1.2 FUNCTIONS AND THEIR PROPERTIES

Learning Targets for 1.2

1. Determine whether a set of numbers or a graph is a function
2. Find the domain of a function given an set of numbers, an equation, or a graph
3. Describe the type of discontinuity in a graph as removable or non-removable
4. For a given function, describe the intervals of increasing and decreasing
5. Label a graph as bounded above, bounded below, bounded, or unbounded.
6. Find all relative extrema on a graph
7. Understand the difference between absolute and relative extrema
8. Describe the symmetry of a graph as odd or even

This section is full of vocabulary (obvious from the list above?). We will be investigating functions, and you will need to answer questions to determine how each of these properties are applied to various functions.

Function Definition and Notation

In the last chapter, we used the phrase "y is a function of x". But what is a function? In Algebra 1, we defined a function as a rule that assigned one and only one (a unique) output for every input. We called the input the *domain* and the output the *range*. Usually, the set of possible x-values is the domain, and the resulting set of possible y-values is the range.

Definition: Function

A **function** from a set D to a set R is a rule that assigns a unique element in R to each element in D.

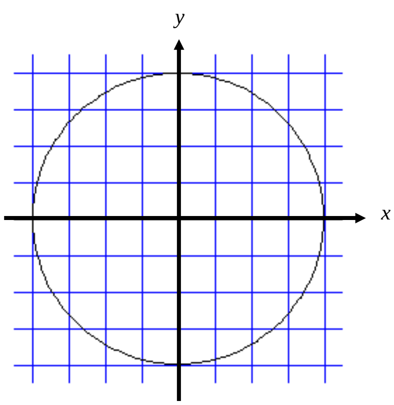
To determine whether or not a graph is a function, you can use the **vertical line test**. If any vertical line intersects a graph more than once, then that graph is NOT a function.

Example 1: A relation is ANY set of ordered pairs. State the domain and range of each relation, then tell whether or not the relation is a function.

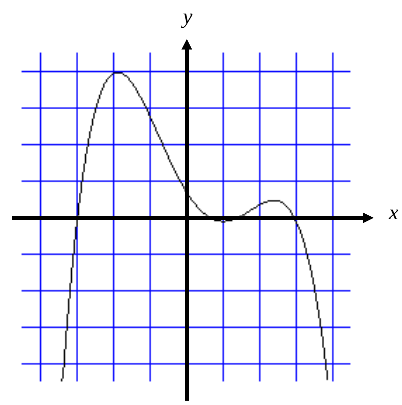
a) $\{(-3,0),(4,2),(2,-6)\}$

b) $\{(4,-2),(4,2),(9,-3)(-9,-3)\}$

c)



d)



For many functions, the domain is all real numbers, or $\mathbb{R}$. We typically start with this, and then see if there are any values of x that cannot be used.

The domain of a function can be restricted for 3 reasons that you need to be aware of in this course:

1. $\sqrt{\text{NO negatives}}$ … no negative numbers inside a square root
2. $\dfrac{**}{\text{NO zeros}}$ … no zeros in the denominator of a fraction
3. $\log(\text{NO negatives ... and NO zeros})$. … no negatives and no zeros inside a logarithm

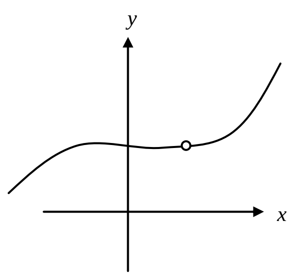
Example 2: Without using a graphing calculator, what is the domain of each of the following functions?

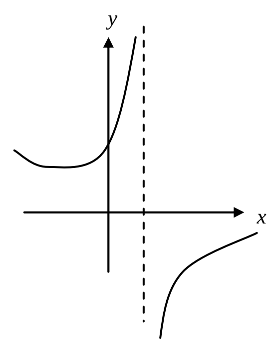
a) $y = \log x$ b) $h(x) = \dfrac{x^2 - 9}{x+3}$ c) $y = \dfrac{\sqrt{x+2}}{x^2 - x - 12}$

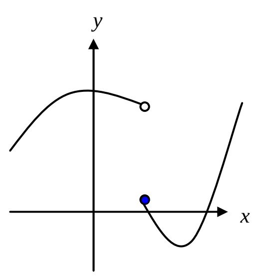
Continuity

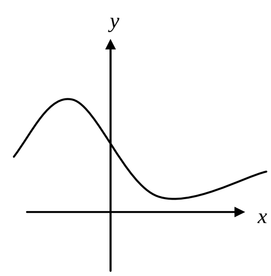
In non – technical terms, a function is continuous if you can draw the function "without ever lifting your pencil".

Example 3: Tell whether the following graphs are continuous or discontinuous. If discontinuous, label each type of discontinuity as removable or non-removable.





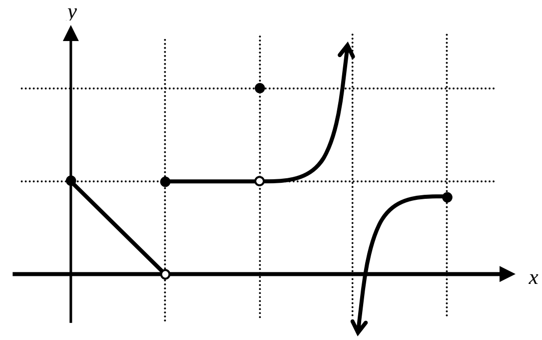




Increasing/Decreasing Functions

While there are technical definitions for increasing and decreasing, just remember to read the graph LEFT to RIGHT.

Example 4: Using the graph below, what intervals is the function increasing? Decreasing? Constant?



Boundedness

You need to understand the difference between the following terms:

BOUNDED BELOW:

BOUNDED ABOVE:

BOUNDED:

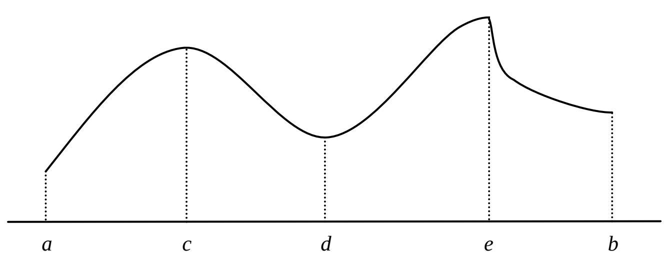
Local and Absolute Extrema

Extrema is the plural form of one extreme value. Extrema is one word that includes maximums and minimums.

Local (or Relative) Extrema are y-values bigger (or smaller) than ____.

Absolute Extrema are y-values bigger (or smaller) than ____.

Example 5: Suppose the following function is defined on the interval $[a, b]$. Label $f(a)$ through $f(e)$ as relative or absolute extrema.



Symmetry

The next topic we concern ourselves with when dealing with functions is the idea of symmetry. Symmetry on a graph means the functions "look the same" on one side as it does on another. We are most concerned with the types of symmetry that can be explored numerically and algebraically in terms of ODD and EVEN functions.

3 Types of Symmetry

Symmetry with respect to the *y*-axis: **EVEN FUNCTIONS**

Graphically Numerically Algebraically

Symmetry with respect to the origin: **ODD FUNCTIONS**

Graphically Numerically Algebraically

Symmetry with respect to the *x*-axis: **NOT A FUNCTION**

Graphically Numerically Algebraically

1.2 FUNCTIONS AND THEIR PROPERTIES

Learning Targets for 1.2
9. Prove a function is odd, even, or neither
10. Given a rational equation, find vertical asymptotes and holes (if they exist)
11. Given a rational equation, describe the end behavior using infinity notation and limit notation.

Odd vs Even Functions

While there are many functions out there that are neither even nor odd, your concern with odd and even functions is twofold …

#1: Identify graphs of functions that are Odd or Even

#2: PROVE a function is Odd or Even

While the first item above can be done graphically, numerically, or algebraically, the second is done ONLY algebraically.

Example 6: Prove each function is odd, even, or neither.

a) $f\left(x\right)=5x^2+5$

b) $g\left(x\right)=x^2+8x+12$

Vertical Asymptotes

Example 7: Graph the function $f\left(x\right)=\dfrac{1}{x+3}$. What happens at $x = -3$? … Why?

Example 8: Graph the function $g\left(x\right)=\dfrac{x^2-9}{x+3}$. What happens at $x = -3$? … Why?

Vertical asymptotes occur in rational functions when ____.

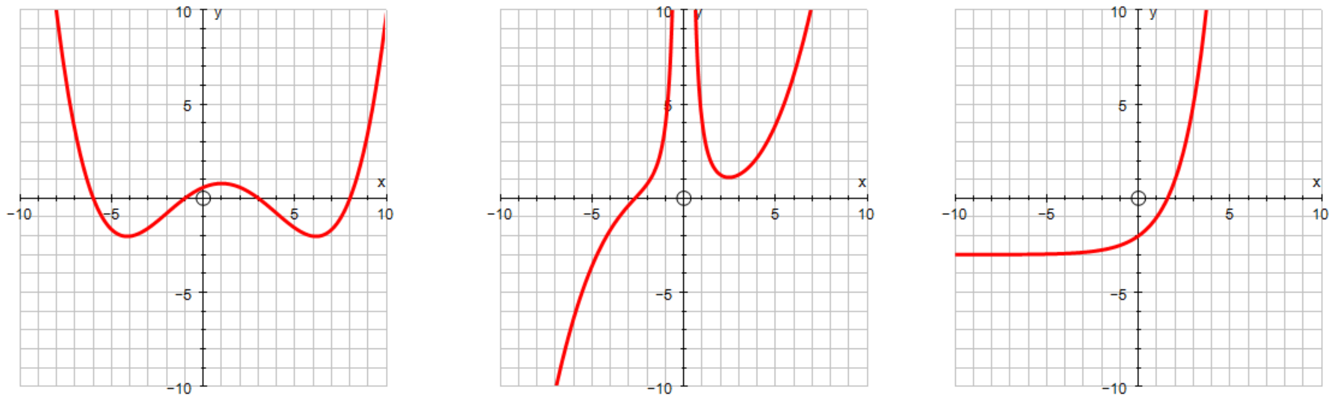
If you plug a point into your function and get _________, you should look for a removable discontinuity … a.k.a. "a hole".

End Behavior

As $x \to \pm\infty$, we say the *end behavior* of the function is a description of what value $f(x)$ approaches.

For now, we will focus on finding end behavior using a graphical approach.

Example 9: Describe the end behavior of each graph shown below.



We can write the end behavior in a more compact form using *limit notation*. There is a specific (a.k.a. correct) way of writing limits that you should learn.

Example 10: Rewrite the given end behavior for $f(x)$ using limit notation. Sketch a possible graph for $f(x)$.

a) As $x \to \infty, f(x) \to -\infty$
As $x \to -\infty, f(x) \to \infty$

b) As $x \to \infty, f(x) \to \infty$
As $x \to -\infty, f(x) \to 0$

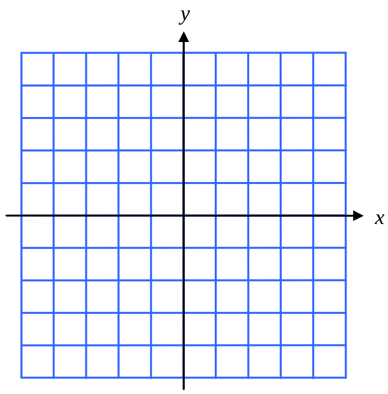
IF $f(x) \to$ "a number" as $x \to \pm\infty$ … meaning the end behavior of the function is a number, we say the function has a *horizontal asymptote* (at that number) … more on this in section 2.7.

1.3 TWELVE BASIC FUNCTIONS

Lesson Targets for 1.3
1. Graph and Identify all 12 parent functions
2. Graph a piecewise function (covered in class)

Parent Function #1: Linear Function (book refers to this as the identity function): Equation: ____________

Graph this function (label 5 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Domain:

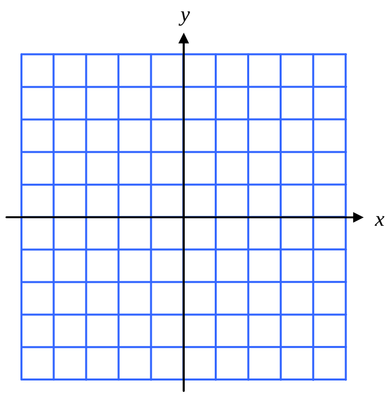
Extrema:

Range:

End Behavior:

Parent Function #2: Quadratic Function (book refers to this as the squaring function): Equation: _____________

Graph this function (label 5 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Domain:

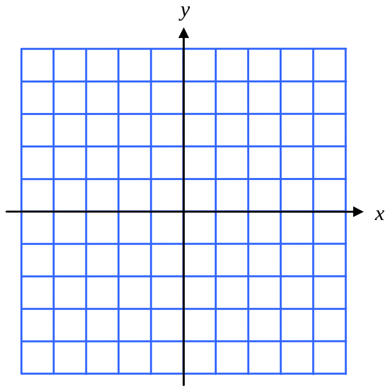
Extrema:

Range:

End Behavior:

Parent Function #3: Cubic Function (the book refers to this as the cubing function) : Equation: _____________

Graph this function (label 3 points)



Domain:

Range:

Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

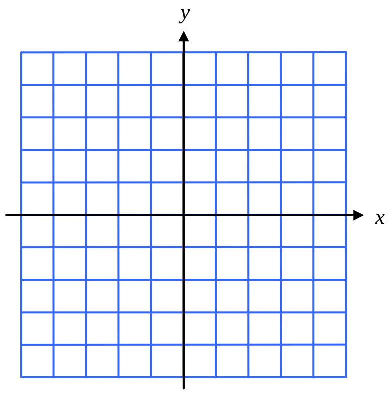
Increasing/Decreasing:

Extrema:

End Behavior:

Parent Function #4: Inverse Linear Function (book refers to this as the Reciprocal Function): Equation: _____________

Graph this function
(label 2 points, a H.A., and a V.A.)



Domain:

Range:

Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

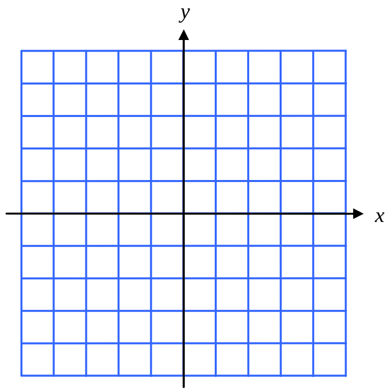
Increasing/Decreasing:

Extrema:

End Behavior:

Parent Function #5: Square Root Function: Equation: _____________

Graph this function (label 3 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Extrema:

End Behavior:

Domain:

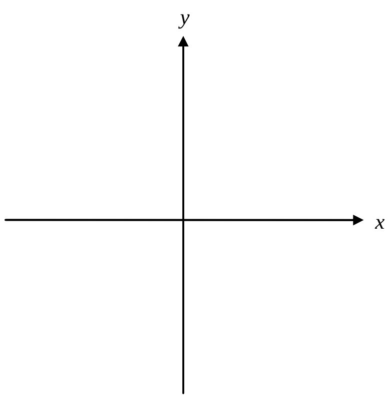
Range:

Parent Function #6: Exponential Function: (the book uses only base *e*)

We will use the equation $f(x) = b^x$,

where $b > 1$ represents ____________, and $0 < b < 1$ represents _______________.

Graph this function (label 2 points and a H.A.)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Extrema:

End Behavior:

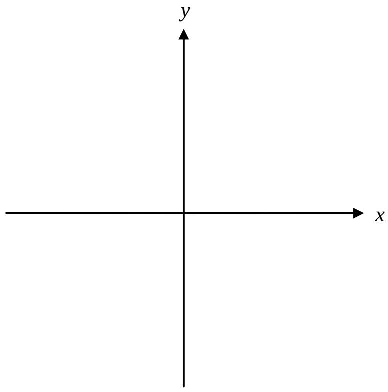
Domain:

Range:

Parent Function #7: Logarithm Function: (the book only uses a natural logarithm)

We would like you to use $f(x) = \log_b x$

Graph this function (label 2 points and a V.A.)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Domain:

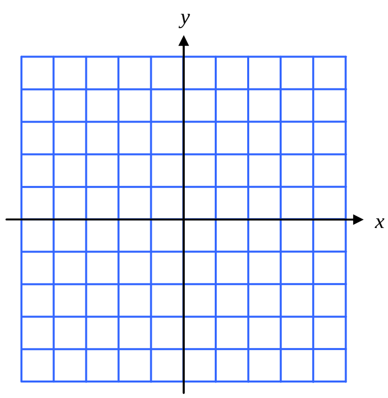
Extrema:

Range:

End Behavior:

Parent Function #8: Absolute Value Function: Equation: ______________

Graph this function (label 5 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Domain:

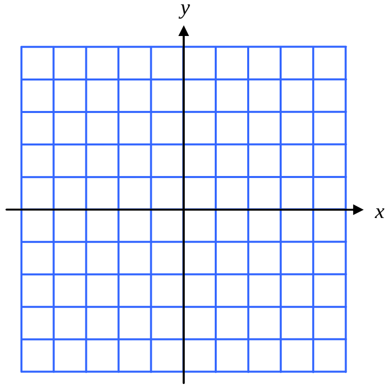
Extrema:

Range:

End Behavior:

Parent Function #9: Greatest Integer Function: Equation: _____________

Graph this function (label at least 6 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Extrema:

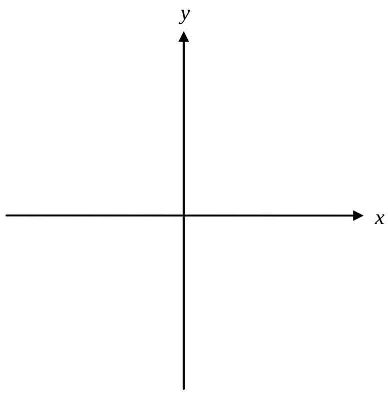
End Behavior:

Domain:

Range:

Parent Function #10: Logistic Function: Equation: ______________

Graph this function (label 1 point and 2 H.A.)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Extrema:

End Behavior:

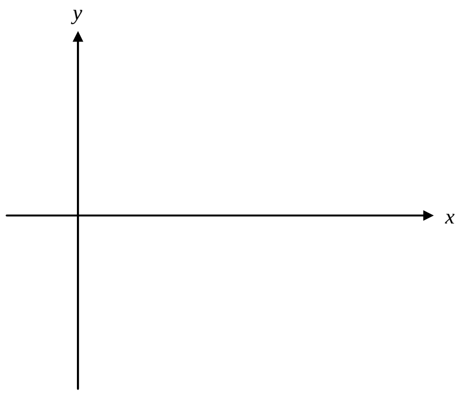
Domain:

Range:

We will spend a large amount of time with the next two during second semester. For now, the parent function is enough.

Parent Function #11: Sine Function: Equation: $y = \sin(x)$

Graph this function



Domain:

Range:

Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

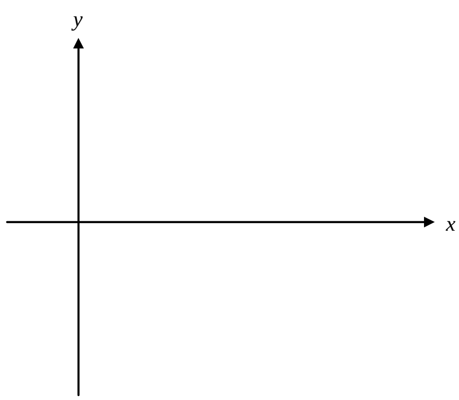
Increasing/Decreasing:

Extrema:

End Behavior:

Parent Function #12: Cosine Function: Equation: $y = \cos(x)$

Graph this function



Domain:

Range:

Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Extrema:

End Behavior:

1.5 GRAPHICAL TRANSFORMATIONS

Learning Targets for 1.5
1. Be able to graph transformations of parent functions
2. Be able to adjust the function when there is a horizontal stretch/shrink AND a left/right movement.

You MUST be able to graph the parent functions from 1.3 in order to successfully transform them.

A transformation is a stretch/shrink, a reflection, or simple movement of a parent function horizontally or vertically.

The general rule … "Inside … x's … opposite" and "Outside … y's … same"

Example 1: Suppose you are given the function $f(x)$. If a, b, c, and d are real numbers, our transformed function is

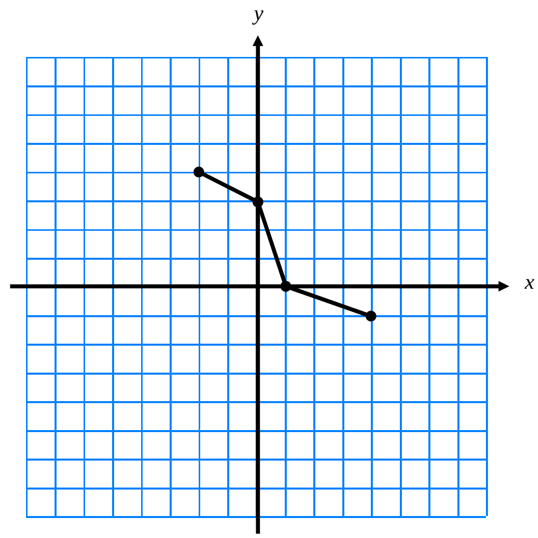
$$a \cdot f\left(b\left(x+c\right)\right)+d$$

The box below summarizes our transformation rules:

	Inside	Outside
$+/-$	$f(x+c)$ $f(x-c)$	$f(x)+d$ $f(x)-d$
$\times/\div$	$f(bx)$ $f\left(\frac{1}{b}x\right)$ $f(-x)$	$a \cdot f(x)$ $\frac{1}{a} \cdot f(x)$ $-f(x)$

♬ : Make sure you ________________________ before you ____________________________.

Example 2: Perform the following transformations on the graph of $f(x)$ below:



a) $f(x-3)+2$

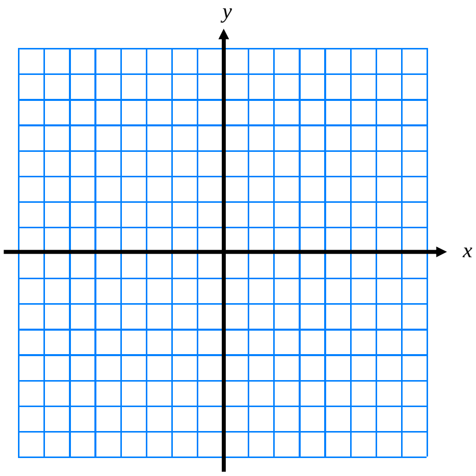
b) $-f\left(\dfrac{x}{2}\right)$

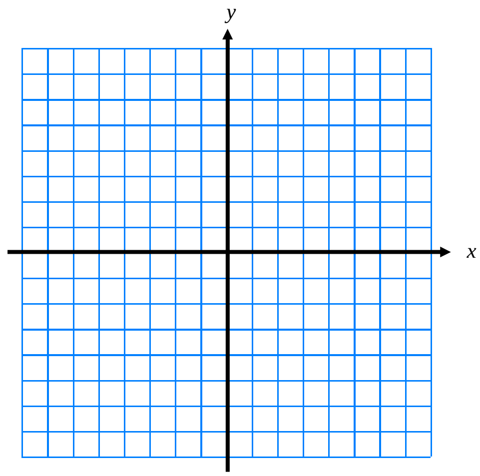
c) $f(2(x+3))$

Example 3: Graph the following functions by transforming their parent functions: **MOST OFTEN MISSED**

a) $f(x)=\sqrt{3x-6}$

b) $g(x)=\log_3(5-x)$





1.4 BUILDING FUNCTIONS FROM FUNCTIONS

Learning Target for 1.4
1. Be able to Add, Subtract, Multiply, and Divide two functions
2. Be able to Compose two functions
3. Find an Inverse Functions Graphically, Numerically, and Algebraically
4. Prove (or Verify) two functions are Inverses

Creating functions from other functions can be done in a variety of ways. We are going to add, subtract, multiply, divide, and compose functions to create new functions.

Example 1: Express in function notation what the following expressions mean:

a) $(f+g)(x)$ b) $(f-g)(x)$ c) $(fg)(x)$ d) $\left(\frac{f}{g}\right)(x)$

Example 2: Consider the two functions $f(x)=6-x$ and $g(x)=\sqrt{x+2}$. What is the Domain of each?

a) Find $(g-f)(x)$. State the Domain of the new function.

b) Find $\left(\frac{f}{g}\right)(x)$. State the Domain of the new function.

<u>*Composite Functions*</u>

When the range of one function is used as the domain of a second function we call the entire function a composite function.

We use the notation $(f \circ g)(x) = f(g(x))$ to describe composite functions.

This is read as "*f* composed with *g*" or "*f* of *g* of *x*".

The function f in the example above is an example of a composite function. The linear function “*x* + 2” is applied first, then the square root function.

Example 3: Suppose $f(x)=1-x^2$ and $g(x)=\sqrt{x}$.

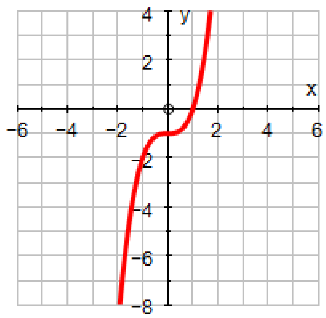
a) Find $g(f(x))$. What is the domain of $g(f(x))$?

b) Find $f(g(x))$. What is the domain of $f(g(x))$?

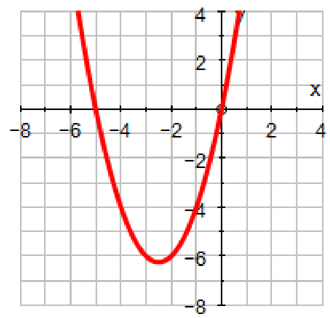
Inverse Functions

A function has an inverse function if and only if the original function passes the Horizontal Line Test. The Horizontal Line Test works just like the Vertical Line Test (it's just horizontal ☺). All an inverse function does is switch the *x* and *y* or the domain and range.

Example 4: The graphs of two functions are shown below. Do they have an inverse? Why or why not?



$h(x) = x^3 - 1$



$k(x) = x^2 + 5x$

Once we know whether a function has an inverse, our next task is to find an equation and/or a graph for the inverse.

Finding the Inverse Graphically: Reflect the graph of the original function over the line $y = x$.

Finding the Inverse Numerically: Plot the reverse of the coordinates.

Finding the Inverse Algebraically (… THIS is our focus): Switch the *x* and *y* in the original equation, then solve the new equation for *y* in order to write *y* as a function of x. Your NEW *y* can be written as $f^{-1}(x)$

Example 5: Let $f(x) = x^3 - 1$, find the inverse algebraically.

Verifying Inverses

It is one thing to find the inverse function (either graphically or algebraically), but it is another to verify that two functions are actually inverses. Whenever you are verifying anything in mathematics, you must go back and use the definition.

> *Definition: Inverse Function*
>
> A function $f(x)$ has an inverse $f^{-1}(x)$ if and only if $f\left(f^{-1}(x)\right) = x = f^{-1}\left(f(x)\right)$

Example 6: According to this definition, how many composite functions must be used to verify inverses?

Example 7: **<u>Verify</u>** that the function you found in example 5 is in fact the inverse function.

Example 8: Find $f^{-1}(x)$ and verify if $f(x)=\dfrac{x+3}{x-2}$. … YEAH … you'll probably need most of this space ☺.

1.6 MODELING WITH FUNCTIONS

Lesson Targets for 1.6

1. Change English statements into mathematical expressions
2. Write equations to model given situations.
3. Use equations to solve percentage and mixture problems.

Example 1: Write a mathematical expression for the quantity described verbally.
(An expression has no equal sign, and can therefore NOT be solved.)

a) A number x decreased by six and then doubled.

b) A salary after a 4.4% increase, if the original salary is x dollars.

Example 2: The diameter of a right circular cylinder equals half its height. Write the volume of the cylinder as a function of its height. The volume of a right circular cylinder is given by $V = \pi r^2 h$.

Example 3: For each statement below, do the following:

1. Write an equation (be sure to define any variables used).
2. Solve the equation, and answer the question.

a) One positive number is twice another positive number. The sum of the two numbers is 390. Find the two numbers.

b) Joe Pearlman received a 3.5% pay decrease. His salary after the decrease was $27,985. What was his salary before the decrease?

c) Investment returns Jackie invests $25,000 part at 5.5% annual interest and the balance at 8.3% annual interest. How much is invested at each rate if Jackie receives a 1-year interest payment of $1571?

d) The chemistry lab at the University of Hardwoods keeps two acid solutions on hand. One is 20% acid and the other is 35% acid. How much 20% solution and how much 35% acid solution should be used to prepare 25 liters of a 26% acid solution?