizontally and vertically. A horizontal translation is called a *phase shift*. For a function in the form $y = a \sin b(x - h) + k$ or $y = a \cos b(x - h) + k$:

- $|a| =$ amplitude
- $\frac{2\pi}{b} =$ period
- $h =$ phase shift — If $h > 0$, the graph moves to the right. If $h < 0$, the graph moves to the left.
- $k =$ vertical shift — If $k > 0$, the graph moves up. If $k < 0$, the graph moves down.

Problem

What are the amplitude, period, and any phase shift or vertical shift in the graph of the function $y = 2 \sin \frac{1}{3}(x + 5)$?

$y = 2 \sin \frac{1}{3}(x - (-5)) + 0$	Write function as $y = a \sin b(x - h) + k$.
$a = 2,\ b = \frac{1}{3},\ h = -5,\ k = 0$	Identify *a*, *b*, *h*, and *k*.
$\lvert a \rvert = \lvert 2 \rvert = 2$	amplitude = 2
$\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{3}} = 6\pi$	period = 6π
$h = -5$	The phase shift is 5 units to the left.
$k = 0$	There is no vertical shift.

Exercises

Determine the amplitude, period, and any phase shift or vertical shift in the graphs of the functions.

1. $y = 6 \cos 3x + 2$

6; $\frac{2\pi}{3}$; 2 units up

2. $y = -\sin \frac{1}{2}(x - \pi)$

1; 4π; π units right

3. $y = 2 \sin 8\left(x - \frac{\pi}{3}\right) - 5$

2; $\frac{\pi}{4}$; $\frac{\pi}{3}$ right; 5 units down

4. $y = \cos 2(x - 1) + 3.4$

1; π; 1 unit right; 3.4 units up

5. $y = \frac{2}{3} \sin (x + 3\pi) - \pi$

$\frac{2}{3}$; 2π; 3π units left; π units down

6. $y = -3 \cos \left(x + \frac{\pi}{4}\right) + 12$

3; 2π; $\frac{\pi}{4}$ units left; 12 units up

Name ______________________ Class ____________ Date ____________

Reteaching (continued)

Translating Sine and Cosine Functions

The graph of a function in the form $y = a \sin b(x - h) + k$ is a translation of the graph of $y = a \sin bx$. The graph of a function in the form $y = a \cos b(x - h) + k$ is a translation of the graph of $y = a \cos bx$.

Problem

What is the graph of $y = 2 \sin 3\left(x - \frac{\pi}{3}\right) + 1$ in the interval from 0 to 2π?

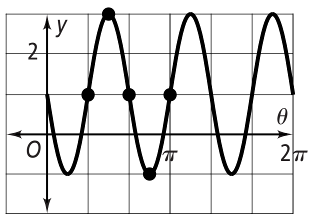
Step 1 Compare the function to $y = a \sin b(x - h) + k$. $a = 2$ and $b = 3$
Find the amplitude, period, h, and k. $|a| = |2| = 2; \frac{2\pi}{3}; h = \frac{\pi}{3}; k = 1$

Step 2 Find the minimum and maximum of the curve before the vertical shift. Because the amplitude is 2, the maximum is 2 and the minimum is -2.

Step 3 Make a table of values. Choose x-values at intervals of one-fourth the period: $\frac{\frac{2\pi}{3}}{4} = \frac{\pi}{6}$. The y-values before the vertical shift cycle through the pattern zero-max-zero-min-zero. Add h to the x-values and add k to the y-values to find the translated points.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$x + \frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	0	2	0	-2	0
$y + 1$	1	3	1	-1	1

Step 4 Plot the translated points from the table.

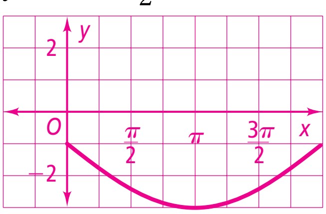


Step 5 Draw a smooth curve through the points. Extend the pattern from 0 to 2π.

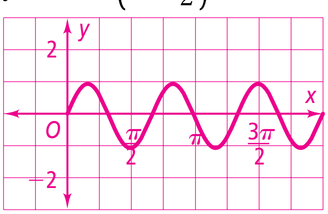
Exercises

Sketch each graph in the interval from 0 to 2π.

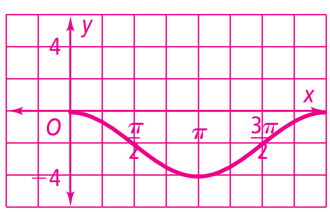
7. $y = -2 \sin \frac{1}{2}x - 1$



8. $y = \cos 3\left(x + \frac{\pi}{2}\right)$



9. $y = -2 \cos(x + \pi) - 2$



Name ____________________ Class __________ Date __________

Additional Vocabulary Support

Reciprocal Trigonometric Functions

There are two sets of cards below that show how to find the exact value of $\cot \frac{2\pi}{3}$. The set on the left explains the thinking. The set on the right shows the steps. Write the thinking and the steps in the correct order.

Think Cards

Find the exact value of $\tan \frac{2\pi}{3}$.

Write the reciprocal of $\cot \frac{2\pi}{3}$.

Use the definition of cotangent.

Substitute and simplify.

Find the point where the unit circle intersects the terminal side of the angle $\frac{2\pi}{3}$ radians.

Write Cards

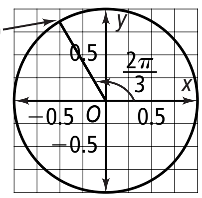
$\cot \frac{2\pi}{3} = \frac{1}{\tan\left(\frac{2\pi}{3}\right)}$

$\tan \frac{2\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$

$\tan \frac{2\pi}{3}$

$\cot \frac{2\pi}{3} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



Think

First, write the reciprocal of $\cot \frac{2\pi}{3}$ ______.

Second, find the point where the unit circle intersects the terminal side of the angle $\frac{2\pi}{3}$ radians.

Next, find the exact value of $\tan \frac{2\pi}{3}$ ______.

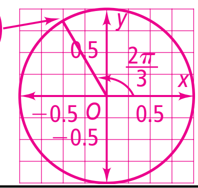
Then, use the definition of cotangent ______.

Finally, substitute and simplify ______.

Write

Step 1 $\tan \frac{2\pi}{3}$

Step 2 $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



Step 3 $\tan \frac{2\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$

Step 4 $\cot \frac{2\pi}{3} = \frac{1}{\tan\left(\frac{2\pi}{3}\right)}$

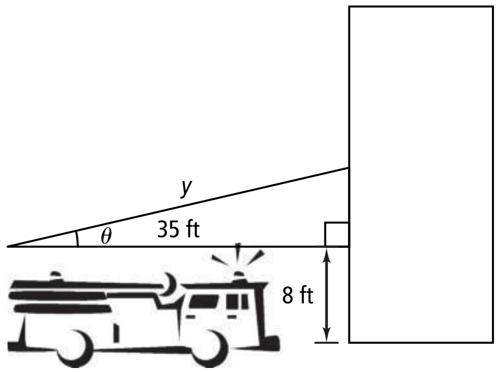
Step 5 $\cot \frac{2\pi}{3} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$

Name ______________________ Class ____________ Date ____________

Think About a Plan

Reciprocal Trigonometric Functions

Indirect Measurement The function $y = 35 \sec \theta$ models the length y in feet of a fire ladder as a function of the measure of the angle θ formed by the ladder and the horizontal when the hinge of the ladder is 35 ft from the building.

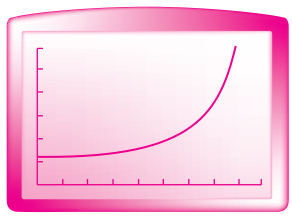


- **a.** Graph the function.
- **b.** In the drawing, $\theta = 13°$. How far is the ladder extended?
- **c.** How far is the ladder extended when it forms an angle of 30°?
- **d.** Suppose the ladder is extended to its full length of 80 ft. What angle does it form with the horizontal? How far up a building can the ladder reach when fully extended? (*Hint:* Use the information in the drawing.)

1. What is a reasonable domain and range for the function?

Answers may vary. Sample: domain: 0° to 90°, range: 0 to 300 ft

2. Graph the function on your graphing calculator. Sketch the graph.



3. How can the graph help you find the length of the ladder?

Answers may vary. Sample: You can use the TABLE feature to find the length for different values of θ.

4. How far is the ladder extended when it forms an angle of 13°? When it forms an angle of 30°? **about 35.9 ft; about 40.4 ft**

5. Write an equation you can solve to find the angle the ladder forms with the horizontal when it is fully extended to 80 ft. **$35 \sec \theta = 80$**

6. How can you use your graphing calculator to solve your equation? What is the solution?

Answers may vary. Sample: Graph the line $y = 80$ with the graph of $y = 35 \sec \theta$ and find their intersection; about 64.1°

7. How can you find the length of the vertical leg of the right triangle in the drawing?

Use the Pythagorean Theorem with $y = 80$

8. How far up a building can the ladder reach when fully extended? **about 79.9 ft**

Name ______________________ Class ______________ Date ______________

Practice

Form G

Reciprocal Trigonometric Functions

Find each value without using a calculator. If the expression is undefined, write *undefined*.

1. $\csc(-\pi)$
undefined

2. $\cot\frac{2\pi}{3}$
$-\frac{\sqrt{3}}{3}$

3. $\sec\left(-\frac{11\pi}{6}\right)$
$\frac{2\sqrt{3}}{3}$

4. $\csc\frac{3\pi}{4}$
$\sqrt{2}$

5. $\cot\left(-\frac{\pi}{2}\right)$
0

6. $\csc 3\pi$
undefined

7. $\sec\frac{\pi}{3}$
2

8. $\cot\left(-\frac{\pi}{6}\right)$
$-\sqrt{3}$

Graphing Calculator **Use a calculator to find each value. Round your answers to the nearest thousandth.**

9. $\cot 42^\circ$
1.111

10. $\csc\frac{\pi}{6}$
2

11. $\csc(-2)$
-1.100

12. $\sec\pi$
-1

13. $\cot(-4)$
-0.864

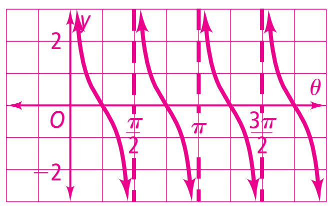
14. $\sec(-35^\circ)$
1.221

15. $\cot\frac{\pi}{3}$
0.577

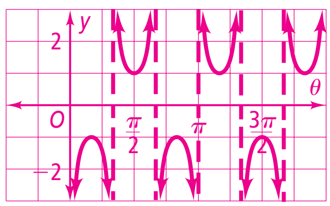
16. $\sec 1.5$
14.137

Graph each function in the interval from 0 to 2π.

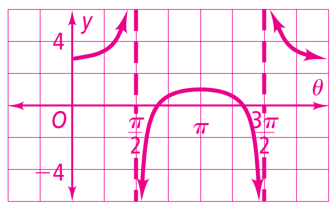
17. $y = \cot 2\theta$



18. $y = -\csc 3\theta$



19. $y = \sec\theta + 2$



Graphing Calculator **Use the graph of the appropriate reciprocal trigonometric function to find each value. Round to four decimal places.**

20. $\cot 30^\circ$
1.7321

21. $\csc 180^\circ$
undefined

22. $\cot 70^\circ$
0.3640

23. $\sec 100^\circ$
-5.7588

24. $\sec 50^\circ$
1.5557

25. $\csc 100^\circ$
1.0154

26. $\cot 0^\circ$
undefined

27. $\sec 125^\circ$
-1.7434

28. A sparrow perches on the ledge of a building. It is 122 ft above the ground. It looks down at a squirrel along a line of sight that makes an angle of θ with the building. The distance in feet of an object on the ground from the sparrow is modeled by the function $d = 122\sec\theta$. How far away are squirrels sighted at angles of 35° and 50°? about 149 ft, about 190 ft

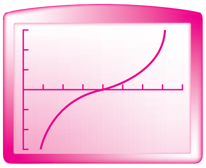
Name ______________________________ Class ____________ Date ____________

Practice (continued) *Form G*

Reciprocal Trigonometric Functions

Graphing Calculator **Graph each function in the interval from 0 to 2π. Describe any phase shift and vertical shift in the graph.**

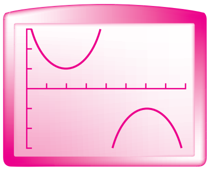
29. $y = -\cot \frac{1}{2}\theta$



x Scale: $\frac{\pi}{4}$
y Scale: 1

no phase or vertical shift

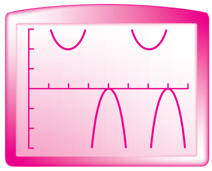
30. $y = \sec\left(\theta - \frac{\pi}{2}\right)$



x Scale: $\frac{\pi}{4}$
y Scale: 1

phase shift: $\frac{\pi}{2}$

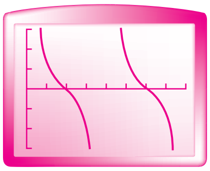
31. $y = \csc 2\theta + 1$



x Scale: $\frac{\pi}{4}$
y Scale: 1

vertical shift: 1

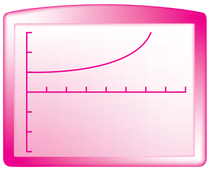
32. $y = \cot(\theta + \pi)$



x Scale: $\frac{\pi}{4}$
y Scale: 1

phase shift: $-\pi$

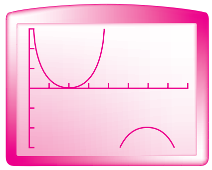
33. $y = \sec \frac{1}{4}\theta$



x Scale: $\frac{\pi}{4}$
y Scale: 1

no phase or vertical shift

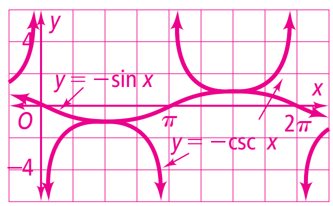
34. $y = \csc\theta - 1$



x Scale: $\frac{\pi}{4}$
y Scale: 1

vertical shift: -1

35. a. Graph $y = -\sin x$ and $y = -\csc x$ on the same axes in the interval from 0 to 2π.



b. State the domain, range, and period of each function.

c. For which values of x does $-\sin x = -\csc x$?

d. Compare and Contrast Compare the two graphs in part (a). How are they alike? How are they different?

e. Reasoning Is the value of $-\csc x$ positive when $-\sin x$ is positive and negative when $-\sin x$ is negative? Justify your answer.

b. $y = -\sin x$: all real numbers; $-1 \le y \le 1$; 2π; $y = -\csc x$: all real numbers except multiples of π; $y \le -1$ and $y \ge 1$; 2π

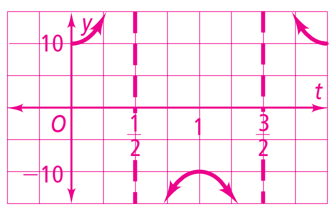
c. $x = \frac{\pi}{2}, \frac{3\pi}{2}$

d. Same period; $y = -\sin x$ is continuous, but $y = -\csc x$ has asymptotes.

e. Yes; because $-\csc x = -\frac{1}{\sin x}$, any x-value that gives a positive value for $-\sin x$ will also give a positive value for $-\csc x$.

36. A fire truck is parked on the shoulder of a freeway next to a long wall. The red light on the top of the truck rotates through one complete revolution every 2 s. The function $y = 10 \sec \pi t$ models the length of the beam in feet to a point on the wall in terms of time t.

a. Graph the function.



b. Find the length at time 1.75 s. **about 14.14 ft**

c. Find the length at time 2 s. **10 ft**

Name ______________________________ Class ____________ Date ____________

Practice

Form K

Reciprocal Trigonometric Functions

Match each of the following items with its reciprocal.

1. sine **A.** secant

2. cosine **B.** cotangent

3. tangent **C.** cosecant

Find each value without using a calculator.

4. sec 2π

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\sec 2\pi = \frac{1}{\cos 2\pi} = \frac{\boxed{1}}{\boxed{1}} = \boxed{1}$$

5. $\csc\frac{5\pi}{6}$ **2**

6. $\cot\frac{\pi}{3}$ $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

7. $\csc\frac{7\pi}{4}$ $-\sqrt{2}$

8. $\sec\frac{4\pi}{3}$ **−2**

9. $\cot\frac{\pi}{4}$ **1**

Use a calculator to find the decimal values of the following expressions. Round your answers to the nearest thousandth. Remember to use the reciprocals of sine, cosine, and tangent.

10. sec 20 **2.450**

11. csc 3.4 **−3.913**

12. cot (−4) **−0.864**

13. csc 62° **1.133**

14. sec 286° **3.628**

15. cot 165° **−3.732**

Name ______________________________ Class ____________ Date ____________

Practice (continued)

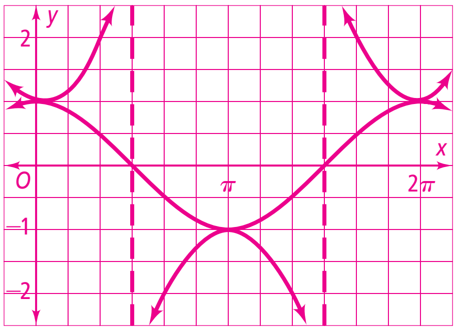
Form K

Reciprocal Trigonometric Functions

Use the table of values to solve the following problem.

16. What are the graphs of $y = \cos x$ and $y = \sec x$ in the interval from 0 to 2π?

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
cos x	1	0.9	0.5	0	−0.5	−0.9	−1	−0.9	−0.5	0	−0.5	−0.9	1
sec x	1	1.2	2	undef	−2	−1.2	−1	−1.2	−2	undef	2	1.2	1



Use the graph of secant, cosecant, or cotangent to find each value. Round your answers to the nearest thousandth.

17. sec 40°
1.305

18. cot 50°
0.839

19. csc 75°
1.035

20. cot 120°
−0.577

21. csc 160°
2.924

22. cot 80°
0.176

Use a graphing calculator to solve the following problem.

23. A spotlight sits on top of a building and shines on a bush. The beam of light shines in a path that makes an angle of 50° with the building. The distance in feet from the spotlight to the bush is modeled by the function $d = 125 \sec \theta$. What is the distance from the bush to the spotlight rounded to the nearest foot?
194 ft

Name ______________________ Class ____________ Date ____________

Standardized Test Prep

Reciprocal Trigonometric Functions

Gridded Response

Solve each exercise and enter your answer in the grid provided.

For Exercises 1–2, let $\cos\theta = \frac{5}{13}$ and $\sin\theta > 0$. Enter each answer as a fraction.

1. What is $\cot\theta$?

2. What is $\csc\theta$?

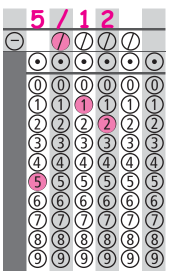
For Exercises 3–5, let $\tan\theta = \frac{12}{5}$ and $-\frac{\pi}{2} \le \theta < \frac{\pi}{2}$. Enter each answer as a decimal rounded to the nearest hundredth.

3. What is $\cot\theta + \cos\theta$?

4. What is $(\sin\theta)(\cot\theta)$?

5. What is $\sec\theta + \tan\theta$?

Answers

1.


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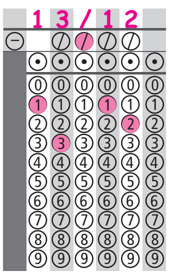

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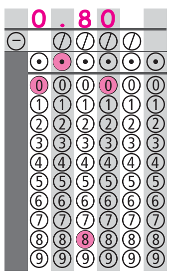

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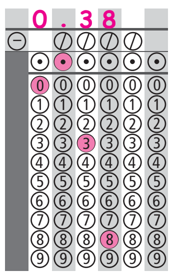

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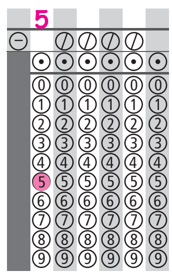


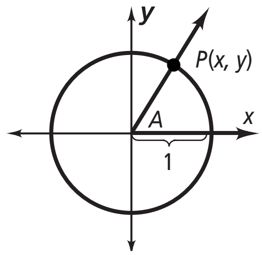
Name ______________________________ Class ____________ Date ____________

Enrichment

Reciprocal Trigonometric Functions

The Trigonometric Form of the Pythagorean Theorem

The unit circle is the circle whose center is the origin and whose radius is 1. Suppose the initial side of a central angle A is the positive x-axis. Let the terminal side of $\angle A$ intersect the unit circle at the point $P(x, y)$.



1. What is the distance from point P to the origin? **1**

2. What are the values of sin A and cos A in terms of x and y?
$\sin A = y$; $\cos A = x$

3. What are the coordinates of point P in terms of $\angle A$?
$(\cos A, \sin A)$

4. Use the coordinates of P from Exercise 3 to write an equation expressing the distance from P to the origin. $\sqrt{(\cos A - 0)^2 + (\sin A - 0)^2} = 1$

5. Simplify your equation. $(\cos A)^2 + (\sin A)^2 = 1$

6. The result, usually written as $\sin^2 A + \cos^2 A = 1$, is known as the trigonometric form of the Pythagorean theorem. Solve the equation for cos A in terms of sin A. $\cos A = \pm\sqrt{1 - \sin^2 A}$

7. Express tan A in terms of sin A and cos A. $\tan A = \frac{\sin A}{\cos A}$

8. How could you express tan A in terms of sin A only? $\tan A = \pm\frac{\sin A}{\sqrt{1 - \sin^2 A}}$

9. How could you express cot A in terms of sin A only? $\cot A = \pm\frac{\sqrt{1 - \sin^2 A}}{\sin A}$

As you can see, the trigonometric form of the Pythagorean theorem allows you to express any one function in terms of the others. Use the trigonometric form of the Pythagorean theorem to express each of the following functions as indicated.

10. sin A in terms of cos A $\sin A = \pm\sqrt{1 - \cos^2 A}$

11. cos A in terms of csc A $\cos A = \pm\sqrt{1 - \frac{1}{\csc^2 A}}$

12. tan A in terms of cos A $\tan A = \pm\frac{\sqrt{1 - \cos^2 A}}{\cos A}$

Name ______________________ Class ____________ Date ____________

Reteaching

Reciprocal Trigonometric Functions

You have already worked with sine, cosine, and tangent functions. The reciprocals of these functions are also trigonometric functions:

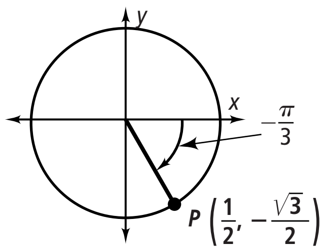
Cosecant	Secant	Cotangent
$\csc\theta = \frac{1}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{1}{\tan\theta}$
$\sin\theta = \frac{1}{\csc\theta}$	$\cos\theta = \frac{1}{\sec\theta}$	$\tan\theta = \frac{1}{\cot\theta}$

Problem

What is the exact value of $\sec\left(-\frac{\pi}{3}\right)$? Do not use a calculator.

Step 1 Find the reciprocal of $\sec\left(-\frac{\pi}{3}\right)$. $\quad \frac{1}{\sec\left(-\frac{\pi}{3}\right)} = \cos\left(-\frac{\pi}{3}\right)$

Step 2 Draw the unit circle.
Draw the terminal side of the angle $-\frac{\pi}{3}$.



Step 3 Label the coordinates of the point where the unit circle intersects the terminal side of the angle $-\frac{\pi}{3}$.

Step 4 Find the exact value of $\cos\left(-\frac{\pi}{3}\right)$.

$\cos\left(-\frac{\pi}{3}\right) = x\text{-coordinate of point } P = \frac{1}{2}$.

Step 5 Use the definition of secant. $\quad \sec\left(-\frac{\pi}{3}\right) = \frac{1}{\cos\left(-\frac{\pi}{3}\right)} = \frac{1}{\frac{1}{2}} = 2$

Exercises

Find the exact value of each expression. Do not use a calculator.

1. $\cot\frac{\pi}{6}$ $\sqrt{3}$

2. $\sec\left(-\frac{3\pi}{4}\right)$ $-\sqrt{2}$

3. $\csc\left(-\frac{\pi}{2}\right)$ -1

4. $\sec\frac{5\pi}{3}$ 2

5. $\csc\frac{\pi}{4}$ $\sqrt{2}$

6. $\cot\frac{2\pi}{3}$ $-\frac{\sqrt{3}}{3}$

7. $\sec(3p)$ -1

8. $\csc\left(-\frac{\pi}{6}\right)$ -2

Name ______________________________ Class ____________ Date ____________

Reteaching (continued)

Reciprocal Trigonometric Functions

The graphs of cosecant, secant, and cotangent functions are related to the graphs of sine, cosine, and tangent functions.

- The graph of a cosecant function has a vertical asymptote where the value of the related sine function is zero.
- The graph of a secant function has a vertical asymptote where the value of the related cosine function is zero.
- The graph of a cotangent function is a reflection across a vertical line of the related tangent function.

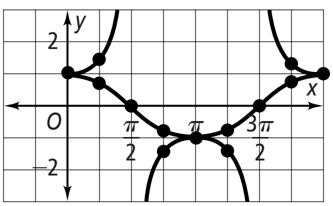
Problem

What are the graphs of $y = \cos x$ and $y = \sec x$ in the interval from 0 to 2π?

Step 1 Make a table of values. Use the fact that $\sec\theta = \frac{1}{\cos\theta}$. The graph of $y = \sec x$ has asymptotes where $\cos x$ is equal to zero.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos\theta$	1	0.707	0	−0.707	−1	−0.707	0	0.707	1
$\frac{1}{\cos\theta}$	$\frac{1}{1}$	$\frac{1}{0.707}$	$\frac{1}{0}$	$\frac{1}{-0.707}$	$\frac{1}{-1}$	$\frac{1}{-0.707}$	$\frac{1}{0}$	$\frac{1}{0.707}$	$\frac{1}{1}$
$\sec\theta$	1	1.414	—	−1.414	−1	−1.414	—	1.414	1

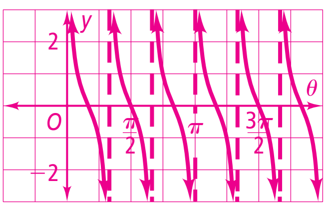
Step 2 Plot the points from the table. Connect the points with smooth curves.



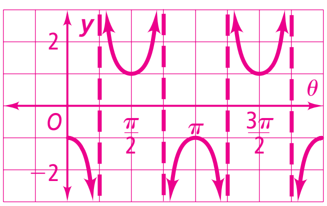
Exercises

Sketch each graph in the interval from 0 to 2π.

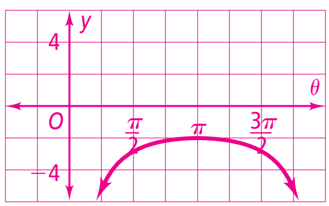
9. $y = \cot 3\theta$



10. $y = -\sec 2\theta$



11. $y = -2\csc\frac{1}{2}\theta$



Name ______________________ Class ____________ Date ____________

Additional Vocabulary Support

Trigonometric Identities

Problem

What is the simplified trigonometric expression for $\cos^2\theta + \tan^2\theta\cos^2\theta$?

Explain	Work	Justify
First, write the equation.	$\cos^2\theta + \tan^2\theta\cos^2\theta$	Original equation
Second, rewrite using a common factor.	$\cos^2\theta\,(1 + \tan^2\theta)$	Factor.
Next, replace $1 + \tan^2\theta$ with $\sec^2\theta$.	$\cos^2\theta\,(\sec^2\theta)$	Pythagorean Identity
Then, replace $\sec^2\theta$ with $\frac{1}{\cos^2\theta}$.	$\cos^2\theta\left(\frac{1}{\cos^2\theta}\right)$	Reciprocal Identity
Then, multiply.	$\frac{\cos^2\theta}{\cos^2\theta}$	Simplify.
Finally, simplify.	1	Simplify.

Solution
1

Exercise

What is the simplified trigonometric expression for $\cot\theta\,(\cot\theta + \tan\theta)$?

Explain	Work	Justify
First, write the expression.	$\cot\theta\,(\cot\theta + \tan\theta)$	Original expression
Second, multiply.	$\cot^2\theta + \cot\theta\tan\theta$	Distributive Property
Next, replace $\cot\theta$ with $\frac{1}{\tan\theta}$.	$\cot^2\theta + \left(\frac{1}{\tan\theta}\right)\tan\theta$	Reciprocal Identity
Then, multiply.	$\cot^2\theta + 1$	Simplify.
Finally, replace $\cot^2\theta + 1$ with $\csc^2\theta$.	$\csc^2\theta$	Pythagorean Identity

Solution
$\csc^2\theta$

Name ______________________ Class ____________ Date ____________

Think About a Plan

Trigonometric Identities

Simplify the trigonometric expression. $\dfrac{\csc\theta}{\sin\theta + \cos\theta\cot\theta}$

Know

1. $\csc\theta = \boxed{\dfrac{1}{\sin\theta}}$ $\cot\theta = \boxed{\dfrac{\cos\theta}{\sin\theta}}$

2. The Pythagorean identity involving the sine and cosine functions is $\boxed{\sin^2\theta + \cos^2\theta = 1}$.

Need

3. To solve the problem I need to:

Answers may vary. Sample: transform the expression into a simpler expression using basic and Pythagorean identities.

Plan

4. Write each function in the expression in terms of sines and cosines.

$$\frac{\csc\theta}{\sin\theta + \cos\theta\cot\theta} = \frac{\boxed{\dfrac{1}{\sin\theta}}}{\sin\theta + \cos\theta\left(\boxed{\dfrac{\cos\theta}{\sin\theta}}\right)}$$

5. How can you eliminate the fractions in the numerator and denominator?

Answers may vary. Sample: Multiply the numerator and denominator by $\sin\theta$.

6. Simplify the trigonometric expression.

$$\frac{\csc\theta}{\sin\theta + \cos\theta\cot\theta} = \frac{\dfrac{1}{\sin\theta}}{\sin\theta + \cos\theta\left(\dfrac{\cos\theta}{\sin\theta}\right)} \cdot \frac{\sin\theta}{\sin\theta} = \frac{1}{\sin^2\theta + \cos^2\theta} = \frac{1}{1} = 1$$

Name ______________________ Class ____________ Date ____________

Practice

Form G

Trigonometric Identities

Verify each identity. Give the domain of validity for each identity. 1.–18. Verifications may vary.

1. $\sin\theta\sec\theta\cot\theta = 1$ real numbers except multiples of $\frac{\pi}{2}$

2. $\csc\theta = \cot\theta\sec\theta$ real numbers except multiples of $\frac{\pi}{2}$

3. $\frac{\sin\theta}{\csc\theta} = \sin^2\theta$ real numbers except multiples of π

4. $\cos\theta\csc\theta\tan\theta = 1$ real numbers except multiples of $\frac{\pi}{2}$

5. $\sin\theta\tan\theta + \cos\theta = \sec\theta$ real numbers except odd multiples of $\frac{\pi}{2}$

6. $\frac{\csc\theta}{\cot\theta} = \sec\theta$ real numbers except multiples of $\frac{\pi}{2}$

7. $\sec\theta = \tan\theta\csc\theta$ real numbers except multiples of $\frac{\pi}{2}$

8. $\tan\theta + \cot\theta = \sec\theta\csc\theta$ real numbers except multiples of $\frac{\pi}{2}$

9. $\tan^2\theta + 1 = \sec^2\theta$ real numbers except odd multiples of $\frac{\pi}{2}$

10. $\cos\theta\cot\theta + \sin\theta = \csc\theta$ real numbers except multiples of π

11. $\frac{\sec\theta}{\csc\theta} = \tan\theta$ real numbers except multiples of $\frac{\pi}{2}$

12. $\sec\theta\cot\theta = \csc\theta$ real numbers except multiples of $\frac{\pi}{2}$

13. $\sec^2\theta - \tan^2\theta = 1$ real numbers except odd multiples of $\frac{\pi}{2}$

14. $\sec\theta = \csc\theta\tan\theta$ real numbers except multiples of $\frac{\pi}{2}$

15. $\frac{\sin\theta + \cos\theta}{\sin\theta} = 1 + \cot\theta$ real numbers except multiples of π

16. $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$ real numbers except odd multiples of $\frac{\pi}{2}$

17. $\cot\theta\sec\theta = \csc\theta$ real numbers except multiples of $\frac{\pi}{2}$

18. $(1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta$ real numbers

Simplify each trigonometric expression.

19. $1 - \sec^2\theta$ $-\tan^2\theta$

20. $\frac{\sec\theta}{\tan\theta}$ $\csc\theta$

Practice (continued)

Form G

Trigonometric Identities

Simplify each trigonometric expression.

21. $\csc\theta\tan\theta$ **sec θ**

22. $\sec\theta\cos^2\theta$ **cos θ**

23. $\csc^2\theta - \cot^2\theta$ **1**

24. $1 - \sin^2\theta$ **$\cos^2\theta$**

25. $\tan\theta\cot\theta$ **1**

26. $\cos\theta\cot\theta + \sin\theta$ **csc θ**

27. $\cos\theta\tan\theta$ **sin θ**

28. $\frac{\sin\theta\cot\theta}{\cos\theta}$ **1**

29. $\sec\theta\tan\theta\csc\theta$ **$\sec^2\theta$**

30. $\sec\theta\cot\theta$ **csc θ**

31. $\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta}$ **1**

32. $\frac{\tan\theta\csc\theta}{\sec\theta}$ **1**

33. $\cot^2\theta - \csc^2\theta$ **-1**

34. $\frac{\cot\theta}{\csc\theta}$ **cos θ**

Express the first trigonometric function in terms of the second.

35. $\csc\theta$, $\sin\theta$ **$\frac{1}{\sin\theta}$**

36. $\cot\theta$, $\tan\theta$ **$\frac{1}{\tan\theta}$**

37. $\sec\theta$, $\cos\theta$ **$\frac{1}{\cos\theta}$**

38. $\cos\theta$, $\sin\theta$ **$\pm\sqrt{1 - \sin^2\theta}$**

39. Writing Which side of the equation below should you transform to verify the identity? Explain. $\frac{\cos^2\theta + \tan^2\theta - 1}{\sin^2\theta} = \tan^2\theta$ **Left; it is easier to break down the left-hand side than to build up the right-hand side.**

Name ______________________ Class ____________ Date ____________

Practice

Form K

Trigonometric Identities

Verify each identity. Give the domain of validity for each identity. **1.–18. Verifications may vary.**

1. $\sin\theta \cot\theta = \cos\theta$

Write the equation in terms of sine and cosine: $\sin\theta\left(\boxed{\frac{\cos\theta}{\sin\theta}}\right) = \cos\theta$

Look at each part to determine where the identity is valid.

$\sin\theta$ and $\cos\theta$ are defined $\boxed{\text{for all real numbers}}$.

$\cot\theta$ is defined for all real numbers except $\boxed{\text{multiples of } \pi}$.

The domain of validity is $\boxed{\text{all real numbers except multiples of } \pi}$.

2. $\dfrac{\cos\theta}{\sec\theta} = \cos^2\theta$ **real numbers except odd multiples of $\frac{\pi}{2}$**

3. $\sin\theta \cot\theta \sec\theta = 1$ **real numbers except multiples of $\frac{\pi}{2}$**

4. $\dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \sec\theta$ **real numbers except odd multiples of $\frac{\pi}{2}$**

5. $\dfrac{\cot\theta}{\csc\theta} = \cos\theta$ **real numbers except multiples of π**

6. $\dfrac{\sec\theta}{\tan\theta} = \csc\theta$ **real numbers except multiples of $\frac{\pi}{2}$**

7. $\sin^2\theta \sec\theta + \cos\theta = \sec\theta$ **real numbers except odd multiples of $\frac{\pi}{2}$**

8. $\cot^2\theta - \csc^2\theta = -1$ **real numbers except multiples of π**

9. $\csc\theta - \sin\theta = \cos\theta \cot\theta$ **real numbers except multiples of π**

10. $\dfrac{\csc\theta}{\cot\theta} = \sec\theta$ **real numbers except multiples of $\frac{\pi}{2}$**

11. $\sin\theta(1 + \cot^2\theta) = \csc\theta$ **real numbers except multiples of π**

12. $\dfrac{1 - \sin^2\theta}{\sin^2\theta} = \cot^2\theta$ **real numbers except multiples of π**

13. $\dfrac{\cos\theta \csc\theta}{\tan\theta} = \cot^2\theta$ **real numbers except multiples of $\frac{\pi}{2}$**

Name ______________________ Class ____________ Date ____________

Practice (continued) *Form K*

Trigonometric Identities

Simplify each trigonometric expression.

14. $\dfrac{\sin\theta\csc\theta}{\cot\theta}$

To start, replace one expression with an expression containing sine, cosine, or tangent:

$\csc\theta = \dfrac{1}{\sin\theta}$

$\dfrac{\sin\theta\csc\theta}{\cot\theta} = \dfrac{\sin\theta \cdot \boxed{\dfrac{1}{\sin\theta}}}{\cot\theta} = \dfrac{\boxed{1}}{\cot\theta}$

Then use a reciprocal identity: $\dfrac{\boxed{1}}{\cot\theta} = \boxed{\tan\theta}$

$\dfrac{\sin\theta\csc\theta}{\cot\theta} = \boxed{\tan\theta}$

15. $\cos^2\theta\sec\theta\csc\theta$ **$\cot\theta$**

16. $\cot\theta\tan\theta - \sec^2\theta$ **$-\tan^2\theta$**

17. $\cos\theta\,(1 + \tan^2\theta)$ **$\sec\theta$**

18. $\dfrac{\tan\theta}{\sec\theta}$ **$\sin\theta$**

19. $\dfrac{\cot\theta\sin\theta}{\cos\theta}$ **1**

20. $\cos\theta + \sin\theta\tan\theta$ **$\sec\theta$**

21. $\cos\theta\cot\theta\sin\theta$ **$\cos^2\theta$**

22. $\dfrac{\csc\theta\cot\theta\cos\theta}{\cot^2\theta}$ **1**

23. Writing Explain the relationship between the Pythagorean Theorem and the Pythagorean identities. **Answers may vary. Sample: You use the unit circle and the Pythagorean Theorem to derive the Pythagorean identities.**

24. Error Analysis A student writes on a quiz that the domain of validity for the expression $\dfrac{\sec\theta}{\sin\theta}$ is all real numbers except multiples of π. What is the student's error? What is the correct domain of validity? **Answers may vary. Sample: The student only found the values for which $\sin\theta$ is zero. The student forgot to consider when $\sec\theta$ is undefined, which is at odd multiples of $\frac{\pi}{2}$. The domain of validity is all real numbers except multiples of $\frac{\pi}{2}$.**

Name ______________________________ Class ____________ Date ____________

Standardized Test Prep

Trigonometric Identities

Multiple Choice

For Exercises 1–5, choose the correct letter.

1. The expression $\csc\theta\sin\theta + \cot^2\theta$ is equivalent to which of the following? **C**

(A) $\sin^2\theta$ (B) $\cot^2\theta$ (C) $\csc^2\theta$ (D) $\cos^2\theta$

2. How can you express $(1 + \sin\theta)(\sec\theta - \tan\theta)$ in terms of $\cos\theta$? **I**

(F) $\frac{1}{\cos\theta}$ (G) $\cos^2\theta$ (H) $1 - \cos^2\theta$ (I) $\cos\theta$

3. Which of the following expressions are equivalent? **D**

I. $\frac{\cos^2\theta}{\cot^2\theta}$ **II.** $\sin^2\theta$ **III.** $1 - \cos^2\theta$

(A) I and II only (C) II and III only

(B) I and III only (D) I, II, and III

4. Which equation is not true? **F**

(F) $\tan\theta = \frac{\cos\theta}{\sin\theta}$ (H) $\sin^2\theta = 1 - \cos^2\theta$

(G) $\csc\theta = \frac{1}{\sin\theta}$ (I) $\csc^2\theta = \cot^2\theta + 1$

5. Which of the following is the expression $\sin\theta\cos\theta(\tan\theta + \cot\theta)$ in simplified form? **B**

(A) $\cos\theta$ (B) 1 (C) $\tan\theta$ (D) $\sin^2\theta$

Short Response

6. Show that $\frac{\csc^2\theta - \cot^2\theta}{1 - \sin^2\theta} = \sec^2\theta$ is an identity.

[2] $\frac{\csc^2\theta - \cot^2\theta}{1 - \sin^2\theta} = \frac{\frac{1 - \cos^2\theta}{\sin^2\theta}}{\cos^2\theta} = \frac{\frac{\sin^2\theta}{\sin^2\theta}}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$

[1] minor errors in answer

[0] incorrect answers and no work shown OR no answer given

Name ______________________ Class ____________ Date ____________

Enrichment

Trigonometric Identities

Reduction to a Canonical Form

When you are confronted with the problem of proving a trigonometric identity, you may ask yourself, "What substitutions should I make?" or "What basic identities should I use to prove or disprove this identity?" A good practice is to reduce both sides of the identity to a simpler, canonical form. **Canonical form** is the simplest form to which a function can be reduced.

One such form involves only terms that are expressed as powers of cosine and/or the first power of sine. Fill in the blanks below to see how each of the other four basic trigonometric functions can be expressed in this canonical form.

$\sec A = \frac{1}{\cos A}$ $\csc A = \frac{1}{\sin A}$

$\tan A = \frac{\sin A}{\cos A}$ $\cot A = \frac{\cos A}{\sin A}$

Whenever an expression contains a power of sine, it can be reduced to this canonical form by using the Pythagorean identity $\sin^2 A = 1 - \cos^2 A$.

1. Use this procedure to change $1 - 2\sin^2 A$ to canonical form. $2\cos^2 A - 1$

2. How can you express any even power of sine ($\sin^{2n} A$) in canonical form? $(1 - \cos^2 A)^n$

3. How could you express any odd power of sine ($\sin^{2n+1} A$) in canonical form? $(1 - \cos^2 A)^n \sin A$

Reduce each of the following trigonometric expressions to the canonical form involving powers of cosine and/or the first power of sine.

4. $\tan A + \sec A$ $\frac{\sin A + 1}{\cos A}$

5. $\cot^2 A - \sec^3 A$ $\frac{\cos^5 A + \cos^2 A - 1}{\cos^3 A - \cos^5 A}$

6. $\sec^3 A \tan^2 A$ $\frac{1 - \cos^2 A}{\cos^5 A}$

7. $\sin^2 A + \cos^3 A$ $1 - \cos^2 A + \cos^3 A$

8. $\tan A + \cot A$ $\frac{1}{\cos A \sin A}$

9. $\sin A - \cos A \cot A$ $\frac{1 - 2\cos^2 A}{\sin A}$

10. $\cot A \sin A - \sin^3 A \sec A$ $\frac{\cos^2 A - \sin A(1 - \cos^2 A)}{\cos A}$

Name ______________________ Class ____________ Date ____________

Reteaching

Trigonometric Identities

A *trigonometric identity* is a trigonometric equation that is true for all values of the variable except those that cause the expressions on either side of the equal sign to be undefined. You can use the trigonometric identities below to replace complicated-looking expressions with much simpler ones.

Reciprocal Identities	$\csc\theta = \frac{1}{\sin\theta}$	$\cot\theta = \frac{1}{\tan\theta}$	$\sec\theta = \frac{1}{\cos\theta}$
Pythagorean Identities	$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$

Problem

What is $\cot\theta\,(\tan\theta + \cot\theta)$ expressed in simplified terms?

Use the identities to rewrite $\cot\theta$ and $\tan\theta$.

$\cot\theta\,(\tan\theta + \cot\theta) = \frac{1}{\tan\theta}\left(\tan\theta + \frac{1}{\tan\theta}\right)$ Write $\cot\theta$ in terms of $\tan\theta$.

$= \frac{1}{\tan\theta}(\tan\theta) + \left(\frac{1}{\tan\theta}\right)^2$ Distribute.

$= 1 + \left(\frac{1}{\tan\theta}\right)^2$ Simplify.

$= 1 + \cot^2\theta$ Reciprocal Identity

$= \csc^2\theta$ Pythagorean Identity

Exercises

Simplify each expression.

1. $\cot\theta\sin\theta$ **$\cos\theta$**

2. $\tan\theta\cos\theta$ **$\sin\theta$**

3. $\csc\theta\sin\theta$ **1**

4. $\cos\theta\sin\theta\sec\theta$ **$\sin\theta$**

5. $\sin\theta + \cot\theta\cos\theta$ **$\csc\theta$**

6. $\csc^2\theta - \cot^2\theta$ **1**

Reteaching (continued)

Trigonometric Identities

To verify an identity, you can transform one side of the equation until it is the same as the other side. Begin by writing all of the functions in terms of sine and cosine.

Once you choose a side of the equation to transform, do not work with the other side of the equation. Raising both sides of the equation to a power or dividing both sides of the equation by a trigonometric expression can introduce extraneous solutions.

Problem

Verify the identity $1 + \cot^2\theta = \csc^2\theta$.

$1 + \cot^2\theta = 1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2$	Cotangent Identity
$= 1 + \frac{\cos^2\theta}{\sin^2\theta}$	Simplify.
$= \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}$	Write the fractions with common denominators.
$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}$	Add.
$= \frac{1}{\sin^2\theta}$	Pythagorean Identity
$= \csc^2\theta$	Reciprocal Identity

Exercises

Verify each identity. **7.–18. Answers may vary.**

7. $\cot\theta\tan\theta = 1$

8. $\cos\theta\sec\theta = 1$

9. $\csc\theta\sin\theta + \cot^2\theta = \csc^2\theta$

10. $\sin\theta(1 + \cot^2\theta) = \csc\theta$

11. $\sec\theta\cot\theta = \csc\theta$

12. $\sec^2\theta - \sec^2\theta\cos^2\theta = \tan^2\theta$

13. $\cot\theta\tan\theta + \tan^2\theta = \sec^2\theta$

14. $\csc^2\theta - \cot^2\theta = 1$

15. $\sin\theta + \cos\theta\cot\theta = \csc\theta$

16. $\frac{\sec\theta - \cos\theta}{\sec\theta} = \sin^2\theta$

17. $\cot\theta\sec\theta\sin\theta = 1$

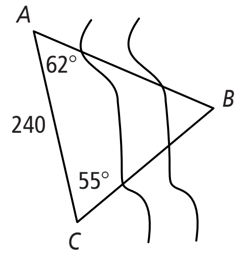
18. $\tan\theta(\sin\theta - \csc\theta) = -\cos\theta$

Name ______________________ Class ____________ Date ____________

Additional Vocabulary Support

Area and the Law of Sines

A surveyor uses the diagram at the right to find the distance *AB*.



You wrote these steps to solve the problem on the note cards, but they got mixed up.

Substitute values into the Law of Sines: m∠*B* = 63°, m∠*C* = 55°, and *b* = 240.

Write the Law of Sines.

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Find the measure of ∠*B*.

AB = 221

Solve for *AB*.

$\frac{\sin 63°}{240} = \frac{\sin 55°}{AB}$

Use the note cards to write the steps in order.

1. First, find the measure of ∠*B*.
2. Second, write the Law of Sines. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
3. Next, substitute values into the Law of Sines: m∠*B* = 63°, m∠*C* = 55°, and *b* = 240.
4. Next, solve for *AB*. $\frac{\sin 63°}{240} = \frac{\sin 55°}{AB}$.
5. Finally, *AB* = 221.

Name ______________________ Class ____________ Date ____________

Think About a Plan

Area and the Law of Sines

Geometry The sides of a triangle are 15 in., 17 in., and 16 in. The smallest angle has a measure of 54°. Find the measure of the largest angle. Round to the nearest degree.

Know

1. The sides of the triangle are **15 in., 17 in., and 16 in.**.

2. The smallest angle has a measure of $\boxed{54^\circ}$.

3. The smallest angle is opposite the $\boxed{\text{smallest}}$ side.

4. The largest angle is opposite the $\boxed{\text{largest}}$ side.

Need

5. To solve the problem I need to find:

Answers may vary. Sample: the measure of the angle that is opposite the side with measure 17 in.

Plan

6. What equation can you use to find the measure of the largest angle?

$$\frac{\sin 54^\circ}{15} = \frac{\sin x}{17}$$

7. Solve your equation for the measure of the largest angle.

$$x = \sin^{-1}\left(\frac{17 \sin 54^\circ}{15}\right)$$

8. What is the measure of the largest angle? **66°**

Name ______________________ Class __________ Date __________

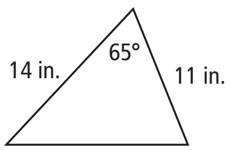
Practice

Form G

Area and the Law of Sines

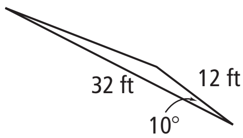
Find the area of each triangle. Round your answers to the nearest tenth.

1.



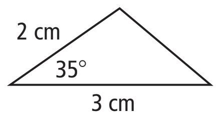
69.8 in.^{2}

2.



33.3 ft^2

3.

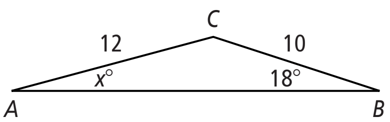


1.7 cm^2

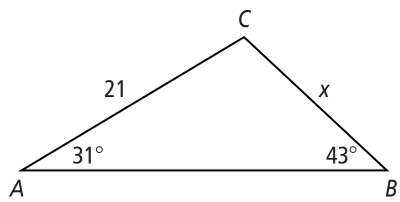
4. A triangle has sides of lengths 15 in. and 22 in., and the measure of the angle between them is 95°. Find the area of the triangle. **164.4 in.^{2}**

Use the Law of Sines. Find the measure *x* to the nearest tenth.

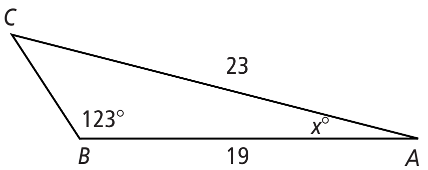
5. **14.9**



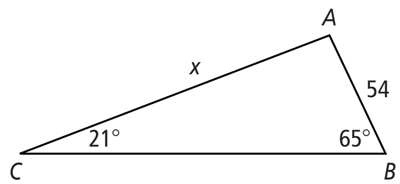
6. **15.9**



7. **13.1**



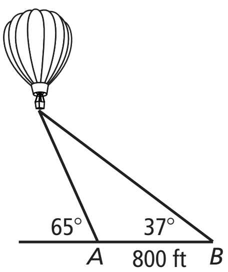
8. **136.6**



9. In $\triangle GHJ$, $m\angle J = 39°$, $h = 36$ cm, and $j = 42$ cm. Find $m\angle H$. **32.6°**

10. In $\triangle MNP$, $m\angle P = 33°$, $m = 54$ ft, and $p = 63$ ft. Find $m\angle M$. **27.8°**

11. A hot-air balloon is observed from two points, *A* and *B*, on the ground 800 ft apart as shown in the diagram. The angle of elevation of the balloon is 65° from point *A* and 37° from point *B*. Find the distance from point *A* to the balloon. **1025.5 ft**



Find the remaining sides and angles of $\triangle PQR$. Round your answers to the nearest tenth.

12. $m\angle Q = 64°$, $m\angle R = 64°$, and $r = 8$

$m\angle P = 52°$, $p = 7.0$, $q = 8.0$

13. $m\angle Q = 64°$, $q = 22$, and $r = 14$

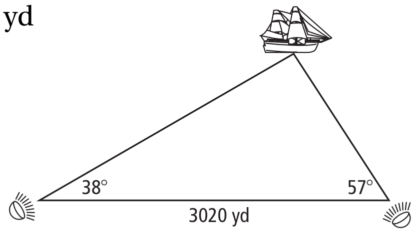
$m\angle P = 81.1°$, $m\angle R = 34.9°$, $p = 24.2$

Name ______________________ Class __________ Date __________

Practice (continued) *Form G*

Area and the Law of Sines

14. Two searchlights on the shore of a lake are located 3020 yd apart as shown in the diagram. A ship in distress is spotted from each searchlight. The beam from the first searchlight makes an angle of 38° with the baseline. The beam from the second light makes an angle of 57° with the baseline. Find the ship's distance from each searchlight. **2542.5 yd and 1866.4 yd**



Find the area of $\triangle ABC$. Round your answer to the nearest tenth.

15. $m\angle B = 28°$, $m\angle C = 70°$, $a = 9.8$
21.4 units^2

16. $m\angle A = 42°$, $a = 4.17$, $c = 5.02$
10.4 units^2

17. $m\angle A = 17°$, $m\angle C = 75°$, $b = 18.1$
46.3 units^2

18. $m\angle C = 81°$, $b = 6.7$, $c = 9.3$
25.1 units^2

In $\triangle ABC$, $m\angle A = 25°$ and $m\angle B = 50°$. Find each value to the nearest tenth.

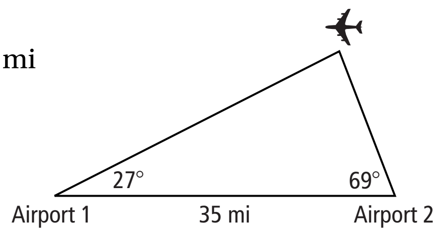
19. Find AC for $BC = 6.2$ in.
11.2 in.

20. Find BC for $AC = 14.9$ cm.
8.2 cm

21. Find AC for $AB = 53.7$ ft.
42.6 ft

22. Find BC for $AB = 27.3$ m.
11.9 m

23. An airplane is flying between two airports that are 35 mi apart. The radar in one airport registers a 27° angle between the horizontal and the airplane. The radar system in the other airport registers a 69° angle between the horizontal and the airplane. How far is the airplane from each airport to the nearest tenth of a mile? **32.9 mi from Airport 1, 16.0 mi from Airport 2**



24. Writing Suppose you know the measures of two sides of a triangle and the measure of the angle between the two sides. Can you use the Law of Sines to find the remaining side and angle measures? Explain. **No; answers may vary. Sample: To use the Law of Sines you must know the measure of one angle and the measure of the side opposite that angle, in addition to one other side or angle measure.**

25. Reasoning How can you find the measures of the angles of $\triangle ABC$ if you know the measures of its sides and its area? **Answers may vary. Sample: Use area $= \frac{1}{2}ab \sin C$ with two of its sides to find $m\angle C$. Then use the Law of Sines to find another angle measure, and subtract the two known angle measures from 180° to find the third angle measure.**

Name ______________________ Class ____________ Date ____________

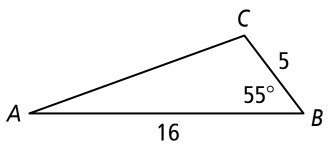
Practice

Form K

Area and the Law of Sines

Find the area of each triangle. Round your answers to the nearest tenth.

1.

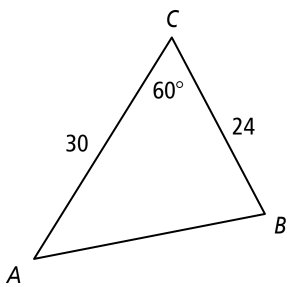


$$\text{Area} = \frac{1}{2}ac \sin B$$

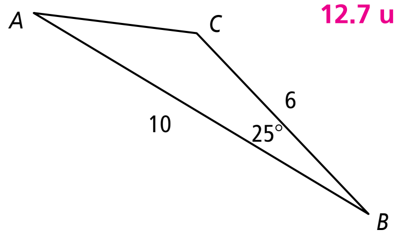
$$= \frac{1}{2}\left(\boxed{5}\right)\left(\boxed{16}\right) \sin B \boxed{55°}$$

$$\approx \boxed{32.8 \text{ units}^2}$$

2. 311.8 units^2

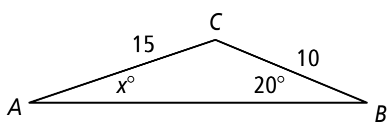


3. 12.7 units^2



Use the Law of Sines. Find the measure *x* to the nearest tenth.

4.



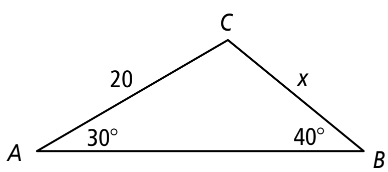
$$\frac{\sin x°}{10} = \frac{\sin 20°}{15}$$

$$15 \sin x° = 10 \sin 20°$$

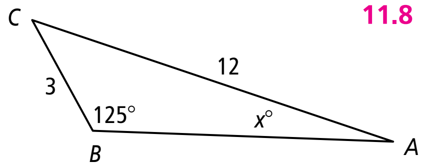
$$\sin x° = \boxed{\frac{10 \sin 20°}{15}}$$

$$x° = \sin^{-1}\left(\boxed{\frac{10 \sin 20°}{15}}\right) \approx \boxed{13.2°}$$

5. 15.6



6. 11.8



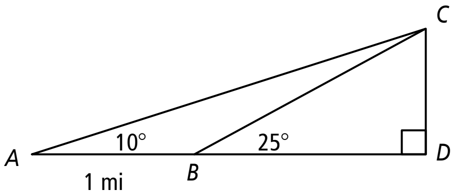
7. In acute $\triangle QRS$, $m\angle Q = 45°$, $q = 30$ cm, and $r = 35$ cm. Find $m\angle R$. 55.6°

Name ______________________________ Class ____________ Date ____________

Practice (continued) *Form K*

Area and the Law of Sines

8. A surveyor measures the angle to the top of a hill from two different points. The angles she measures and the distance between the valley points are shown in the diagram at the right. What is the distance from point B to the top of the hill? Round your answer to the nearest hundredth of a mile. **0.67 mi**



Find the remaining sides and angles of $\triangle ABC$. Round your answers to the nearest tenth.

9. $m\angle B = 50°$, $m\angle C = 50°$, and $c = 12$

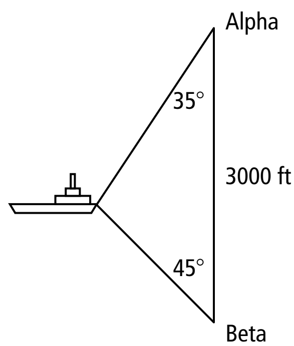
$m\angle A = 80°$, $b = 12$, $a = 15.4$

10. $m\angle B = 50°$, $b = 20$, and $c = 15$

$m\angle A = 94.9°$, $m\angle C = 35.1°$, $a = 26.0$

11. Two Coast Guard ships, the Alpha and the Beta, are 3000 ft apart. The angles from a line between the Coast Guard ships to a disabled ship are shown in the diagram at the right. How far is the disabled ship from each Coast Guard ship? Round your answers to the nearest foot.

Alpha: 2154 ft, Beta: 1747 ft



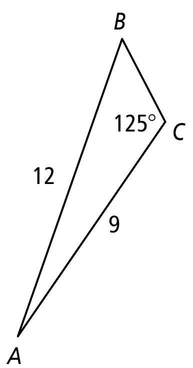
In $\triangle ABC$, $m\angle A = 30°$ and $m\angle B = 80°$. Find each value to the nearest tenth.

12. Find AC for $AB = 50$ ft. **52.4 ft**

13. Find BC for $AB = 30$ m. **16.0 m**

14. Error Analysis A student finds the measure of angle A in the triangle at the right. What mistake does he make? What is the correct measure?

Answers may vary. Sample: The student substituted the wrong side for a and actually found $m\angle B$; 17.1°



$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin A}{9} = \frac{\sin 125}{12}$$

$$A = \sin^{-1}\left(\frac{9 \sin 125}{12}\right) \approx 37.9°$$

Name ______________________ Class ____________ Date ____________

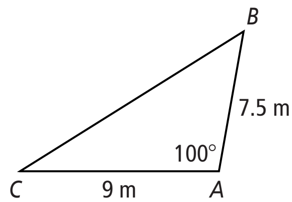
Standardized Test Prep

Area and the Law of Sines

Gridded Response

Solve each exercise and enter your answer in the grid provided.

1. What is the area of the triangle at the right? Round your answer to the nearest tenth of a meter.



2. A triangle has side lengths 202 ft and 201.5 ft, and the measure of the angle between them is 82.5°. What is the area of the triangle? Round your answer to the nearest square foot.

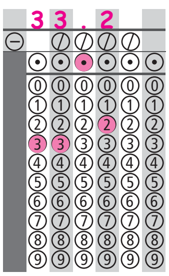
3. In $\triangle ABC$, $m\angle A = 38°$, $m\angle C = 32°$, and $BC = 40$ cm. What is the length of AC? Round your answer to the nearest tenth of a centimeter.

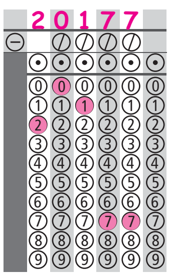
For Exercises 4 and 5, use $\triangle XYZ$, where $x = 10$, $z = 6$, and $m\angle X = 115°$.

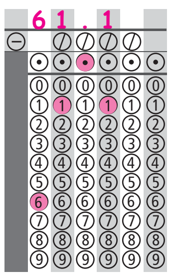
4. What is $m\angle Y$? Round your answer to the nearest tenth.

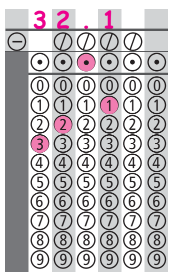
5. What is the value of sin Z? Round your answer to the nearest thousandth.

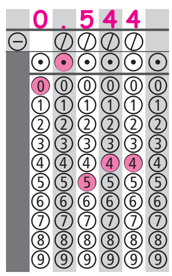
Answers

1. 

2. 

3. 

4. 

5. 

Name ______________________ Class ____________ Date ____________

Enrichment

Area and the Law of Sines

Triangulation is a technique for pinpointing the location of an object. Suppose a searchlight is rotated until an object is illuminated by its beam. Although the angle that the beam makes with the ground or baseline is known, the location of the object is not.

Now suppose a second searchlight, placed a known distance from the first, is rotated until it also illuminates the object. Because the angle it makes with the ground is also known, the Law of Sines can be applied to find the distance to the object. A variation of this method, known as Very Long Baseline Interferometry (VLBI), is used to determine the distance to objects in space.

Use triangulation to solve each of the following problems. Round your answers to two decimal places.

1. Two searchlights on the shore of a lake are located 2200 yd apart at points A and B. A ship in distress is spotted from each searchlight. The beam from the first searchlight makes an angle of 43° with $\overline{AB}$. The beam from the second light makes an angle of 51° with $\overline{AB}$. How far is the ship from the first searchlight? **1713.90 yd**

How far is the ship from the second searchlight?
1504.06 yd

2. Two observatories are located 54 mi apart at points P and Q. In an experiment to measure the speed of light, a mirror is built parallel to the line between the two observatories. A laser is aimed from one observatory to the mirror, where it bounces back and hits a receiver at the second observatory. If the angle between the laser beam and $\overline{PQ}$ at the first observatory is 76°, how far is the mirror from the first observatory? **111.61 mi**

How far is the mirror from the second observatory?
111.61 mi

3. A ship at sea is sailing parallel to the coast. It has recently passed a lighthouse whose beam makes an angle of 55° with the coastline when the light shines on the ship. A pier lies ahead on the coast, and a light from the ship to the pier makes an angle of 61° with the ship's course. If the distance from the lighthouse to the pier is 8 mi, how far is the ship from the lighthouse? **7.78 mi**

How far is the ship from the pier?
7.29 mi

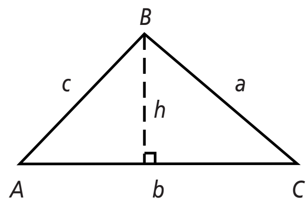
Name ______________________ Class ____________ Date ____________

Reteaching

Area and the Law of Sines

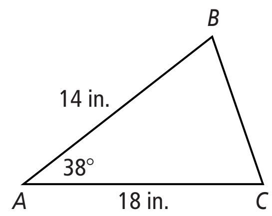
For $\triangle ABC$ shown, there are three ways to calculate the area K.

$K = \frac{1}{2}bc \sin A$ $\qquad$ $K = \frac{1}{2}ac \sin B$ $\qquad$ $K = \frac{1}{2}ab \sin C$



Problem

What is the area of $\triangle ABC$ to the nearest tenth of a square inch?



In the triangle, b is the side opposite $\angle B$ and c is the side opposite $\angle C$. So, $b = 18$ in., $c = 14$ in., and $m\angle A = 38°$.

Because you know b, c, and $\angle A$, use the formula $K = \frac{1}{2}bc \sin A$.

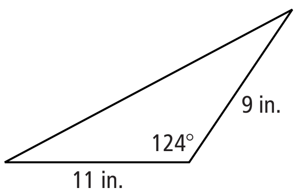
$$K = \frac{1}{2}(18)(14)\sin(38°)$$

$$\approx 77.6$$

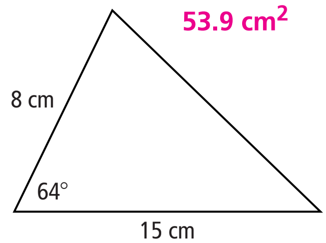
The area of the triangle is about 77.6 in.^{2}.

Exercises

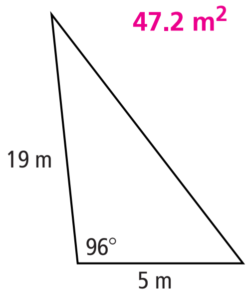
Find the area of each triangle. Round your answer to the nearest tenth.

1. 

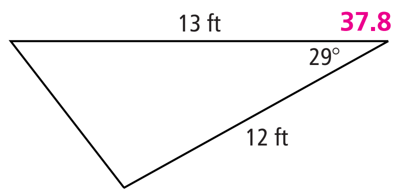
41.0 in.^{2}

2. 

53.9 cm^2

3. 

47.2 m^2

4. 

37.8 ft^2

Name ______________________ Class __________ Date __________

Reteaching (continued)

Area and the Law of Sines

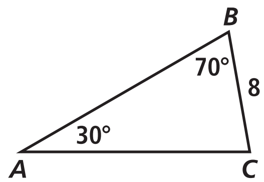
Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Use the Law of Sines when you are given the measure of two angles of a triangle and the length of any side or the measure of two sides and the measure of the angle opposite one of them.

Problem

In $\triangle ABC$, $m\angle A = 30°$, $m\angle B = 70°$, and $BC = 8$. What is AC?

$A = 30°$, $B = 70°$, $BC = 8$ Because we know two angles and a side, use the Law of Sines.



Draw a triangle, and label A, B, and BC.

$\frac{\sin 30°}{8} = \frac{\sin 70°}{AC}$ Substitute values into the Law of Sines formula.

$\frac{0.5}{8} = \frac{0.9397}{AC}$ Use a calculator to find sin 30° and sin 70°.

$0.5AC = 7.52$ Multiply each side by $8AC$.

$AC = 15.04$ Solve for AC.

Exercises

Find the remaining side measure or angle for each triangle. Round your answers to the nearest tenth.

5. Find a if $A = 18°$, $B = 28°$, and $b = 100$.
65.8

6. Find c if $B = 18°$, $C = 152°$, and $b = 4$.
6.1

7. Find a if $C = 16°$, $A = 92°$, and $c = 32$.
116.0

8. Find B if $C = 95°$, $b = 5$, and $c = 6$.
56.1°

9. Find B if $A = 40°$, $b = 6$, and $a = 12$.
18.7°

10. Find c if $A = 50°$, $C = 60°$, and $a = 36$.
40.7

11. Find c if $B = 110°$, $C = 40°$, and $b = 18$.
12.3

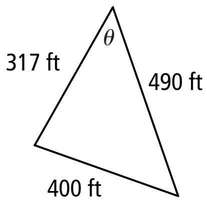
12. Find a if $A = 5°$, $C = 125°$, and $c = 510$.
54.3

Name ______________________ Class ____________ Date ____________

Additional Vocabulary Support

The Law of Cosines

The column on the left shows the steps used to solve a problem using the Law of Cosines. Use the column on the left to answer each question in the column on the right.

Problem **Solve using the Law of Cosines** Three sides of a triangular plot of land measure 317 ft, 490 ft, and 400 ft. What is the measure of the angle formed by the sides that measure 317 ft and 490 ft?	**1.** Read the title of the problem. What process are you going to use to solve the problem? **Answers may vary. Sample: Law of Cosines**
Sketch a picture.  	**2.** How do you know that you can use the Law of Cosines to solve this problem? **Answers may vary. Sample: Three sides are given.**
Choose the form of the Law of Cosines that has cos *A* on one side. $a^2 = b^2 + c^2 - 2bc\cos A$	**3.** Could the other forms of the Law of Cosines be used? Explain. **Answers may vary. Sample: Yes; because the missing angle could have been labeled *B* or *C*, any form will work.**
Substitute. $(400)^2 = (317)^2 + (490)^2 - 2(317)(490)(\cos\theta)$	**4.** Why is 400 substituted for *a*? **Answers may vary. Sample: The value for *a* must be the side opposite the missing angle.**
Simplify and solve. $160{,}000 = 100{,}489 + 240{,}100 - 310{,}660(\cos\theta)$ $160{,}000 = 340{,}589 - 310{,}660(\cos\theta)$ $-180{,}589 = -310{,}660(\cos\theta)$ $0.5813 = \cos\theta$ $\theta = 54.5^\circ$ The angle measures about 54.5°.	**5.** Why is 310,660 not subtracted from 240,100? **Answers may vary. Sample: Because 310,660 and cos θ are multiplied, it cannot be subtracted.** **6.** What operation is used to solve for θ in the last step? **$\cos^{-1}$**

Name ______________________ Class ____________ Date ____________

Think About a Plan

The Law of Cosines

Navigation A pilot is flying from city A to city B, which is 85 mi due north. After flying 20 mi, the pilot must change course and fly 10° east of north to avoid a cloud bank.

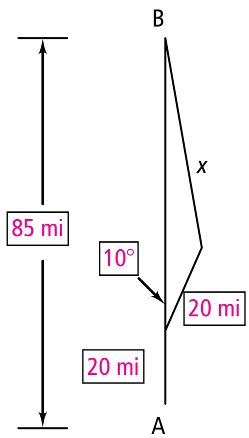
a. If the pilot remains on this course for 20 mi, how far will the plane be from city B?

b. How many degrees will the pilot have to turn to the left to fly directly to city B? How many degrees from due north is this course?

1. How can a diagram help you solve this problem?

Answers may vary. Sample: A diagram can help me understand the angles and side lengths of the triangles in the problem.

2. Fill in the missing information in the diagram.



3. How can you find the length of the other unknown side of the triangle?

Answers may vary. Sample: I can subtract 20 mi from 85 mi.

4. How can the Law of Cosines help you find the length x?

Answers may vary. Sample: I can use the lengths of the other two sides and the 10° angle between them in the Law of Cosines.

5. Find x, the distance the plane is from city B. 45.4 mi

6. How can you find the number of degrees the pilot will have to turn to the left to fly directly to city B?

Answers may vary. Sample: Use the Law of Sines with the value for x, the side with length 65 mi, and the 10° angle. There are two possible triangles, so the solution we want is the angle that is greater than 90°. The angle found will be the supplement of the angle representing the number of degrees the pilot will have to turn.

7. How many degrees will the pilot have to turn to the left to fly directly to city B? 14.4°

8. How can you find the number of degrees this course is from due north?

Answers may vary. Sample: The other angle in the triangle represents the number of degrees the course is from due north. Subtract the sum of the two angles I know from 180°.

9. How many degrees from due north is this course? 4.4°

Name ______________________ Class ____________ Date ____________

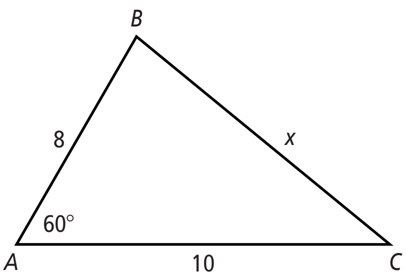
Practice

Form G

The Law of Cosines

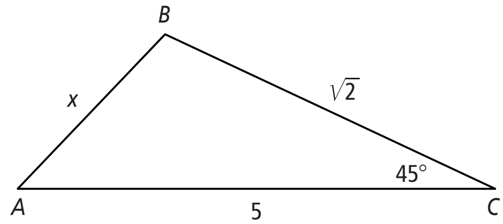
Use the Law of Cosines. Find length x to the nearest tenth.

1.



$x = \sqrt{84} \approx 9.2$

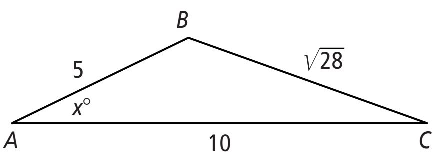
2.



$x = \sqrt{17} \approx 4.1$

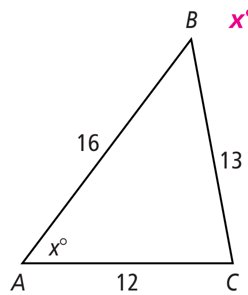
Use the Law of Cosines. Find measure x to the nearest degree.

3.



$x° = 14°$

4.



$x° = 53°$

5. In $\triangle XYZ$, $x = 4$ cm, $y = 7$ cm, and $z = 10$ cm. Find $m\angle X$. **18.2°**

6. In $\triangle FGH$, $f = 32$ in., $g = 79$ in., and $h = 86$ in. Find $m\angle G$. **66.7°**

7. In $\triangle ABC$, $a = 3$ ft, $b = 2.9$ ft, and $c = 4.6$ ft. Find $m\angle C$. **102.4°**

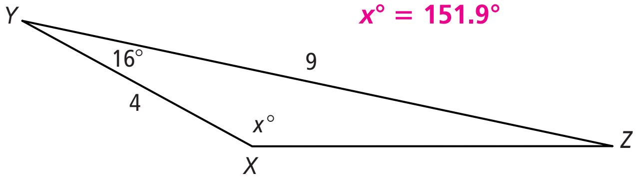
8. In $\triangle FGH$, $f = 34$ m, $g = 18.9$ *m, and* $h = 21.5$ m. Find $m\angle G$. **30.4°**

9. In $\triangle ABC$, $a = 14$ yd, $b = 16$ yd, and $c = 18$ yd. Find $m\angle C$. **73.4°**

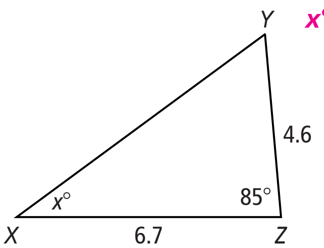
Practice (continued) *Form G*

The Law of Cosines

For Exercises 10–13, use the Law of Cosines and the Law of Sines. Find *x* to the nearest tenth.

10. 

$x° = 151.9°$

11. 

$x° = 36.0°$

12. In $\triangle ABC$, $b = 8$ cm, $c = 7$ cm, and $m\angle A = 149°$. Find $m\angle C$. **14.4°**

13. In $\triangle FGH$, $f = 7$ yd, $g = 22$ yd, and $m\angle H = 85°$. Find $m\angle F$. **18.1°**

14. The sides of a triangular lot are 158 ft, 173 ft, and 191 ft. Find the measure of the angle opposite the longest side to the nearest tenth of a degree. **70.3°**

15. A car travels 50 mi due west from point *A*. At point *B*, the car turns and travels at an angle of 35° north of due east. The car travels in this direction for 40 mi, to point *C*. How far is point *C* from point *A*? **28.7 mi**

16. **a.** In $\triangle ABC$, $m\angle A = 84.1°$, $b = 4.8$, and $c = 7.2$. Use the Law of Cosines to find *a* and then use the Law of Sines to find the measure of angles *B* and *C*. Round to the nearest tenth. **$a = 8.2$; $m\angle B = 35.4°$; $m\angle C = 60.5°$**

b. **Error Analysis** Your classmate says that this triangle does not exist. You say that it does. Who is correct? Explain. **You; the angles of this triangle add up to 180°, which is equal to the sum of the angles of any triangle.**

Name ______________________ Class ____________ Date ____________

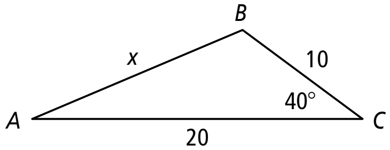
Practice

Form K

The Law of Cosines

Use the Law of Cosines. Find length x to the nearest tenth.

1.



To start, list the known information.
$a = 10$, $b = 20$, $m\angle C = 40°$
Use the form of the Law of Cosines that has c^2 on one side:

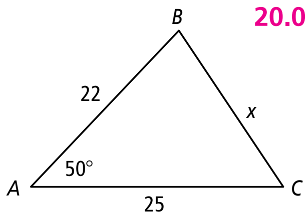
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= \left(\boxed{10}\right)^2 + \left(\boxed{20}\right)^2 - 2\left(\boxed{10}\right)\left(\boxed{20}\right)\cos 40°$$

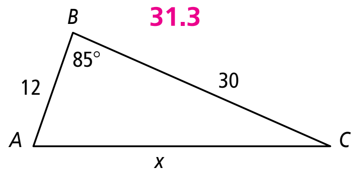
$$\approx \boxed{193.58}$$

$$c \approx \sqrt{\boxed{193.58}} \approx \boxed{13.9}$$

2. 20.0

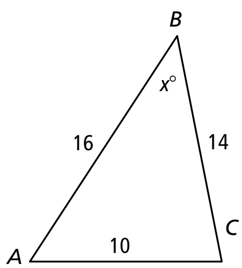


3. 31.3



Use the Law of Cosines. Find measure x to the nearest tenth of a degree.

4.



To start, list the known information. $a = 14$, $b = 10$, $c = 16$

Use the form of the Law of Cosines that contains $\angle B$:
$b^2 = a^2 + c^2 - 2ac\cos B$

$$\boxed{10}^2 = \left(\boxed{14}\right)^2 + \left(\boxed{16}\right)^2 - 2\left(\boxed{14}\right)\left(\boxed{16}\right)\cos B$$

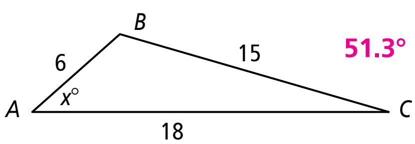
$$\boxed{100} = \boxed{196} + \boxed{256} - \boxed{448}\cos B$$

$$\boxed{-352} = \boxed{-448}\cos B$$

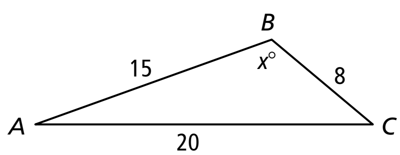
$$\cos B = \boxed{\frac{352}{448}}$$

$$m\angle B = \cos^{-1}\left(\boxed{\frac{352}{448}}\right) \approx \boxed{38.2°}$$

5. 51.3°



6. 117.5°

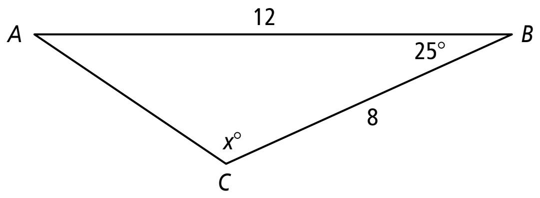


Name ______________________ Class ____________ Date ____________

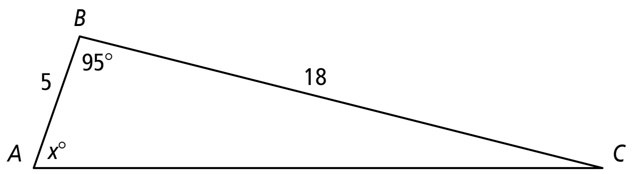
Practice (continued) *Form K*

The Law of Cosines

Use the Law of Cosines and the Law of Sines. Find *x* to the nearest tenth.

7. 

119.6°

8. 

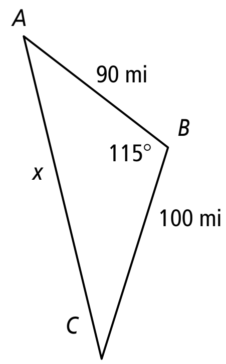
69.9°

9. In $\triangle PQR$, $p = 7$ ft, $q = 9$ ft, and $m\angle R = 55°$. Find $m\angle Q$. **76.0°**

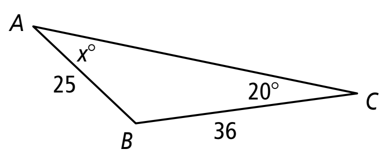
10. In $\triangle DEF$, $e = 30$ m, $f = 50$ m, and $m\angle D = 20°$. Find $m\angle F$. **134.8°**

11. The lengths of the sides of a triangular garden are 10 m, 11 m, and 13 m. Find the measure of the angle opposite the longest side to the nearest tenth of a degree. **76.3°**

12. A map of a county's airports is shown in the diagram to the right. A pilot flies from her home airport at point *A* to an airport at point *B*, and then to an airport at point *C*. The pilot wants to know the distance back to her home airport to decide if she has enough fuel. How far is point *C* from the home airport? Round your answer to the nearest tenth. **160.3 mi**



13. Error Analysis To find $m\angle A$ in $\triangle ABC$ below, a student says that he must first use the Law of Cosines. Is he correct? Explain your reasoning.



Answers may vary. Sample: No; since the known angle is not between the known sides, he cannot use the Law of Cosines. He can directly use the Law of Sines to find $m\angle A$.

Name ______________________ Class ____________ Date ____________

Standardized Test Prep

The Law of Cosines

Multiple Choice

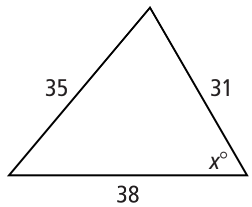
For Exercises 1–5, choose the correct letter.

1. What is length x to the nearest tenth? **A**

(A) 10.7 (C) 39.8

(B) 21.9 (D) 113.9

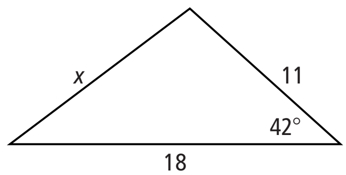
2. What is measure x to the nearest tenth? **H**



(F) 7.7° (H) 59.9°

(G) 50.1° (I) 70.0°

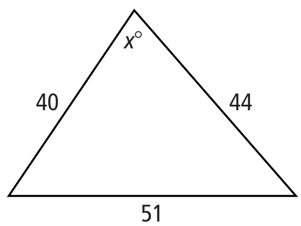
3. What is length x to the nearest tenth? **D**



(A) 27.2 (C) 13.4

(B) 17.3 (D) 12.3

4. What is measure x to the nearest tenth? **I**



(F) 36.5° (H) 56.3°

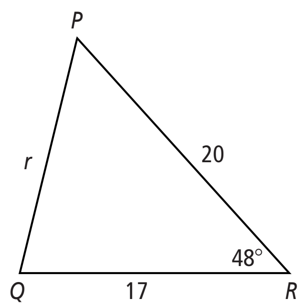
(G) 49.1° (I) 74.6°

5. In $\triangle ABC$, $a = 20$, $b = 12$, and $c = 30$. What is the area of $\triangle ABC$ to the nearest tenth? **B**

(A) 75.5 units^2 (B) 80.5 units^2 (C) 85 units^2 (D) 95 units^2

Short Response

6. Find the remaining side and angles in the triangle.



[2] $r = 15.3$, $m\angle P \approx 55.7°$, $m\angle Q \approx 76.3°$

[1] incorrect length for r OR incorrect measures for $\angle P$ and $\angle Q$

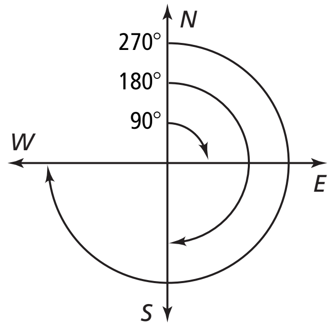
[0] no answers given

Name ______________________ Class __________ Date __________

Enrichment

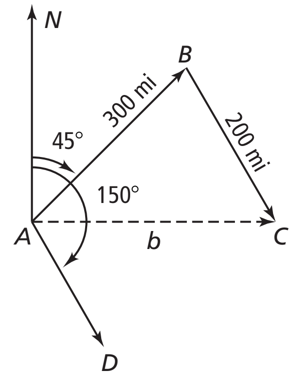
The Law of Cosines

Flight Paths



By using degree measurements to represent compass directions, you can describe the heading, or direction, in which a plane is traveling. In this system, 0° (360°) corresponds to due north, 90° corresponds to due east, 180°corresponds to due south, and 270° corresponds to due west.

Angles are measured in a *clockwise* direction. This is different from measuring angles in standard position on a coordinate system.



With this method you can describe a flight path in terms of distances and headings. For example, suppose a plane flies 300 mi on a heading of 45°; then the plane changes course and flies 200 mi on a heading of 150°.

If you could determine the angle between the two "legs" of the trip ($\angle B$), you could then use the Law of Cosines to find how far the plane has traveled from its point of departure (b).

Because $\overline{BC}$ is parallel to $\overline{AD}$, $\angle B$ is supplementary to $\angle BAD$, which is 105°. Thus $\angle B = 180° - 105° = 75°$. Using the law of cosines, $b^2 = 300^2 + 200^2 - 2(300)(200)\cos 75°$, and $b = 314.6$ mi.

Determine the heading on which the plane would have to travel to return to point *A*.

1. First use the Law of Sines to find $\angle BAC$, and then add 45°.
$m\angle BAC = 37.9°$; $m\angle BAC + 45° = 82.9°$

2. Add 180° to find the return heading.
return heading is 262.9°

Suppose a plane flies 240 mi on a heading of 35°. Then the plane changes course and flies 160 mi on a heading of 160°.

3. Determine the heading on which the plane would have to travel to return to its point of origin. **return heading is 256.5°**

Name ________________________________ Class ____________ Date ____________

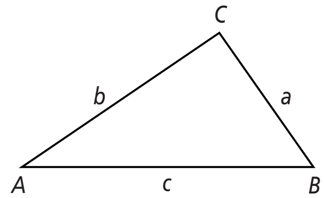
Reteaching

The Law of Cosines

You can use the Law of Cosines to solve word problems involving triangles.

Law of Cosines
In $\triangle ABC$,
a is the length of the side opposite $\angle A$,
b is the length of the side opposite $\angle B$, and
c is the length of the side opposite $\angle C$.

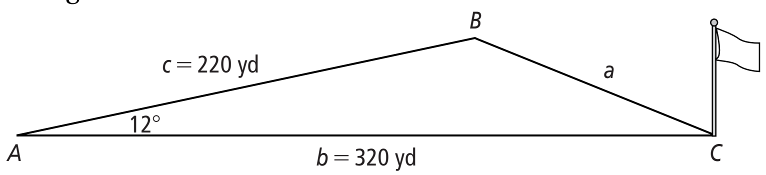
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$



Problem

A golfer hits a golf ball 220 yd on a hole that is 320 yd long. His shot is 12° off of his target. What is the distance from the golf ball to the hole?

Step 1 Draw a diagram.



Step 2 Determine which measure you want to find. The distance from the golf ball to the hole is a.

Step 3 Decide which form of the Law of Cosines to use.

$a^2 = b^2 + c^2 - 2bc \cos A$ — Use the form that has a^2 on one side.

$a^2 = (320)^2 + (220)^2 - 2(320)(220) \cos 12°$ — Substitute: $b = 320$, $c = 220$, $m\angle A = 12°$

$a \approx 114$ — Simplify.

The golf ball is about 114 yd from the hole.

Exercises

1. A golfer hits a golf ball 180 yd on a hole that is 240 yd long. His shot is 8° off of his target. What is the distance from the golf ball to the hole? **about 66.6 yd**

2. After a strong storm, a sapling is leaning at an angle of 97° with the ground. A nursery worker attaches a guy wire to a strap around the tree at a height of 4 ft. He attaches the guy wire to a stake and drives it into the ground 6 ft from the base of the tree. How long is the guy wire? **about 7.6 ft**

Name ______________________ Class ____________ Date ____________

Reteaching (continued)

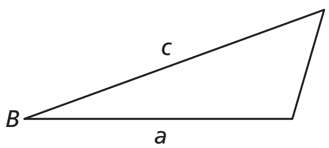
The Law of Cosines

- To determine whether to use the Law of Sines or the Law of Cosines, look at the given information.
- If given the measure of two sides for a triangle and the angle between them or the measure of all three sides, use the Law of Cosines.
- If given the measure of two angles of a triangle and the length of any side or the measure of two sides and the measure of the angle opposite one of them, use the Law of Sines.

Problem

The length of two sides of a triangle are 120 and 100, and the measure of the angle between them is 20°. What is the length of the third side?

$B = 20°$, $a = 120$, $c = 100$ — Because *B* is between *a* and *c*, use the Law of Cosines.



Draw a triangle and label *a*, *c*, and *B*.

$b^2 = 120^2 + 100^2 - 2(120)(100)\cos 20°$ — Insert the values into the Law of Cosines formula. Note that the side you are finding the length of and that side's opposite angle are on opposite ends of the equation.

$b^2 = 14{,}400 + 10{,}000 - (24{,}000)(0.93969)$ — Use a calculator to find cos 20°.

$b^2 = 1847.4$ — Simplify.

$b = 42.98$ — Find the square root of each side.

Exercises

Find each value to the nearest tenth.

3. Find a if $A = 18°$, $c = 72$, and $b = 100$.
38.6

4. Find c if $a = 15$, $C = 152°$, and $b = 4$.
18.6

5. Find A if $B = 45°$, $a = 9$, and $c = 19$.
26.7°

6. Find a if $A = 16°$, $b = 92$, and $c = 32$.
61.9

7. Find B if $a = 9$, $b = 3$, and $c = 11$.
12.9°

8. Find c if $C = 30°$, $a = 15$, and $b = 15$.
7.8

9. Find C if $B = 95°$, $a = 5$, and $c = 6$.
47.3°

10. Find B if $a = 18$, $b = 26$, and $c = 15$.
103.6°